A LOCALLY COMPACT CONNECTED GROUP ACTING ON THE PLANE
HAS A CLOSED ORBIT

BY

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The theorem of the title has its origin in a question concerning topological semigroups: Suppose $S$ is a topological semigroup with identity $1$ on a manifold. It is known that the set $H(1)$ of all elements having an inverse with respect to $1$ is a Lie group $[5]$. Let $G$ be the component of the identity of $H(1)$ and let $L$ be the boundary of $G$. The question arises whether $L$ (if non-empty) necessarily contains an idempotent. This was shown to be so in $[5]$ if $S$ is a plane. We had recently shown that this is so if $S$ is Euclidean three-space and $L$ is topologically a plane. For each case, use was made of the Lemma 2.5 in $[5]$ that if $G$ has a closed left orbit and a closed right orbit in $L$ then $L$ contains an idempotent. The question was thus raised whether a connected Lie group can act on the plane without a closed orbit. Using a technique developed to prove the result in the second of the two cases above, together with the result of Professor Hofmann in $[2]$ of this journal, we prove that it cannot. Finally, we include an argument sent to us by Professor Hofmann which extends this theorem to locally compact connected groups.

A result which we use repeatedly is Theorem 2 of $[3]$. This asserts that if a one-parameter group $P$ acts as a transformation group on the plane and an orbit $Pz$ is unbounded in both directions (that is, $Pz$ is topologically a line and neither component of $Pz \setminus \{x\}$ has compact closure) then $Pz = z$ for all $z \in (Pz) \setminus Pz$. In fact, this result is needed under the assumption that $P$ acts, not on the entire plane $E$, but only on a closed subset of $E$. An examination of the proof in $[3]$ reveals that this is actually what is proved. This theorem allows us to obtain a closed orbit when an orbit exists which is unbounded in both directions. In case every orbit $Pz$ has one of its ends bounded we apparently need the following lemma. (By an "end" of $Pz$ is meant one of the components of $Pz \setminus \{x\}$.)

**Lemma 1.** Let $S$ be a subset of the plane and suppose that the multiplicative group $P$ of positive real numbers acts as a transformation group of $S$. Let $R$ denote one of the components of $P \setminus \{1\}$. Suppose $x, y, z$ are points of $S$ such that

$$y \in (Rx) \setminus Pz \quad \text{and} \quad z \in (Ry) \setminus Py.$$ 

Then $Pz = z$.

**Proof.** Order $P$ so that $p > 1$ if and only if $p \in R$. Assume $Pz \neq z$. Then there exists an interval $A = [a, b], a < 1 < b$, such that the map $p \to pw$ is a

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homeomorphism on $A$ for each $w$ sufficiently close to $z$. Since $z \in (Ry) \setminus Py$, there is an unbounded sequence $\{p_n\}$, $p_n \in R$, such that $p_n \rightarrow z$. Let $D$ be a small disc about $z$. We may suppose that each of the arcs $Az$ and $Ap_n$, $n = 1, 2, \cdots$, cuts $D$. There exists a sub-arc of $Az$ in $D$ which contains $z$ and separates $D$ into exactly two components. One of these will contain an infinite number of the points $p_n$. Let $E$ denote such a component and assume, without loss of generality that each $p_n \in E$.

Choose a point $q$ in the boundary of $E$ but not in $Az$. Let $B$ be an arc from $z$ to $q$ lying, except for its end points, in the interior of $E$. Say that an arc goes across $E$ if it has its end points on the boundary of $E$ but not in $Az$, if these end points are on opposite sides of $B$ and if, except for its end points, it is contained in the interior of $E$.

Consider the collection of components of the intersection of $Ry$ with the interior of $E$. The closure of each member of this collection is a sub-arc of $Ry$. Let $C$ denote the collection of these arcs which go across $E$. It is clear that $C$ can be linearly ordered so that later terms of $C$ are nearer $Az$ than earlier terms. Even more: each member of $C$ has an immediate successor and the members of $C$ may be arranged in a sequence $C_1, C_2, \cdots, C_n, \cdots$ so that in general $C_{n+1}$ is the successor of $C_n$. Now choose a (possibly new) sequence $r_n \in Ry$ so that $r_n \rightarrow z$ and $r_n \in C_n$ for each $n$. Since each $r_n \in (Ry) \setminus Py$, there exists a sequence $\{q_n\}$, $q_n \in R$, such that $q_n \rightarrow z$ and $Ar_n$ and $Ar_{n+1}$ and so that $q_n \rightarrow z$. The collection of arcs $Az$, $\{Ar_n\}$ and $\{Aq_n\}$ taken together forms an equicontinuous collection of arcs (see the proof of Theorem 1 of [2], for example). Therefore, by [1], we may assume that each of these arcs is a straight line segment.

