GENERALIZED GROUP ALGEBRAS

BY
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1. Introduction

E. Hewitt and H. Zuckerman have shown how to define convolution multiplication in a very general context [7]. In particular, using their multiplication it is possible to make the conjugate space of the complex Banach space of all bounded complex-valued functions defined on a semigroup into a Banach algebra. This algebra has been studied previously by M. M. Day [2] in case the semigroup is left amenable. This algebra, however, seems ill-suited to the study of harmonic analysis due both to its size and to the lack of available analytical machinery.

We propose to continue the study of "harmonic analysis" in the context of left amenable groups but with two inovations. Firstly, we utilize the Stone-Čech compactification of the discrete semigroup to place our study in the context of regular Borel measures on a compact Hausdorff space, cf. [3]. Secondly, we restrict our attention to the L_2 space of the measure "associated" with a left invariant mean. We show that this is also a Banach algebra under convolution multiplication and this is the generalized group algebra referred to in the title. One of our interests in this group algebra results from its connection with several questions we raised in [3]. The relation of our work with these questions is discussed in §5.

Our utilization of the Stone-Čech compactification is given in §2 along with other preliminaries. In §3 the convolution multiplication is defined and some properties of it are derived. The generalized group algebra is defined in §4 and some of its structure (including the determination of its Jacobson radical) is derived in §5. We conclude with some remarks in §6.

2. Preliminaries

Let Σ be a semigroup. We shall denote by $\mathfrak{B}(\Sigma)$ the complex Banach space of bounded complex-valued functions on Σ in which

$$\|f\| = \sup \{|f(\sigma)| \mid \sigma \in \Sigma\}.$$

An element L of $\mathfrak{B}(\Sigma)^*$ (the conjugate space of $\mathfrak{B}(\Sigma)$) is said to be a left invariant mean on Σ if (1) $\|L\| = 1$; (2) $Lf \geq 0$ for $f \geq 0$; and (3) $L({}_\sigma f) = Lf$ for each $\sigma \in \Sigma$, where $({}_\sigma f)(\tau) = f(\sigma\tau)$ for $\tau \in \Sigma$. Not every semigroup possesses a left invariant mean; a semigroup that does is said to be left amenable. An abelian semigroup is always left amenable. For this result and further information on left invariant means see [6, §17].

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Let Σ be a semigroup. We shall denote by $\beta\Sigma$ the Stone-Čech compactification of the discrete space Σ. The complex Banach space $\mathfrak{C}(\beta\Sigma)$ of complex-valued continuous functions on $\beta\Sigma$ (with the supremum norm) is isometrically isomorphic to $\mathfrak{B}(\Sigma)$. If π is the inclusion map of Σ into $\beta\Sigma$, then the induced map π_* from $\mathfrak{C}(\beta\Sigma)$ to $\mathfrak{B}(\Sigma)$ (for $f \in \mathfrak{C}(\beta\Sigma)$ and $\sigma \in \Sigma$ we have $(\pi_* f)(\sigma) = f(\pi\sigma))$ is an onto, isometrical isomorphism [4, 10.2 and 10.3].

Using the representation theorem of F. Riesz, we can identify $\mathfrak{C}(\beta\Sigma)^*$ (conjugate space) with the space of finite complex-valued regular Borel measures on $\beta\Sigma$, which we will denote by $\mathbf{M}(\beta\Sigma)$. Further, let π^* be the adjoint map of π_*, that is, π^* is the map defined from $\mathfrak{B}(\Sigma)^*$ to $\mathfrak{C}(\beta\Sigma)^*$ such that for $L \in \mathfrak{B}(\Sigma)^*$ and $f \in \mathfrak{C}(\beta\Sigma)$ we have $(\pi^* L)f = L(\pi_* f)$. Then π^* is an onto, isometrical isomorphism because π_* is. Composing π_* and the identification of $\mathfrak{C}(\beta\Sigma)^*$ and $\mathbf{M}(\beta\Sigma)$ we obtain the (canonical) isometrical isomorphism from $\mathfrak{B}(\Sigma)^*$ onto $\mathbf{M}(\beta\Sigma)$. Let us denote this composite map by Φ. We can define Φ more succinctly as follows: for $L \in \mathfrak{B}(\Sigma)^*$, ΦL is the unique measure $\mu \in \mathbf{M}(\beta\Sigma)$ such that $\int_{\beta\Sigma} f\, d\mu = L(\pi_* f)$ for every $f \in \mathfrak{C}(\beta\Sigma)$.

Each of the spaces $\mathfrak{B}(\Sigma)^*$ and $\mathbf{M}(\beta\Sigma)$ also possesses a natural partial ordering with respect to which the "subspace of real-valued elements" forms a vector lattice. For real-valued functionals L and M in $\mathfrak{B}(\Sigma)^*$, we define $L \geq M$ if $Lf \geq Mf$ for every positive $f \in \mathfrak{B}(\Sigma)$, while for real-valued measures μ and ν in $\mathbf{M}(\beta\Sigma)$ we define $\mu \geq \nu$ if $\mu(E) \geq \nu(E)$ for every Borel subset of $\beta\Sigma$. With respect to these orderings, the map Φ is an order isomorphism.

There is a natural analogue of "absolute value" in each of $\mathfrak{B}(\Sigma)^*$ and $\mathbf{M}(\beta\Sigma)$; that is also preserved by Φ. For $M \in \mathfrak{B}(\Sigma)^*$ and $0 \leq f \in \mathfrak{B}(\Sigma)$, we define $|M|$ at f as follows:

$$|M|(f) = \sup\{|Mg| \mid |g| \leq f\}.$$

We will also denote the absolute value of a measure μ by $|\mu|$. One obvious fact which we shall need is that for each $L \in \mathfrak{B}(\Sigma)^*$ and $\nu \in \mathbf{M}(\beta\Sigma)$, the following identities hold: $\||L|\| = \|L\|$ and $\||\nu|\| = \|\nu\|$.

