OPERATOR REPRESENTATION THEOREMS[1]

BY

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We consider representations of bounded, compact and weakly compact linear operators from a Banach space to a space $BC(S)$, where S is an *arbitrary* topological space and $BC(S)$ is the space of bounded continuous scalar-valued functions on S with the sup norm. With the use of our theorems, one can quickly and easily deduce numerous operator representation theorems, many of which are new. For example, taking as domain space a space with a well-known conjugate space and range space as either c_0 or m, one fills in quite a few blanks in Tables VI A, B and C in [4]. Our proofs for range $BC(S)$, S arbitrary, are appreciably simpler than those found in the literature for range $C(S)$, S a compact Hausdorff space.

The spaces of bounded, compact and weakly compact linear maps from a B-space X to a B-space Y will be denoted, respectively, by $B[X, Y]$, $K[X, Y]$ and $W[X, Y]$. Unexplained terminology and notation will be found in [4].

Phillips [9] represented the general bounded operator from X to $B(S)$. Gelfand [5] represented the general bounded and compact operator from a B-space X to $C[0, 1]$ while Sirvint [11], [12] represented the general weakly compact operator. More recently, Bartle [1, Theorem 10.2] represented these three types of operators mapping X into the space $C(S)$ of continuous functions on a *compact Hausdorff* space S. However, Bartle's theorem is stated for $BC(S)$ with S an arbitrary topological space and it is wrong in this generality for compact and weakly compact maps. A counter-example is found by taking S to be an infinite set with the discrete topology (thus metrizable and locally compact), that is by taking the range to be $B(S)$. The compactness of S seems to be needed in the last part of the second sentence of the proof in [1].

A representation theorem for bounded operators from X into certain subspaces of $B(S)$ is given in [7]. Taylor [14] gives still more general representation theorems for bounded operators. In particular, he shows [13, Theorem 4.51-B] that Phillips' representation theorem is valid for the general bounded operator from X to $B(S, Z)$, where X and Z are Banach spaces and $B(S, Z)$ is the Banach space of bounded functions from the set S to Z with the sup norm. Wada [16] extended theorem 10.2 of [1] to the case where X is a barrelled locally convex space and $BC(S)$ is replaced by $C_K(S)$, where S is a completely regular Hausdorff space, K is a collection of compact sets which cover S, and $C_K(S)$ is the locally convex topological linear space of all real-valued continuous functions on S, equipped with the topology of uniform con-

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vergence on sets in K. In the weakly compact case, it is further assumed that S is a k_0-space.

We begin with the following lemma, which is given in [3, page 368]. It is given there as a corollary to a result obtained using integration with respect to a vector-valued measure. The proof given here is much more elementary and direct. For an arbitrary nonvoid set S, $l(S)$ is the B-space of scalar-valued functions x on S, having countable support, and such that the norm of x, given by $|x| = \sum \{|x(s)| : s \in S, x(s) \neq 0\}$, is finite.

LEMMA 1. *For an arbitrary nonvoid set S, let $T : l(S) \to Y$ be a linear map and let $\{e_s : s \in S\}$ be the collection of characteristic functions of one-point sets. Then*

(a) *If T is bounded, then $\{Te_s : s \in S\}$ is bounded and*

$$|T| = \sup \{|Te_s| : s \in S\}.$$

Conversely, given any bounded subset $\{Te_s : s \in S\} \subseteq Y$, a unique bounded linear operator T is defined by $T(\sum x(s)e_s) = \sum x(s)Te_s$ and the norm of T is as above.

(b) *The operator T is weakly compact if and only if $\{Te_s : s \in S\}$ is weakly conditionally compact.*

(c) *The operator T is compact if and only if $\{Te_s : s \in S\}$ is conditionally compact.*

Proof. Part (a) is clear, using the completeness of Y. To establish (b), note that the closed unit sphere K in $l(S)$ is the weakly closed convex balanced hull [13, page 132]) $\overline{ccb}^w(\{e_s : s \in S\})$ of the set $\{e_s : s \in S\}$. Thus

$$T(K) = T(\overline{ccb}^w(\{e_s : s \in S\})) \subseteq \overline{T(ccb(\{e_s : s \in S\}))}^w = \overline{ccb(\{Te_s : s \in S\})}^w.$$

Now if $\{Te_s : s \in S\}$ is weakly conditionally compact, it follows that $T(K)$ is weakly conditionally compact by using [4, V.6.4], the fact that the balanced hull of a weakly conditionally compact set is weakly conditionally compact, and by using the preceding relation. Hence, T is weakly compact. Conversely, if T is weakly compact, then $\{Te_s : s \in S\} \subseteq T(K)$ is weakly conditionally compact. This establishes part (b) of the lemma.