The discussion is now facilitated somewhat by thinking of $Az$ as lying along the X-axis with $z$ at the origin and $az$ "to the left" of the origin. Let $a_1, b_1 \in P$ be such that $a < a_1 < 1 < b_1 < b$. Let $L_1$ be the line perpendicular to $Az$ at $a_1z$ and let $L_2$ be the line perpendicular to $Az$ at $b_1z$. There is no loss in generality in assuming that each $ar_n$ and each $aq_n$ lies to the left of $L_1$ and each $br_n$ and each $bq_n$ lies to the right of $L_2$.

An arc will be said to cross $L_1 L_2$ if it has one end point on $L_1$, the other end point on $L_2$ and except for these points is contained between $L_1$ and $L_2$. An arc will be said to cross $L_1 L_2$ in the right direction if it is a sub-arc of an arc having the form $Av$ for some $v \in E$, if it crosses $L_1 L_2$ and if in the order it inherits from $A$, the smallest point is on $L_1$ and the largest point is on $L_2$. If for such an arc the largest point is on $L_1$ and the smallest point is on $L_2$, it will be said to cross $L_1 L_2$ in the wrong direction.

Let $c_n$ denote the intersection of $L_1$ and $Ar_n$ and let $d_n$ denote the intersection of $L_2$ and $Ar_n$. Let $Q_n$ denote the quadrilateral whose vertices are $c_n$, $d_n$, $c_{n+1}$ and $d_{n+1}$. There are now two cases according to whether $c_n > c_{n+1}$ or $c_{n+1} > c_n$. We consider the first case. The argument is similar in the second case and is omitted. Since $C_{n+1}$ is the successor of $C_n$, no sub-
arc of \([d_n, d_{n+1}]y\) crosses \(L_1 L_2\) between \(Ar_n y\) and \(Ar_{n+1} y\). Furthermore there exist arbitrarily large integers \(n\) such that \(q_n x\) is between \(Ar_n y\) and \(Ar_{n+1} y\). Choose such an integer. By [4, p. 173] there is an arc \(T\) joining \(d_n y\) to \(d_{n+1} y\) which, except for its end points, is contained in \(Q_n\), and which has only end points in common with \([d_n, d_{n+1}]y\). Similarly there is an arc \(S\) joining \(c_n y\) to \(c_{n+1} y\) which, except for its end points, is contained in \(Q_n\), which misses not only \([d_n, d_{n+1}]y\) but \(T\) as well. The arcs \(S\) and \(T\) can be chosen to be polygonal and to intersect \(Ag_n x\) in one point each. Let \(C\) be the simple closed curve formed by joining \(T\) and \([d_n, d_{n+1}]y\). Then \(Ag_n x\) has points on opposite sides of \(C\). Since \(Ag_n x\) intersects \(C\) in only one point, its end points are on opposite sides of \(C\). Certainly the intersection of \(Ag_n x\) and \(S\) is on the opposite side of \(C\) from \(bq_n x\). Since \(S\) has no points in common with \(C\), \(c_n y\) and \(bq_n x\) lie on opposite sides of \(C\). Again, since \([ar_n, c_n]y\) has no points in common with \(C\), \(ar_n y\) and \(bq_n x\) lie on opposite sides of \(C\). Since \(ar_n y \in (Rx)^- \setminus Px\), there exist numbers \(r, s \in R\) such that \(s > r > bq_n\) and such that \(r x\) is on \(L_2\) while \(sx\) is on \(L_1\). Thus there is a sub-arc of \([r, s]x\) which crosses \(L_1 L_2\) in the wrong direction.

We sketch a proof that all points in \(S\) sufficiently near \(z\) must lie on arcs which cross \(L_1 L_2\) in the right direction. Since the preceding result contradicts this fact, we conclude that \(Pz = z\) and the proof of the lemma will be complete.