Corresponding to the semigroup $\{T'_\sigma \mid \sigma \in \Sigma\}$ of left translation operators on $\Sigma(T'_\sigma \tau = \sigma\tau)$, there is the semigroup $\{T_\sigma \mid \sigma \in \Sigma\}$ of homeomorphisms of $\beta\Sigma$, where each T_σ is the unique extension of T'_σ [4, 0.12 and 6.5]. Let U_σ denote the automorphism of $\mathfrak{C}(\beta\Sigma)$ induced by T_σ, that is, for $f \in \mathfrak{C}(\beta\Sigma)$ we have

$$(U_\sigma f)x = f(T_\sigma x)$$

for each $x \in \beta\Sigma$. Finally, let V_σ denote the automorphism of $\mathbf{M}(\beta\Sigma)$ induced by U_σ; for $\mu \in \mathbf{M}(\beta\Sigma)$ we have that $V_\sigma \mu$ is the unique measure in $\mathbf{M}(\beta\Sigma)$ such that

$$\int_{\beta\Sigma} f\, d(V_\sigma \mu) = \int_{\beta\Sigma} (U_\sigma f)\, d\mu$$

for each $f \in \mathfrak{C}(\beta\Sigma)$. For any measure $\mu \in \mathbf{M}(\beta\Sigma)$ a straight forward argument will show that $\Phi^{-1}\mu$ is a left invariant mean on Σ if and only if μ is a probability measure (positive measure of mass one) that is invariant relative to the semigroup of automorphisms $\mathcal{V} = \{V_\sigma \mid \sigma \in \Sigma\}$. We shall refer to such a measure as a left invariant measure relative to Σ.

Let $\mathcal{I}$ denote the set of all left invariant measures/Σ; then $\mathcal{I}$ is a w^*-compact convex subset of $\mathbf{M}(\beta\Sigma)$ and thus it follows from the Kreĭn-Mil'man Theorem that $\mathcal{I}$ is the w^*-closed convex hull of its extreme points. An extreme point of $\mathcal{I}$ will be referred to as an (elementary) left invariant measure/Σ.

3. Convolution algebras

Let Σ be a semigroup. It is possible to define a convolution multiplication on $\mathfrak{B}(\Sigma)^*$ with respect to which it is a Banach algebra. This definition of convolution is due originally to Hewitt and Zuckerman [7] in a slightly more general context, but seems to have been studied first in the context of left amenable semigroups by Day [2]. The definition of Day was a specialization of Aren's definition of multiplication in the second conjugate of a Banach algebra [1].

DEFINITION 3.1. Let L and M be functionals in $\mathfrak{B}(\Sigma)^*$. For $f \in \mathfrak{B}(\Sigma)$, let $M \circ f$ denote the function in $\mathfrak{B}(\Sigma)$ defined such that $(M \circ f)\sigma = M({}_\sigma f)$ for each $\sigma \in \Sigma$. The convolution of L and M, written $L * M$, is then that functional in $\mathfrak{B}(\Sigma)^*$ for which $(L * M)f = L(M \circ f)$ for every $f \in \mathfrak{B}(\Sigma)$.

LEMMA 3.2. *If L and M are functionals in $\mathfrak{B}(\Sigma)^*$, then*

$$|L| * |M| \geq |L * M|.$$

If further $L \geq 0$ and $M \geq 0$, then $L * M \geq 0$ and $\|L\| \cdot \|M\| = \|L * M\|$.

Proof. Suppose $L, M \in \mathfrak{B}(\Sigma)^*$ and $0 \leq f \in \mathfrak{B}(\Sigma)$. Then

$$\begin{aligned} |L * M|(f) &= \sup\{|(L * M)(g)| \mid |g| \leq f\} \\ &= \sup\{|L(M \circ g)| \mid |g| \leq f\} \\ &\leq \sup\{|L|(|M \circ g|) \mid |g| \leq f\} \\ &\leq \sup\{|L|(|M| \circ |g|) \mid |g| \leq f\} \\ &\leq |L|(|M| \circ f) = (|L| * |M|)f \end{aligned}$$

and thus $|L * M| \leq |L| * |M|$.

If $L \geq 0$ and $M \geq 0$, then $L * M = |L| * |M| \geq |L * M| \geq 0$ and thus $L * M$ assumes its norm at the constant function 1. The proof is completed with the following computation:

$$\|L * M\| = (L * M)1 = L(M \circ 1) = L(\|M\| \cdot 1) = \|L\| \, \|M\|.$$

PROPOSITION 3.3. *The space $\mathfrak{B}(\Sigma)^*$ is a Banach algebra under convolution multiplication.*

Proof. From [6, §19] it follows that $\mathfrak{B}(\Sigma)^*$ is an algebra. For the norm inequality, let us suppose that $L, M \in \mathfrak{B}(\Sigma)^*$; then the preceding lemma shows that

$$\|L * M\| = \||L * M|\| \leq \||L| * |M|\| = \|L\| * \|M\|,$$

and thus $\mathfrak{B}(\Sigma)^*$ is a Banach algebra.

The convolution algebra $\mathfrak{B}(\Sigma)^*$ was studied by Day in [2] with particular emphasis on the case in which Σ is left amenable. Among other things, Day established that the assumption that Σ is abelian does not always imply that $\mathfrak{B}(\Sigma)^*$ is commutative. More specifically, he showed that the only abelian *groups* for which $\mathfrak{B}(\Sigma)^*$ is commutative are the finite ones. We now state a lemma due to Day [2, Thm. 1, p. 530] and offer a proof for completeness.

LEMMA 3.4. *If L is a left invariant mean on Σ and M is an arbitrary functional in $\mathfrak{B}(\Sigma)^*$, then $M * L = M1 \cdot L$.*

Proof. For each function $f \in \mathfrak{B}(\Sigma)$, we have

$$\begin{aligned}(M * L)f = M(L \circ f) &= M(L({}_\sigma f)) \\ &= M(Lf \cdot 1) = M1 \cdot Lf\end{aligned}$$

and thus $M * L = M1 \cdot L$.