To establish (c), note that K is the closed convex balanced hull

$$\overline{ccb}(\{e_s : s \in S\})$$

of $\{e_s : s \in S\}$. Thus

$$T(K) \subseteq \overline{ccb(\{Te_s : s \in S\})}.$$

If $\{Te_s : s \in S\}$ is conditionally compact, so is its closed convex balanced hull, and hence $T(K)$. Thus T is compact. Clearly, if T is compact, $\{Te_s : s \in S\} \subseteq T(K)$ is conditionally compact. This completes the proof of the lemma.

We will now establish the representation for the special case where the range is $B(S)$. It is interesting that the general result follows immediately from this special case.

THEOREM 2. (a) *If T is a bounded linear operator from a B-space X into $B(S)$, then there exists a unique bounded map p of S into X^* such that*

(1) $[Tx](s) = [p(s)](x)$, *x in X and s in S;*
(2) $|T| = \sup \{|p(s)| : s \text{ in } S\}$.

Conversely, given any such p and defining $T : X \to B(S)$ by (1), one obtains a linear operator with norm given by (2).

(b) *The operator T is compact if and only if $p(S)$ is conditionally compact.*

(c) *The operator T is weakly compact if and only if $p(S)$ is conditionally compact in the weak topology.*

Proof. The proof of (a) is clear. To establish (b), suppose T is compact. We define the map π from S into $B(S)^*$ by $\pi(s)f = f(s)$. Since T is compact, T^* is compact and thus $T^*\pi(S) = p(S)$ is conditionally compact. Conversely, we suppose that $p(S)$ is conditionally compact and therefore it is totally bounded. Hence given $\varepsilon > 0$ there exists $\{y_1^*, \cdots, y_n^*\} \subseteq X^*$ such that

$$\min \{|p(s) - y_j^*| : 1 \leqq j \leqq n\} < \varepsilon, \qquad s \in S.$$

For each $s \in S$, let $y_{j(s)}^* \in \{y_1^*, \cdots, y_n^*\}$ be such that $|p(s) - y_{j(s)}^*| < \varepsilon$. Define $T_\varepsilon : X \to B(S)$ by $[T_\varepsilon(X)](s) = y_{j(s)}^*(x)$. Note that $|T_\varepsilon - T| < \varepsilon$ and that T_ε has a null manifold of finite co-dimension, hence is compact. Thus T is in the uniform closure of the compact operators, so it is compact.

For part (c), suppose that T is weakly compact. Then T^* is weakly compact and so $T^*\pi(S) = p(S)$ is conditionally compact in the weak topology. On the other hand, suppose that $p(S)$ is weakly conditionally compact. We will now take advantage of the natural correspondence between $B[X, Y^*]$ and $B[Y, X^*]$ given by $T \leftrightarrow T^*J_Y$, where J_Y is the canonical embedding of Y into Y^{**}. Define $L : X^{**} \to B(S)$ by $[L(x^{**})](s) = x^{**}(p(s))$. Denote the canonical map of X to X^{**} by J_X. Then

$$[L(J_X(x))](s) = [J_X(x)](p(s)) = [p(s)](x) = [T(x)](s),$$

x in X and s in S, so $LJ_X = T$. Now L is the conjugate M^* of the map $M : l(S) \to X^*$ defined by $M(e_s) = p(s)$, for s in S, because

$$[L(x^{**})](s) = x^{**}(Me_s) = [M^*(x^{**})](e_s) = [M^*x^{**})](s)$$

for s in S and x^{**} in X^{**}. But M is weakly compact by Lemma 1 and the weak conditional compactness of $p(S)$. Thus $T = LJ_X = M^*J_X$ is weakly compact.

COROLLARY 3. (a) [1] *If T is a bounded linear operator from a B-space X to $BC(S)$, then there exists a unique bounded continuous map p of S into X^* with* *the X-topology such that*

(1) $[Tx](s) = [p(s)](x)$, *x in X and s in S;*
(2) $|T| = \sup \{|p(s)| : s \text{ in } S\}$.