Let \(D_1, D_2\) be two discs whose radii are slightly larger than one-half the distance from \(L_1\) to \(L_2\) and whose centers are at \(a_1 z\) and \(b_1 z\) respectively. Let \(cz\) be a point in the intersection of the interiors of \(D_1\) and \(D_2\). It is sufficient for our purposes to assume \(az \in D_1\) and \(bz \in D_2\) so that \([a, c]z \subset D_1\) and \([c, b]z \subset D_2\). Corresponding to each \(t \in [a, c]\) there is an interval \(V\) containing \(t\) and a neighborhood \(W\) of \(z\) such that \(VW \subset D_1\). By compactness, there exist a finite number of intervals \(V_1, \ldots, V_n\) which cover \([a, c]\) and an open set \(W_1\), containing \(z\) such that \(V_i W_1 \subset D_1\) for each \(i = 1, \ldots, n\). We may evidently assume \(c \in V_n\) and that in fact \(V_n W_1 \subset D_1 \cap D_2\). Similarly there exist open intervals \(V_1, \ldots, V_n\) covering \([c, b]\) and an open set \(W_2\) containing \(z\) such that \(V_i W_2 \subset D_2\) for \(i = 1, \ldots, m\). If \(w \in W_1 \cap W_2\) then, since \(Aw = [a, c]w \cup [c, b]w\), \(Aw\) contains no sub-arc which crosses \(L_1 L_2\) in the wrong direction. The proof of the lemma is complete.

**Lemma 2.** Let \(G\) be a connected Lie group acting as a transformation group on a space \(M\). Let \(x \in M\) and let \(P\) be a one-parameter subgroup of \(G\). Then the following are equivalent.

1. \(Gx \neq \{x\}\) and \(Gx = Px\);
2. \(\dim Gx = 1\) and \(P\) has no conjugate in the isotropy subgroup \(G_z\) of \(G\).

**Proof.** Suppose \(Gx \neq \{x\}\) and \(Gx = Px\). Then \(\dim Gx = 1\). If \(gPg^{-1} \subset G_z\) for some \(g \in G\) then \(P \subset g^{-1}G_z g\) so \(P g^{-1}x = g^{-1}x\). However \(g^{-1}x = px\) for some \(p \in P\) so \(Ppx = px\). On the other hand \(Pp = P\) while \(Px = Gx \neq px\). Thus \(P\) has no conjugate in \(Gx\).
Suppose \( \dim Gx = 1 \) and that \( P \) has no conjugate in \( G_x \). Let \( G \) act on the left coset space \( G/G_x \) by left multiplication. Let \( m = G_x \). Now \( Gx = Px \) provided \( Gm = Pm \) under this action. Since \( G/G_x \) is a one-dimensional manifold, it is a line or a circle. Therefore, if \( Gm \neq Pm \) then there exists \( v \in (Pm) \setminus Pm \) and \( Pm = v \) since otherwise \( Pm \cap Pm = \emptyset \). There is \( g \in G \) such that \( v = gm \). Since the isotropy group of \( v \) is \( gG_x g^{-1} \), \( P \subset gG_x g^{-1} \), so \( g^{-1}Pg \subset G_x \) which is a contradiction. Obviously, if \( \dim Gx = 1 \) then \( Gx \neq \{x\} \).

Hereafter, if \( G \) is a transformation group on a space \( M \) and \( x \in M \) then \( G \) will denote the component of the identity of the isotropy group of \( x \). We have already observed that Theorem 2 of [3] is true for arbitrary closed subsets of the plane. The same observation holds for Theorem 1 and we use it in this form without further mention. That theorem asserts that if \( P \) operates as a transformation group on the plane then every orbit of \( P \) is either a point, a simple closed curve or topologically a line.

**Lemma 3.** Let \( G \) be a connected Lie group acting as a transformation group on a closed subset \( S \) of the plane. If for some \( x \in S \), \( Gx \) is a line and \( G_x \) is a normal subgroup then \( G \) has a closed orbit.

**Proof.** There exist one-parameter subgroups \( P_1, \ldots, P_n \) such that \( P_1P_2 \cdots P_n \) generates \( G \) and no \( P_i \) is contained in \( G_x \), \( i = 1, 2, \ldots, n \). Since \( G_x \) is normal, no \( P_i \) has a conjugate in \( G_x \) for each \( i = 1, 2, \ldots, n \) by the previous lemma. Furthermore, since \( G_x \) is normal, \( G_x y = y \) for all \( y \in (Gx)^{-} \). Therefore, if \( y \in (Gx)^{-} \) then either \( \dim Gy = 0 \) and \( Gy = y \) or \( \dim Gy = 1 \) and \( Gy = G_x \). It follows that \( Gy = P_i y \) for \( i = 1, 2, \ldots, n \) and \( y \in (Gx)^{-} \). If \( Gx \) is unbounded in both directions and \( y \in (Gx)^{-} \setminus Gx \) then \( P_i y = y \) for each \( i \) by Theorem 2 of [3]. Since \( P_1P_2 \cdots P_n \) generates \( G \), \( Gy = y \) so \( G \) has a closed orbit.