The study of $\mathfrak{B}(\Sigma)^*$ is complicated mainly for two reasons. Firstly, there is the fact that most of the analytical tools used in studying the more classical convolution algebras seem not to be available (e.g., the Fubini Theorem), and secondly, there is the sheer size of $\mathfrak{B}(\Sigma)^*$ (e.g., the assumption that $\mathfrak{B}(\Sigma)^*$ is separable implies that Σ is finite). We propose to overcome these difficulties as follows: we will restrict our attention to a "subalgebra" of $\mathfrak{B}(\Sigma)^*$, and then study this object as the corresponding Banach algebra of measures in $\mathbf{M}(\beta\Sigma)$. Toward this end we need to transfer the convolution multiplication from $\mathfrak{B}(\Sigma)^*$ to $\mathbf{M}(\beta\Sigma)$.

PROPOSITION 3.5. *For ξ and ν in $\mathbf{M}(\beta\Sigma)$, we define*

$$\xi * \nu = \Phi[\Phi^{-1}(\xi) * \Phi^{-1}(\nu)].$$

The following propositions follow immediately from 3.2–3.4.

PROPOSITION 3.6. *$\mathbf{M}(\beta\Sigma)$ is a Banach algebra under convolution multiplication.*

Let $\mathbf{M}^+(\beta\Sigma)$ denote the set of positive measures in $\mathbf{M}(\beta\Sigma)$.

PROPOSITION 3.7. *If $\xi, \nu \in \mathbf{M}^+(\beta\Sigma)$, then $\xi * \nu \in \mathbf{M}^+(\beta\Sigma)$ and*

$$(\xi * \nu)(\beta\Sigma) = \xi(\beta\Sigma) \cdot \nu(\beta\Sigma).$$

PROPOSITION 3.8. *If ν is a left invariant measure/Σ and $\xi \in \mathbf{M}(\beta\Sigma)$, then*

$$\xi * \nu = \xi(\beta\Sigma)\nu.$$

While the convolution multiplication on $\mathbf{M}(\beta\Sigma)$ is well defined, we might hope to be able to give a more direct definition of it. That is the purpose of the following proposition.

PROPOSITION 3.9. *If* ξ, $\nu \in \mathbf{M}(\beta\Sigma)$ *and* E *is an open and closed subset of* $\beta\Sigma$, *then*

$$(\xi * \nu)(E) = \int_{\beta\Sigma} \pi_*^{-1}[\nu(T_\sigma^{-1}E)]\, d\xi.$$

Remark 3.10. If $\beta\Sigma$ were an arbitrary compact Hausdorff space, Proposition 3.9 would be of little value in determining the convolution multiplication on $\mathbf{M}(\beta\Sigma)$. The open and closed subsets of $\beta\Sigma$ (being the Stone-Čech compactification of a discrete space), however, form a basis for the topology of $\beta\Sigma$, and thus a regular Borel measure is completely determined by its values on these sets. Thus the formula given in Proposition 3.9 does provide an alternative definition of convolution multiplication.

Proof of Proposition 3.9. Since E is an open and closed subset of $\beta\Sigma$, the characteristic function C_E of E is a continuous function in $\mathfrak{C}(\beta\Sigma)$ and $\pi_* C_E \in \mathfrak{B}(\Sigma)$. Further, if $M = \Phi^{-1}(\nu)$ and $K = \Phi^{-1}(\xi)$, we have

$$(M \circ \pi_* C_E)\sigma = M[_\sigma(\pi_* C_E)] = M[\pi_*(U_\sigma C_E)] = \nu(T_\sigma^{-1}E)$$

and thus

$$\begin{aligned}(\xi * \nu)(E) &= \int_{\beta\Sigma} C_E\, d(\xi * \nu) = (K * M)(\pi_* C_E)\\ &= K(M \circ \pi_* C_E) = K[\nu(T_\sigma^{-1}E)] = \int_{\beta\Sigma} \pi_*^{-1}[\nu(T_\sigma^{-1}E)]\, d\xi.\end{aligned}$$

4. Group algebras

Let Σ be a fixed left amenable semigroup and L be a left invariant mean on Σ. We shall continue to use the terminology and notation of §§2 and 3.

Let $\mu = \Phi L \in \mathbf{M}(\beta\Sigma)$; we will show first that the subspace $\Lambda_1(\mu)$ of measures absolutely continuous with respect to μ is a left ideal in $\mathbf{M}(\beta\Sigma)$.

THEOREM 4.1. $\Lambda_1(\mu)$ *is a closed left ideal in* $\mathbf{M}(\beta\Sigma)$.

Proof. It is well known that $\Lambda_1(\mu)$ is a closed subspace of $\mathbf{M}(\beta\Sigma)$, which can be identified with $\mathfrak{L}_1(\mu)$ using the Radon-Nikodym Theorem. Moreover, to show that $\Lambda_1(\mu)$ is a left ideal in $\mathbf{M}(\beta\Sigma)$, it is clearly sufficient to prove that for $\xi \in \mathbf{M}^+(\beta\Sigma)$ and $0 \leq \nu \leq \mu$, it follows that $\xi * \nu \in \Lambda_1(\mu)$. This is so because linear combinations of positive measures give all of $\mathbf{M}(\beta\Sigma)$, while linear combinations of positive measures dominated by μ are dense in $\Lambda_1(\mu)$. (Equivalently, the bounded measurable functions are dense in $\mathfrak{L}_1(\mu)$.)