Conversely, given any such p and defining $T : X \to BC(S)$ by (1), *one obtains a linear operator with norm given by* (2).

(b) *The operator T is compact if and only if in addition $p(S)$ is conditionally compact.*

(c) *The operator T is weakly compact if and only if in addition $p(S)$ is weakly conditionally compact.*

Proof. Let $T : X \to BC(S)$ be continuous. Note that $\pi : S \to BC(S)^*$ is continuous, where $BC(S)^*$ has the weak* (or $BC(S)$) topology and π is defined by $[\pi(s)]f = f(s)$, f in $BC(S)$, s in S. Now $T^* : BC(S)^* \to X^*$ is continuous with respect to the weak* topologies since it is a conjugate map and so the map $p : S \to X^*$ ($p = T^*\pi$) is continuous with the X-topology on X^*. Conversely, we note that if $p : S \to X^*$ is continuous with respect to the X-topology on X^* then T defined as in (1) does indeed have range in $BC(S)$ and is continuous.

If T is (weakly) compact, then $p(S)$ is (weakly) conditionally compact. If $p : S \to X^*$ is continuous with the X-topology on X and is such that $p(S)$ is (weakly) conditionally compact, then T considered as a map into $B(S)$ is (weakly) compact by Theorem 2. But $BC(S)$ is a (weakly) closed subspace of $B(S)$ whence T is (weakly) compact as a map into $BC(S)$.

Remark. If $T : X \to Y \subseteq Z$ and $T : X \to Z$ is compact, $T : X \to Y$ need not be compact if Y is not closed. See [6], [15].

It is evident that the effect of the requirement that p be weak* continuous is precisely to guarantee that T maps into $BC(S)$; the theorem for $B(S)$ is obtained by omitting this condition. We can in fact generalize Theorem 2 to the case where $BC(S)$ is replaced by any B-space $Y \subseteq B(S)$ and where we add a restriction on p which is necessary and sufficient to guarantee that the range of T is contained in Y. Conversely, we could determine which subspaces $Y \subseteq B(S)$ correspond to various "natural" restrictions on p (for example, continuity of p with respect to various standard topologies on X^*).

More generally, we might determine which subspaces of $B[X, B(S)]$ correspond to various "natural" restrictions on p. For example, if W is a subspace of X, a bounded operator $T : X \to B(S)$ maps W into $BC(S)$ if and only if the associated map $p : S \to X^*$ is continuous with the W-topology on X^*.

Proof. Suppose we have $T(W) \subseteq BC(S)$. Then for each $w \in W$, $[T(w)](s) = [p(s)]w$ is a continuous function of s, hence p is continuous with respect to the W-topology. Conversely, if p is continuous with respect to the W-topology, then for each w, $[p(\cdot)]w = [T(w)](\cdot)$ is an element of $BC(S)$.

Analogously, it follows that a bounded operator $T : X \to B(S, Z)$ maps W

into $BC(S, Z)$ if and only if the associated map $p : S \to B[X, Z]$ is continuous with the W-strong operator topology on $B[X, Z]$, where the W-strong operator topology on $B[X, Z]$ is defined for $W \subseteq X$ by the basic set of neighborhoods

$$N(T; A, \varepsilon) = \{R : R \in B[X, Z], |(T - R)(w)| < \varepsilon, w \in A\}$$

where A is an arbitrary finite subset of W and $\varepsilon > 0$ is arbitrary (compare [4, VI.1.2]).

From Lemma 1 and Theorem 2 we obtain

COROLLARY 4. *If S is an infinite set,*
(a) $B[l(S), Y] = K[l(S), Y]$ *iff Y is finite-dimensional;*
(b) $B[l(S), Y] = W[l(S), Y]$ *iff Y is reflexive;*
(c) $B[X, B(S)] = K[X, B(S)]$ *iff X is finite-dimensional;*
(d) $B[X, B(S)] = W[X, B(S)]$ *iff X is reflexive.*

From the special case of Corollary 3 in which the range space is c, we can find the representation for maps into c_0 by exploiting the natural isomorphism between c and c_0 ; a direct proof is also easy. The result is