Suppose \( Gx \) is not unbounded in both directions. Let \( C \) be a component of \( Gx \setminus \{x\} \) such that \( C^{-} \) is compact. For each \( i = 1, 2, \ldots, n \), let \( R_i \) be the component of \( P_i \setminus \{1\} \) such that \( R_i x = C \). Let \( y \in C^{-} \setminus Gx \). If \( Gy \) is closed, there is nothing further to prove. Otherwise, \( Gy \) is a line contained in \( C^{-} \). Furthermore, \( R_i y = R_j y \) for \( i, j = 1, 2, \ldots, n \). For since \( Gx \) is a line, \( G \setminus G_x \) is the union of two components \( A \) and \( B \). Since \( G_x \) is normal, if \( s \) belongs to one of these components then \( s^{-1} \) belongs to the other and each component is a subsemigroup of \( G \). It follows that for each \( i, j \), \( R_i \) and \( R_j \) belong to the same component of \( G \setminus G_x \). For if \( R_i \) and \( R_j \) belong to different components then there exist elements \( s \in A \) and \( y \in B \) such that \( sx = tx \). Hence \( t^{-1}s \in G_x \). However, both \( t^{-1} \) and \( s \) belong to \( A \) which is impossible since \( A \) is closed under multiplication. Now if \( R_i \) and \( R_j \) belong to a common component of \( G \setminus G_x \) then \( R_i y = R_j y \). To see this, recall that \( P_i y = P_j y \) so \( R_i y = S_j y \) where \( S_j \) is the component of \( P_j \setminus \{1\} \) which belongs to the component of \( G \setminus G_x \) which contains \( R_i \). Since this is \( R_j \), it follows that \( R_i y = R_j y \) for all \( i, j \).
Since $R_1y$ is contained in a compact set, there exists an element 

$$z \in (R_1y)^{-}\backslash P_1y.$$ 

Since $R_1y = R_iy, z \in (R_iy)^{-}\backslash P_iy$ for each $i$. By Lemma 1, $P_iz = z$ for each $i$. Hence $Gz = z$ and it follows that $G$ has a closed orbit.

**Theorem 1.** Let $G$ be a connected Lie group acting as a transformation group on a closed connected subset $S$ of the plane. Then there exists $w \in S$ such that $Gw$ is closed.

**Proof.** If there exists $x \in S$ with $Gx = S$, there is nothing to prove. Suppose there exists $v \in S$ with $\dim Gv = 2$ but $Gv \neq S$. Then $Gv$ has a boundary point $x$ and for every such point, $\dim Gx < 2$. For if $\dim Gx = 2$ then, since $Gx$ is homogeneous $Gx$ is open and hence $Gx \cap Gv \neq \emptyset$ which is impossible. Now if $\dim Gy = 0$ for any $y \in S, Gy = y$ so $G$ has a closed orbit. The proof of the theorem has thus been reduced to the following situation: Either $G$ already has a closed orbit, or there exists $x \in S$ such that $Gy$ is a line for all $y \in (Gx)^{-}$. Furthermore, by the previous lemma, we may assume that $Gx$ is not normal.

Let $\mathfrak{g}$ be the sub-algebra of the Lie algebra of $G$ corresponding to $Gx$. Let $N$ be the normal subgroup of $G$ which is contained in $Gx$ and which corresponds to the ideal $\mathfrak{z}$ of Theorem I of [2]. Of course, $N$ may not be closed, but $N^{-}$ is still normal and contained in $Gx$. Thus there is no loss in generality in assuming that $G/N$ is either abelian, locally isomorphic to $sl(2)$ or isomorphic to the non-commutative group on the plane.

Let $H = G/N$. Since $Ny = y$ for all $y \in (Gx)^{-}$, $H$ operates on $(Gx)^{-}$ according to the rule

$$(gN)y = gy.$$ 

Furthermore, every orbit of $H$ on $(Gx)^{-}$ is an orbit of $G$. Hence if we prove that $H$ has a closed orbit it follows that $G$ has a closed orbit.