If ξ and ν have the properties stated above, then $\mu - \nu \geq 0$ and thus $\xi * (\mu - \nu) \geq 0$ by Proposition 3.7 or $\xi * \mu \geq \xi * \nu \geq 0$. The identity $\xi * \nu = \xi(\beta\Sigma)\cdot\nu$ obtained from Proposition 3.8 implies

$$\xi(\beta\Sigma)\mu \geq \xi * \nu \geq 0.$$

Hence, if E is a Borel set of $\beta\Sigma$ for which $\mu(E) = 0$, then

$$0 \leq (\xi * \nu)(E) \leq \xi(\beta\Sigma)\mu(E) = 0$$

or $(\xi * \nu)(E) = 0$. Therefore $\xi * \nu \in \Lambda_1(\mu)$ and the proof is complete.

COROLLARY 4.2. *$\Lambda_1(\mu)$ is a closed subalgebra of $\mathbf{M}(\beta\Sigma)$.*

Remark 4.3. If we define

$$\Lambda_\infty(\mu) = \{\nu \in \mathbf{M}(\beta\Sigma) \mid |\nu| \leq N\mu \text{ for some } N > 0\}$$

and set $\|\nu\|_\infty = \inf\{N \mid |\nu| \leq N\mu\}$, then $\Lambda_\infty(\mu)$ is easily seen to be a complex Banach space and the preceding proof can be used, almost without change, to show that $\Lambda_\infty(\mu)$ is a Banach algebra under convolution multiplication.

We now turn our attention to $\Lambda_2(\mu)$, the object we wish to study. We choose to study $\Lambda_2(\mu)$ and not $\Lambda_1(\mu)$, because in addition to being a Banach algebra it has a Hilbert space structure, with respect to which "translations are unitary operators".

DEFINITION 4.4. $\Lambda_2(\mu) = \{\nu \in \mathbf{M}(\beta\Sigma) \mid \nu = \int f\, d\mu \text{ for } f \in \mathfrak{L}_2(\mu)\}$ and $\|\nu\|_2 = \|f\|_2$.

We must first provide an alternate characterization of $\Lambda_2(\mu)$. This characterization will be used only to show that $\Lambda_2(\mu)$ is a Banach algebra.

Let δ denote a finite partition of $\beta\Sigma$ into open and closed subsets, that is, $\delta = \{E_i\}_{i=1}^n$ with each E_i an open and closed subset of $\beta\Sigma$ such that $\beta\Sigma = \bigcup_{i=1}^n E_i$ and $E_k \cap E_j = \emptyset$ for $i \neq j$. Let Δ denote the set of all such partitions. For $\nu \in \mathbf{M}(\beta\Sigma)$ and $\delta \in \Delta$, set

$$A(\nu, \delta) = \sum_{i=1}^n |\nu(E_i)|^2 \mid \mu(E_i),$$

where $a/0$ is taken to be 0 if $a = 0$ and ∞ if $a \neq 0$.

LEMMA 4.5. *The measure $\nu \in \Lambda_2(\mu)$ if and only if $\sup_{\delta\in\Delta} A(\nu, \delta) < \infty$. Moreover, if $\nu \in \Lambda_2(\mu)$, then*

$$\|\nu\|_2^2 = \sup\{A(\nu, \delta) \,/\, \delta \in \Delta\}.$$

Proof. If $f \in \Lambda_2(\mu)$ and $d\nu = f\,d\mu$, then for $\{E_i\}_{i=1}^n = \delta \in \Delta$, we have

$$A(\nu, \delta) = \sum_{i=1}^n \left| \int_{E_i} f\, d\mu \right|^2 / \mu(E_i) \leq \sum_{i=1}^n \int_{E_i} |f|^2\, d\mu = \|f\|_2^2$$

and hence $\|\nu\|_2^2 \geq \sup\{A(\nu, \delta) \mid \delta \in \Delta\}$.

Suppose $\nu \in \mathbf{M}(\beta\Sigma)$ is not absolutely continious relative to μ. Then there exists a Borel subset F of $\beta\Sigma$ so that $\mu(F) = 0$ and $|\nu(F)| = \varepsilon > 0$. Since μ and ν are regular Borel measures there exists for each integer n an open set O_n containing F for which $\mu(O_n) < \varepsilon^2/n$ and $|\nu(O_n)| > \varepsilon/2$. Further, because the topology of $\beta\Sigma$ is generated by the open and closed subsets of $\beta\Sigma$, there exists for each integer n an open and closed subset U_n contained in O_n

for which $\mu(U_n) < \varepsilon^2/n$ and $|\nu(U_n)| > \varepsilon/3$. Using the partition $\delta_n = \{U_n, \beta\Sigma - U_n\}$ we find that

$$A(\nu, \delta_n) \geq |\nu(U_n)|^2/\mu(U_n) \geq \frac{\varepsilon^2/9}{\varepsilon^2/n^2} = \frac{n^2}{9}$$

and thus that $\sup\{A(\nu, \delta) \mid \delta \in \Delta\} = \infty$.

Hence, if we assume for $\nu \in \mathbf{M}(\beta\Sigma)$ that

$$\sup\{A(\nu, \delta) \mid \delta \in \Delta\} \leq M,$$

it follows from the Radon-Nikodym Theorem that there exists $f \in \mathfrak{L}_1(\mu)$ so that $d\nu = f\,d\mu$. It remains to prove that $f \in \mathfrak{L}_2(\mu)$ and $\|f\|_2^2 \leq M$. To do this it is sufficient to show that for each finite partition $\{F_i\}_{i=1}^n$ of $\beta\Sigma$ into Borel sets and each $\varepsilon > 0$ there exists $\{E_i\}_{i=1}^n = \delta \in \Delta$ so that

$$\textstyle\sum_{i=1}^n |\nu(E_i)|^2/\mu(F_i) \leq \{\sum_{i=1}^n |\nu(E_i)|^2/\mu(E_i)\} + \varepsilon.$$