COROLLARY 5. *Let $T : X \to c_0$ be a linear map. Then $T(x) = \{x_i'(x)\}$ where the x_i' are linear functionals which are not necessarily continuous but which have the property that $x_i'(x)$ converges to zero for each x in X. Then*
(a) *The map T is continuous iff $x_i' \in X'$ and in this case the x_i' converge to zero in the weak* topology. Note* $\| T \| = \sup \| x_i' \|$.
(b) *The map T is compact iff the x_i' converge to zero in norm.*
(c) *The map T is weakly compact iff the x_i' converge to zero in the weak topology.*

COROLLARY 6. (a) *All the bounded linear maps from X to c_0 (or c) are compact iff weak* and norm sequential convergence are equivalent in X'.*
(b) *All the bounded linear maps from X to c_0 (or c) are weakly compact iff weak* and weak sequential convergence are equivalent in X'.*

The condition in part (b) of the above corollary is satisfied by reflexive spaces and the space $B(S)$ [2, page 109]. We know of no space which is infinite dimensional and satisfies condition (a). Elton Lacey has pointed out that if X satisfies condition (a) then any map from X with separable range is compact, and so X' cannot contain an infinite dimensional reflexive subspace.

To derive the usual form of Corollary 3 in the special case where S is compact, we will use the following lemma.

LEMMA 7. *Let S and X be nonvoid topological spaces. Let X have Hausdorff topologies t_1 and t_2 with t_2 stronger than t_1. Suppose that $p : S \to X$ is t_1-continuous and $p(S)$ is t_2-conditionally compact. Then p is t_2-continuous.*

Proof. The proof follows from standard topological arguments.

COROLLARY 8. *If S is a nonvoid compact topological space, then the condition* *for boundedness of T in Corollary* 3(a) *may be reduced to "p is weak* continuous", the condition for compactness in Corollary* 3(b) *may be replaced by "p is continuous", and the condition for weak compactness in Corollary* 3(c) *may be replaced by "p is weakly continuous".*

Lemma 7 is closely related to Lemma VI, page 27 in [8]. In fact, each of these lemmas is readily deducible from the other. However, the utility of Lemma 7 seems to have been generally unnoticed. It can be used to give quicker simpler proofs of a number of well-known theorems, as we now show. The next corollary appears as [4, VI.5.6], at which point we know an operator is compact if and only if its adjoint is compact [4, VI.5.2].

COROLLARY 9. *An operator in $B[X, Y]$ is compact if and only if its adjoint sends bounded generalized sequences which converge in the Y-topology of Y^* into generalized sequences which converge in the metric of X^*.*

Proof. Suppose T, and hence T^*, is compact. Let $\{y_\alpha^*\}$ be a bounded weak* generalized sequence converging to y. Since T^* is weak* continuous [4, VI.2.3], and $\{T^*y_\alpha^*\}$ is norm conditionally compact, Lemma 7 shows at once that Ty_α^* is norm-convergent to Ty. Conversely, if T^* has the given property, and S^* is the unit sphere in Y^*, then since S^* is weak* compact, T^*S^* is norm conditionally compact.

The next corollary resembles [4, VI.4.7].

COROLLARY 10. *An operator in $B[X, Y]$ is weakly compact if and only if its adjoint is continuous with respect to the BY-topology on Y^* and the X^{**}-topology on X^*.*

The next corollary is a theorem of Ringrose's [10, Theorem 3.5].

COROLLARY 11. *Let X and Y be Banach spaces and $T \in B[X, Y]$. Then T is compact if and only if it is continuous as a mapping from the unit sphere of X with the X^*-topology, into Y.*

Proof. If T is compact, Lemma 7 shows that it has the stated continuity property. Conversely suppose T has this property. Now

$$J_X : X \to J_X(X) \subseteq X^{**}$$

is a homeomorphism, where X and X^{**} both have the X^* topology. If S is the unit sphere in X, $J_X(S)$ is a weak* conditionally compact subset of $J_X(X)$, hence is weak* totally bounded. Therefore S is weakly totally bounded. It follows that $T(S)$ is norm totally bounded and hence that T is compact.

Note that T in $B[X, Y]$ continuous as a map from the unit sphere of X with the weak topology into Y is equivalent to being continuous as a map from X with the BX^* topology to Y [2, page 41] which is equivalent to T mapping bounded weakly convergent generalized sequences into norm-convergent generalized sequences.

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