We consider the possibilities for $H$ separately. If $H$ is abelian then $H$ has a closed orbit by Lemma 3 since every subgroup of $H$ is normal. Suppose $H$ is locally isomorphic to $sl(2)$. Thus $H$ is isomorphic to the quotient of the covering group $K$ of $sl(2)$ modulo a discrete central subgroup. Now $Hx$ is a planar subgroup. Let $P$ be a one-parameter group of $H$ which is the image under the natural map of a one-parameter subgroup of $K$ which intersects the center non-trivially. Such $P$ can have no conjugate in any planar subgroup of $H$. Hence, $Hx = Px$. In fact, $Hy = Py$ for all $y \in (Hx)^{-}$ since under our present assumptions, $Hx$ is a planar group for each such $y$. If $Hx$ is unbounded in both directions then $Hx$ is closed since if $y \in (Hx)^{-}\backslash Hx, Py = y$ by [3] (as extended to arbitrary closed sets in the plane). This is a contradiction. Therefore suppose that $Hx$ is not unbounded in both directions and let $R$ denote a component of $P\backslash\{1\}$ such that $(Rx)^{-}$ is compact. Choose $y \in (Rx)^{-}\backslash Px$. We may suppose that $Py$ is not closed. Thus $Rx$ is not closed so there exists $z \in (Rx)^{-}\backslash Py$. But then by Lemma 1, $Pz = z$ which is
a contradiction. We have proved that if $G/N$ is locally isomorphic to $sl(2)$ then $G$ has a closed orbit.

Finally, suppose $H$ is the non-commutative group on the plane. If $H_x$ is the normal one-parameter subgroup $Q$ of $H$ there is nothing further to prove in virtue of Lemma 3. Otherwise, $H x = Q x$ since then $Q$ is the only one-parameter subgroup of $H$ having no conjugate in $H_x$. If $H x$ is unbounded in both directions we may assume that $H x$ is closed since otherwise there exists $z \in (H x)^- \backslash H x$ and for such $z$, $Q z = z$ as we know. Hence by Lemma 3 again, $H$ has a closed orbit. Thus suppose one end $R x$ of $Q x$ has compact closure and let $y \in (R x)^- \backslash Q x$. Unless $H y$ is closed, there is $z \in (R y)^- \backslash Q y$. But then by Lemma 1, $Q z = z$ so by Lemma 3 again, $H$ has a closed orbit. All cases have now been considered and the proof of the theorem is complete.

In virtue of Lemma 2.5 of [5] mentioned in the beginning we have the following:

**Corollary.** Let $S$ be a topological semigroup with identity $1$. Assume that the set $G$ of all elements in $S$ which have an inverse with respect to $1$ is a connected Lie group. Let $L$ be a connected non-empty ideal in $S$ which is homeomorphic to a closed subset of the plane. Then $L$ contains an idempotent.

Professor Hofmann has observed to us that Theorem 1 actually holds for any locally compact group $G$ such that $G/G_0$ is compact, where $G_0$ is the component of the identity in $G$. His observations run as follows: Under this assumption $G$ is the projective limit of Lie groups (cf. [6]). Let $M$ be a compact normal subgroup such that $G/M$ is a Lie group. Let $N$ be the subgroup of all $g \in G$ such that $g x = x$ for all $x \in S$. Then $N$ is closed and normal. So is $M \cap N$. Now $M/M \cap N$ is a compact group acting effectively on the plane; hence the component of the identity of $M/M \cap N$ is a Lie group [6, p. 259] and hence $M/M \cap N$ is finite-dimensional. If $M/M \cap N$ is not itself a Lie group it contains a totally disconnected non-discrete subgroup [6, p. 237] which contradicts the fact that no such group can act effectively on the plane [6, p. 249]. Thus $M/M \cap N$ is a Lie group. But $G/M$ is a Lie group with a finite number of components, so $G/M \cap N$ is a Lie group with a finite number of components. Therefore $G/N$ is a Lie group. But $G/N$ is the group which actually acts in the plane. Thus the following theorem is proved:

**Theorem 2.** (Hofmann). Let $G$ be a locally compact group acting as a group of homeomorphisms on the plane and let $G_0$ be its identity component. If $G/G_0$ is compact then $G$ has a closed orbit.

**References**


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