That this is sufficient follows from the fact that for K positive so that $\int_{\beta\Sigma} |f|^2\, d\mu > K$, there exists a partition $\{F_i\}_{i=1}^n$ of $\beta\Sigma$ into Borel sets so that

$$\textstyle\sum_{i=1}^n |\nu(F_i)|^2/\mu(F_i) \geq K.$$

Let $\{F_i\}_{i=1}^n$ be a finite partition of $\beta\Sigma$ into Borel sets and $\varepsilon > 0$. Using a two-step approximation of the F_i, $i = 2, 3, \cdots n$, first by open sets $O_i \supset F_i$, $i = 2, \cdots n$ and second by open and closed subsets $U_i \subset O_i$, $i = 2, \cdots n$, we can make the quantities

$$|\mu(U_i) - \mu(F_i)| \quad \text{and} \quad \big|\, |\nu(U_i)|^2 - |\nu(F_i)|^2 \,\big| \quad \text{for} \quad i = 1, 2, \cdots n$$

sufficiently small enough to imply that

$$\textstyle\sum_{i=1}^n |\nu(F_i)|^2/\mu(F_i) \leq \{\sum_{i=1}^n |\nu(U_i)|^2/\mu(U_i)\} + \varepsilon.$$

Since this is a standard ε–δ argument, we will content ourselves with the sketch of the argument just given and thus the Lemma is proved.

To prove that $\Lambda_2(\mu)$ is a Banach algebra, we need to assume that the semigroup is actually a group. Whether or not this is necessary, we do not know. As a mnemonic device, we shall let G denote the group, while continuing to use μ to denote a left invariant measure relative to it.

THEOREM 4.6. *$\Lambda_2(\mu)$ is a left ideal in $\mathbf{M}(\beta G)$. Moreover for $\xi \in \mathbf{M}(\beta G)$ and $\nu \in \Lambda_2(\mu)$, the norm inequality $\|\xi * \nu\|_2 \leq \|\xi\| \cdot \|\nu\|_2$ is obtained.*

Proof. Suppose $\xi \in \mathbf{M}(\beta G)$, $\nu \in \Lambda_2(\mu)$ and $\delta = \{E_i\}_{i=1}^n \in \Delta$. Then from Proposition 3.9 it follows that

$$\begin{aligned} A((\xi * \nu), \delta) &= \sum_{i=1}^n |(\xi * \nu)(E_i)|^2/\mu(E_i) \\ &= \sum_{i=1}^n \left| \int_{\beta G} \pi_*^{-1}[\nu(T_\sigma^{-1} E_i)]\, d\xi \right|^2 / \mu(E_i) \end{aligned}$$

and using the Cauchy-Schwarz inequality that

$$\begin{aligned}
A(\xi * \nu, \delta) &\leq \sum_{i=1}^{n} \|\xi\| \int_{\beta G} |\pi_*^{-1}[\nu(T_\sigma^{-1}E_i)]|^2 \, d|\xi|/\mu(E_i) \\
&\leq \sum_{i=1}^{n} \|\xi\| \int_{\beta G} \pi_*^{-1}\{|\nu(T_\sigma^{-1}E_i)|^2/\mu(E_i)\} \, d|\xi| \\
&\leq \|\xi\| \int_{\beta G} \pi_*^{-1}\left\{\sum_{i=1}^{n} |\nu(T_\sigma^{-1}E_i)|^2/\mu(T_\sigma^{-1}E_i)\right\} d|\xi| \\
&\leqq \|\xi\| \int_{\beta G} \pi_*^{-1}\{\|\nu\|_2^2\} \, d|\xi| \leq \|\xi\|^2 \|\nu\|_2^2.
\end{aligned}$$

Hence it follows from Lemma 4.5 that $\xi * \nu \in \Lambda_2(\mu)$ and

$$\|\xi * \nu\|_2 \leq \|\xi\| \, \|\nu\|_2 .$$

COROLLARY 4.7. *$\Lambda_2(\mu)$ is a Banach algebra.*

Proof. Because μ is a probability measure, we have for $\xi \in \Lambda_2(\mu)$, that $\|\xi\| \leq \|\xi\|_2$. Thus for ξ and ν in $\Lambda_2(\mu)$, the norm inequality

$$\|\xi * \nu\|_2 \leq \|\xi\| \, \|\nu\|_2 \leq \|\xi\|_2 \, \|\nu\|_2$$

is obtained and $\Lambda_2(\mu)$ is seen to be a Banach algebra.

Remark 4.8. The norm inequality obtained in Theorem 4.6 further shows that the left regular representation of $\mathbf{M}(\beta\Sigma)$ on $\Lambda_2(\mu)$ is norm-decreasing. It will not in general be an isomorphism because $\mathbf{M}(\beta\Sigma)$ is known to be not semi-simple in many interesting cases [5].

Remark 4.9. If Definition 4.4 is changed in the obvious way to yield a definition of $\Lambda_p(\mu)$ for $1 < p < \infty$, then a slight change in the previous proof will yield that $\Lambda_p(\mu)$ is also a Banach algebra.

5. The radical of $\Lambda_2(\mu)$.

Let G be an abelian group[2] and μ be a fixed invariant measure. Further, let Γ denote the abstract character group of G. Each $\chi \in \Gamma$ is a bounded complex-valued function defined on G and thus $\chi \in \mathfrak{B}(G)$. Let ν_χ denote the measure in $\mathbf{M}(\beta G)$ defined $d\nu_\chi = \pi_*^{-1}(\chi) \, d\mu$ and set $X = \{\nu_\chi \mid \chi \in \Gamma\}$. It is easy to verify that X is a subset of $\Lambda_2(\mu)$ and only a bit harder to show that X is an orthonormal subset of $\Lambda_2(\mu)$, cf. [3]. When μ is not an elementary invariant measure, it is shown in [3] that X is not a complete orthonormal subset of $\Lambda_2(\mu)$. Whether or not X is complete when μ is elementary is not known. Our immediate interest in $\Lambda_2(\mu)$ is directed toward the resolution of this question. Although some of our results are slightly more general, *we shall assume from now on that μ is an elementary invariant measure/G.*

[2] We use additive notation for the group operation in G.

Let $\mathfrak{M}$ denote the subspace of $\Lambda_2(\mu)$ spanned by X and $\mathfrak{N} = \mathfrak{M}^{\perp}$.

LEMMA 5.1. *The subspace $\mathfrak{N}$ is a two-sided ideal in $\Lambda_2(\mu)$ that contains the Jacobson radical of $\Lambda_2(\mu)$.*

Proof. We shall prove this by showing that the characters of G can be used to define complex homomorphisms on $\Lambda_2(\mu)$ just as in the case of a locally compact abelian group.

For $\chi \in \Gamma$, we define the functional ψ_χ at $\nu \in \Lambda_2(\mu)$ as follows:

$$\psi_\chi(\nu) = \langle \nu, \nu_\chi \rangle$$

or equivalently

$$\psi_\chi(\nu) = \langle \nu, \nu_\chi \rangle = \int_{\beta G} \frac{d\nu}{d\mu} \frac{d\nu_{\chi^\sim}}{d\mu} \, d\mu$$
$$= \int_{\beta G} \pi_*^{-1}(\chi^\sim) \, d\nu = [\Phi^{-1}(\nu)](\chi^\sim),$$

where $^\sim$ denotes complex conjugation. That ψ_χ is multiplicative follows easily: for ξ and ν in $\Lambda_2(\mu)$, we have

$$\psi_\chi(\xi * \nu) = [\Phi^{-1}(\xi) * \Phi^{-1}(\nu)](\chi^\sim)$$
$$= \Phi^{-1}(\xi)[\Phi^{-1}(\nu) \circ \chi^\sim]$$
$$= \Phi^{-1}(\xi)(\chi^\sim)\Phi^{-1}(\nu)(\chi^\sim) = \psi_\chi(\xi)\psi_\chi(\nu)$$

because $\Phi^{-1}(\nu) \circ \chi^\sim = [\Phi^{-1}(\nu)(\chi^\sim)]\chi^\sim$. Thus the subspace of $\Lambda_2(\mu)$ orthogonal to ν_χ is a maximal ideal. Hence the intersection of these subspaces, $\mathfrak{N}$, is a two-sided ideal that contains the Jacobson radical and the lemma is proved.

LEMMA 5.2. *Suppose $\nu \in \Lambda_2(\mu)$ and $f \in \mathfrak{B}(G)$ such that $V_\sigma \nu = f(\sigma)\nu$ for each $\sigma \in G$. Then $f \in \Gamma$ and there exists a complex number c such that $\nu = c\, \nu_{f^\sim}$.*

Proof. Since $V_\sigma \mu = \mu$ for $\sigma \in G$, it follows that V_σ acts on $\Lambda_2(\mu)$ as a unitary operator and thus $|f(\sigma)| = 1$ for $\sigma \in G$. Further, the identity $V_\sigma V_\tau = V_{(\sigma+\tau)}$ for σ and τ in G implies that $f(\sigma)f(\tau) = f(\sigma + \tau)$ and hence that f is a character on G or that $f \in \Gamma$.

Let ξ be the measure defined $d\xi = \pi_*^{-1}(f)\, d\nu$. Since

$$\nu \in \Lambda_2(\mu) \quad \text{and} \quad \pi_*^{-1}(f) \in \mathfrak{C}(\beta G),$$

it follows that $\xi \in \Lambda_2(\mu)$. Moreover, for $\sigma \in G$ and $g \in C(\beta G)$ we have

$$\int_{\beta G} g \, d[V_\sigma \xi] = \int_{\beta G} (U_\sigma g)\pi_*^{-1}(f) \, d\nu = f(\sigma)^\sim \int_{\beta G} U_\sigma[g\pi_*^{-1}(f)] \, d\nu$$
$$= f(\sigma)^\sim \int_{\beta G} g\pi_*^{-1}(f) \, d[V_\sigma \nu] = \int_{\beta G} g\pi_*^{-1}(f) \, d\nu = \int_{\beta G} g \, d\xi$$

and thus ξ is invariant relative to the group $\mathcal{V}$. Now, if

$$\xi = (\xi_1 - \xi_2) + i(\xi_3 - \xi_4)$$

is a Jordan decomposition of ξ, then each of the measures ξ_1, ξ_2, ξ_3, and ξ_4 is a nonnegative measure that is invariant relative to $\mathcal{V}$ and absolutely continuous with respect to μ. If some ξ_i is not a multiple of μ, then for some $\lambda > 0$, the measure

$$\tfrac{1}{2}\{(\mu - \lambda\xi_i) + |\mu - \lambda\xi_i|\}$$

is a positive measure that is not a multiple of μ and yet is invariant relative to $\mathcal{V}$ and dominated by μ. But then μ would not be an extreme point of $\mathcal{G}$ which is a contradiction. Thus there exists a complex number c so that $\xi = c\mu$ and then

$$c\, d\nu_{f_\sim} = c\pi_*^{-1}(f^\sim)\, d\mu = \pi_*^{-1}(f^\sim)\, d\xi = \pi_*^{-1}(f^\sim)\pi_*^{-1}(f)\, d\nu = d\nu \quad \text{or} \quad \nu = c\cdot\nu_{f_\sim}.$$

LEMMA 5.3. *The subspace $\mathfrak{M}$ is a two-sided ideal in $\Lambda_2(\mu)$ which annihilates $\mathfrak{N}$ and is contained in the center of $\Lambda_2(\mu)$.*

Proof. We prove this by verifying the following formula for the product of $\xi \in \Lambda_2(\mu)$ and $\nu_\chi \in \mathrm{X}$:

$$\xi * \nu_\chi = \nu_\chi * \xi = \psi_\chi(\xi)\nu_\chi = \langle \xi, \nu_\chi\rangle\nu_\chi .$$

If E is an open and closed subset of βG, then from Proposition 3.9 it follows that

$$(\xi * \nu_\chi)(E) = \int_{\beta G} \pi_*^{-1}[\nu_\chi(T_\sigma^{-1}E)]\, d\xi = \nu_\chi(E)\int_{\beta G} \pi_*^{-1}(\chi^\sim)\, d\xi$$
$$= \psi_\chi(\xi)\nu_\chi(E),$$

since

$$\nu_\chi(T_\sigma^{-1}E) = \int_{\beta G} U_\sigma C_E\, d\nu_\chi = \int_{\beta G} C_E\, d[V_\sigma \nu_\chi] = \chi(\sigma)^\sim \nu_\chi(E).$$

Thus we have shown that $\xi * \nu_\chi = \psi_\chi(\xi)\nu_\chi$.

Secondly, we prove that $\nu_\chi * \xi$ is a multiple of ν_χ by showing that

$$V_\rho(\nu_\chi * \xi) = \chi(\rho)^\sim(\nu_\chi * \xi)$$

for $\rho \in G$. Assume again that E is an open and closed subset of βG; using Proposition 3.9 we find that

$$[V_\rho(\nu_\chi * \xi)](E) = (\nu_\chi * \xi)(T_\rho^{-1}E)$$
$$= \int_{\beta G} \pi_*^{-1}[\xi(T_{(\sigma+\rho)}E)]\, d\nu_\chi$$
$$= \int_{\beta G} \pi_*^{-1}[\chi\cdot\xi(T_{(\sigma+\rho)}^{-1}E)]\, d\mu$$

$$= \chi(\rho)^{\sim} \int_{\beta G} U_\rho\{\pi_*^{-1}[\chi \cdot \xi(T_\sigma^{-1}E)]\} \, d\mu$$

$$= \chi(\rho)^{\sim} \int_{\beta G} \pi_*^{-1}[\xi(T_\sigma^{-1}E)] \, d\nu_\chi$$

$$= \chi(\rho)^{\sim}(\nu_\chi * \xi)(E)$$

and thus that $V_\rho(\nu_\chi * \xi) = \chi(\rho)^{\sim}(\nu_\chi * \xi)$.

Using the previous lemma we have that $\nu_\chi * \xi = c\nu_\chi$ for some c. The proof is completed by the application of ψ_χ as follows:

$$c = \psi_\chi(c\nu_\chi) = \psi_\chi(\nu_\chi * \xi) = \psi_\chi(\xi * \nu_\chi) = \psi_\chi(\nu_\chi) = 1.$$

Thus,

$$c = \nu_\chi * \xi = \nu_\chi = \xi * \nu_\chi$$

for $\xi \in \Lambda_2(\mu)$ and $\nu_\chi \in \mathrm{X}$ and the lemma follows.

LEMMA 5.4. *Let $\mathfrak{A}$ be a closed translation invariant subalgebra of $\Lambda_2(\mu)$ containing $\mathfrak{M}$ and let ψ be a continuous homorphism from $\mathfrak{A}$ to the complex numbers. Then there exists $\chi \in \Gamma$ so that $\psi = \psi_\chi \mid \mathfrak{A}$.*

Proof. Since ψ is a bounded linear functional on the Hilbert space $\mathfrak{A}$, there exists $\xi \in \mathfrak{A} \subset \Lambda_2(\mu)$ so that $\psi(\nu) = \langle \nu, \xi \rangle$ for $\nu \in \mathfrak{A}$. For $\nu \in \mathfrak{A}$ so that $\psi(\nu) \neq 0$ and $\tau \in G$ we define

$$f_\nu(\tau) = \psi(V_\tau \nu)/\psi(\nu).$$

Then for ν_1 and ν_2 in $\mathfrak{A}$ so that $\psi(\nu_1) \neq 0$ and $\psi(\nu_2) \neq 0$, we have

$$f_{\nu_1}(\tau) = \frac{\psi(V_\tau \nu_1)}{\psi(\nu_1)} = \frac{\psi[(V_\tau \nu_1) * \nu_2]}{\psi(\nu_1)\psi(\nu_2)} = \frac{\psi[\nu_1 * (V_\tau \nu_2)]}{\psi(\nu_1)\psi(\nu_2)} = \frac{\psi(V_\tau \nu_2)}{\psi(\nu_2)} = f_{\nu_2}(\tau)$$

since $(V_\tau \nu_1) * \nu_2 = \nu_1 * (V_\tau \nu_2)$. This last identity can be proved using Proposition 3.9 as follows; for E an open and closed subset of βG

$$[(V_\tau \nu_1) * \nu_2](E) = \int_{\beta G} \pi_*^{-1}[\nu_2(T_\sigma^{-1}E)] \, d(V_\tau \nu_1)$$

$$= \int_{\beta G} U_\tau\{\pi_*^{-1}[\nu_2 T_\sigma^{-1}E)]\} \, d\nu_1$$

$$= \int_{\beta G} \pi_*^{-1}[\nu_2(T_{(\sigma+\tau)}^{-1} E)] \, d\nu_1$$

$$= \int_{\beta G} \pi_*^{-1}[(V_\tau \nu_2)(T_\sigma^{-1}E)] \, d\nu_1$$

$$= [\nu_1 * (V_\tau \nu_2)](E).$$

Thus, the definition of f_ν is independent of ν and we denote this function on

G by f. Further, for $\nu \in \mathfrak{A}$ so that $\psi(\nu) \neq 0$ and $\sigma \in G$, we have

$$\psi(V_{\sigma^{-1}}\nu) = f(\sigma^{-1})\psi(\nu)$$

and since

$$\langle V_{\sigma^{-1}}\nu, \xi\rangle = \langle \nu, V_\sigma \xi\rangle,$$

then

$$\langle \nu, V_\sigma \xi\rangle = \langle V_{\sigma^{-1}}\nu, \xi\rangle = \psi(V_{\sigma^{-1}}\nu) = f(\sigma^{-1})\psi(\nu) = \langle \nu, f(\sigma^{-1})\xi\rangle.$$

After observing that $\psi(\nu) = 0$ implies $\psi(V_\sigma \nu) = 0$, we have that $V_\sigma \xi = f(\sigma^{-1})\xi$ for $\sigma \in G$. Thus it follows from Lemma 5.2 that $f \in \Gamma$ and that $\xi = \nu_{f\sim}$ or $\psi = \psi_{f\sim} \mid \mathfrak{A}$.

THEOREM 5.5. *The algebra* $\Lambda_2(\mu) = \mathfrak{M} \oplus \mathfrak{N}$, *where*

(1) $\mathfrak{M}$ *is a two-sided ideal contained in the center of* $\Lambda_2(\mu)$,

(2) $\mathfrak{N}$ *is the Jacobson radical of* $\Lambda_2(\mu)$,

(3) $\mathfrak{M}$ *annihilates* $\mathfrak{N}$, *and*

(4) $\mathfrak{N}$ *is the set of topologiually nilpotent elements of* $\Lambda_2(\mu)$.

Proof. Let $\nu \in \mathfrak{N}$ and $\mathfrak{A}$ be the closed translation invariant subalgebra generated by ν and $\mathfrak{M}$. Then $\mathfrak{A}$ is commutative because $\mathfrak{M}$ is contained in the center of $\Lambda_2(\mu)$ by Lemma 5.3 and translations of ν commute (see the proof of the preceding lemma). Thus Lemma 5.4 applies and the Jacobson radical of $\mathfrak{A}$ is seen to be $\mathfrak{N} \cap \mathfrak{A}$. Now since $\mathfrak{A}$ is commutative and ν is in the radical of $\mathfrak{A}$, it follows that ν is topologically nilpotent. Hence $\mathfrak{N}$ is a topologically nil two-sided ideal and is contained in the radical of $\Lambda_2(\mu)$ [9, pp. 56–57]. Using Lemma 5.1, we conclude that $\mathfrak{N}$ is the radical and the proof is complete.

Remark 5.6. If $\Lambda_2(\mu)$ were known to be commutative then Theorem 5.5 would follow directly from Lemma 5.4. Cf. [9, 2.3.6, p. 57].

The fact that $\mathfrak{N}$ is the Jacobson radical of $\Lambda_2(\mu)$ follows immediately from Lemmas 5.1 and 5.4. We use this less direct proof to also obtain (4).

6. Concluding remarks

As we stated in the preceding section we have been unable to determine for any infinite abelian group G whether the subspace $\mathfrak{N}$ is trivial or nontrivial. In any case the orthogonal complement $\mathfrak{M}$ of the radical of the generalized group algebra $\Lambda_2(\mu)$ of an elementary invariant measure μ is isometrically isomorphic to $\mathfrak{L}_2(\bar{\mu}, \bar{G})$, where $\bar{G}$ is the Bohr compactification of G and $\bar{\mu}$ the normalized Haar measure on $\bar{G}$ [10, §1.8].

We also do not know whether or not $\Lambda_2(\mu)$ is commutative. In case $\mathfrak{N} = \{0\}$, then it follows trivially that $\Lambda_2(\mu)$ is commutative. It is of course not true, however, that $\mathfrak{N} \neq \{0\}$ would imply that $\Lambda_2(\mu)$ was not commutative. In fact it seems quite possible that if $\mathfrak{N} \neq \{0\}$, then the multiplication on $\mathfrak{N}$ is trivial, that is, $\eta * \xi = 0$ for $\eta, \xi \in \mathfrak{N}$.

We conclude with a result showing that if $\mathfrak{N} \neq \{0\}$, then it is infinite-dimensional. For $\nu \in \mathfrak{N}$, let $\mathfrak{T}(\nu)$ denote the closed translation invariant subspace of $\Lambda_2(\mu)$ generated by ν. Then $\mathfrak{T}(\nu)$ is contained in $\mathfrak{N}$.

THEOREM 6.1. *If $\nu \in \mathfrak{N}$, then $\mathfrak{T}(\nu)$ is infinite-dimensional.*

Proof. The group $\mathcal{V} = \{V_\sigma \mid \sigma \in G\}$ restricted to $\mathfrak{T}(\nu)$ is a unitary representation of G and hence if finite-dimensional is discretely decomposable. Then since G is abelian it follows that $\mathfrak{T}(\nu)$ contains a one-dimensional subspace that is invariant with respect to $\mathcal{V}$ [8, Thm. 6, 418–419]. If ξ is a nonzero element of this subspace, then Lemma 5.2 tells us that $\xi = c\nu_\chi$ for some $\nu_\chi \in X$. The element ν_χ is orthogonal to $\mathfrak{N}$, however and so this is a contradiction. Thus, $\mathfrak{T}(\nu)$ is infinite-dimensional.

BIBLIOGRAPHY

1. R. ARENS, *The adjoint of a bilinear operator*, Proc. Amer. Math. Soc., vol. 2 (1951), pp. 839–848.
2. M. M. DAY, *Amenable semigroups*, Illinois J. Math., vol. 1 (1957), pp. 509–544.
3. R. G. DOUGLAS, *On the measure theoretic structure of an invariant mean*, Proc. Amer. Math. Soc., vol. 16 (1965) pp. 30–36.
4. L. GILMAN AND M. JERISON, *Rings of continuous functions*, New York, Van Nostrand, 1960.
5. E. GRANIRER, *A theorem on amenable semigroups*, Trans. Amer. Math. Soc., vol. 111 (1964), pp. 367–379.
6. E. HEWITT AND K. A. ROSS, *Abstract harmonic analysis I*, Grundlehren der Mathematischen Wissensschaften, Bd. 115, Berlin-Gottingen-Heidelberg, Springer, 1963.
7. E. HEWITT AND H. S. ZUCKERMAN, *Finite-dimensional convolution algebras*, Acta Math., vol. 93 (1955) pp. 76–119.
8. M. A. NAIMARK, *Normed rings*, Groningen, Noordoff, 1959.
9. C. E. RICHART, *Banach algebras*, New York, Van Nostrand, 1960.
10. W. RUDIN, *Fourier analysis on groups*, New York, Interscience, 1962.

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