# FURTHER REMARKS ON NONLINEAR FUNCTIONAL EQUATIONS

BY
FELIX E. BROWDER

## Introduction

In three preceding papers under a similar title, [5], [6], [7], the writer has studied mappings $T$ from a reflexive complex Banach space $X$ to its dual $X^*$ which we shall call *complex-monotone*. If $(w, u)$ is the sesquilinear pairing between $w$ in $X^*$ and $u$ in $X$, we shall call $T$ *complex-monotone* if it satisfies the two conditions:

(I) *For each positive integer $N$, there exists a continuous, strictly increasing real function $c_N$ on $R^1$ with $c_N(0) = 0$ such that*

$$|(Tu - Tv, u - v)| \geq c_N(\|u - v\|) \tag{1}$$

*for all $u$ and $v$ with $\|u\| \leq N$, $\|v\| \leq N$.*

(II) *There exists a real function $c$ on $R^1$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $u$,*

$$|(Tu, u)| \geq c(\|u\|)\|u\|. \tag{2}$$

It is the object of the present paper to sharpen and extend these results in several significant respects.

In the first place, in [5], [6], and [7], we discussed operators of two types, either $T = T_0 + C$ or $T = L + T_0 + C$, where $T_0$ is a nonlinear operator continuous from the strong topology of $X$ to the weak topology of $X^*$, (demicontinuous), $C$ is a nonlinear completely continuous operator from $X$ to $X^*$, and $L$ is a closed densely defined linear operator from $X$ to $X^*$ such that $L^*$ is the closure of its restriction to $D(L) \cap D(L^*)$. As compared with the best results in the theory of monotone operators from $X$ to $X^*$ where comparable assumptions are made on Re $(Tu - Tv, u - v)$ and Re $(Tu, u)$, (cf. [9]), these classes of operators seem too narrow in at least two respects. The continuity requirement on $T_0$ ought to be reduced to the assumption that $T_0$ is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$. In addition, the perturbing completely continuous operator $C$ should be allowed to intertwine itself with $T_0$ in a suitable sense rather than be merely an additional summand.

In Section 1, we carry through this weakening of requirements to obtain the following results:

THEOREM 1. *Let $T$ be a nonlinear complex-monotone mapping of the reflexive complex Banach space $X$ into its dual space $X^*$. Suppose that $T$ is continuous*

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*from finite-dimensional subspaces of $X$ to the weak topology of $X^*$. Then $T$ is a one-to-one mapping of $X$ onto $X^*$ with a continuous inverse from $X^*$ to $X$.*

THEOREM 2. *Let $T$ be a nonlinear mapping of the dense linear subset $D(T)$ of $X$ into $X^*$ such that $T = L + T_0$, where $L$ is a closed densely defined linear operator from $X$ to $X^*$ such that $L^*$ is the closure of its restriction to $D(L) \cap D(L^*)$, $T_0$ is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$ and maps bounded sets of $X$ into bounded sets of $X^*$. Suppose that $T$ is complex-monotone on $D(T)$. Then $T$ is a one-to-one mapping of $D(T)$ onto $X^*$ and has an inverse $T^{-1}$ mapping $X^*$ continuously into $X$.*

THEOREM 3. *Let $T$ be a mapping of the reflexive space $X$ into $X^*$, ($\dim X \geq 2$) where $T(u) = S(u, u)$ for a mapping $S$ of $X \times X$ into $X^*$ for which:*

(a) *For fixed $v$ in $X$, $S(\cdot, v)$ satisfies condition* (I) *with function $c_N$ independent of $v$ for $\| v \| < N$.*

(b) *For fixed $u$ in $X$, $S(u, \cdot)$ is completely continuous from $X$ to $X^*$ (i.e. continuous on bounded subsets of $X$ from the weak topology of $X$ to the strong topology of $X^*$) uniformly for $u$ on bounded subsets of $X$.*

(c) *$T$ is demicontinuous and satisfies condition* (II).

*Then $T$ maps $X$ onto $X^*$.*

THEOREM 4. *Let $T$ be a mapping of the dense linear subset $D(T)$ of $X$ into $X^*$, where $X$ is a reflexive complex Banach space of dimension $\geq 2$. Suppose that $T = T_0 + L$, where:*

(a) *$L$ is a densely defined closed linear operator from $X$ to $X^*$ such that $L^*$ is the closure of its restriction to $D(L) \cap D(L^*)$.*

(b) *For each $u$ in $X$, $T_0(u) = S(u, u)$, where $S$ is a mapping of $X \times X$ into $X^*$ such that for fixed $v$ in $X$, $S(v, \cdot)$ is completely continuous from $X$ to $X^*$ uniformly for $v$ on bounded subsets of $X$. $T_0$ is demicontinuous and maps bounded subsets of $X$ into bounded subsets of $X^*$.*

(c) *There exists a continuous strictly increasing function $c_N$ on $R^1$ for each $N > 0$ such that for all $u$ and $v$ in $D(T)$ with $\| u \|, \| v \| \leq N$,*

$$| (Lu - Lv + S(u, v) - S(v, v), u - v) | \geq c_N(\| u - v \|).$$

(d) *$T$ satisfies condition* (II) *for all $u$ in $D(T)$.*

*Then $T$ maps $D(T)$ onto $X^*$.*

In Section 2, we give the following analogue of a theorem for monotone operators established by the writer [8] and T. Kato [10]:

THEOREM 5. *Let $T$ be a mapping from the reflexive complex Banach space $X$ to its dual $X^*$. Suppose that $T$ is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$ and that there exists a demicontinuous mapping $U$ of $X$ into $X^*$ (i.e. continuous from the strong topology of $X$ to the weak topology* *of* $X^*$) *such that* $T + U$ *is complex-monotone and* $T + U$ *is locally bounded* (*i.e. maps some neighborhood of each point into a bounded set*).

*Then* $T$ *is demicontinuous from* $X$ *to* $X^*$.

A related partial result in Hilbert space is given in Theorem 2 of Petryshyn [14].

In Section 3, we discuss the computability of the solution $u$ of the equation $Tu = w$ under the hypotheses of Theorems 1 and 2 in separable spaces $X$, obtaining stronger forms for convergence results of [14] in the more general context of Banach spaces. (Sequential approximations to such solutions in Hilbert space were discussed in detail by Zarantonello in [15] for the case of operators $T$ satisfying one-sided Lipschitz conditions.)

In Section 4, we discuss an interesting application in Hilbert space of the writer's result in [6] given by Petryshyn in [14] to yield a nonlinear generalization of the theory of the Friedrichs extension. We reformulate and reprove Petryshyn's result and give an analogous result for Banach spaces $X$.

The writer is indebted to W. Petryshyn for having made a preliminary draft of [14] available to him.

## Section 1

We proceed to the proofs of Theorems 1 through 4 as stated in the introduction.

*Proof of Theorem* 1. Let $\Lambda$ be the directed set of finite-dimensional subspaces of $X$ ordered by inclusion. For each $F$ in $\Lambda$, let $j_F$ be the injection map of $F$ into $X$, $j_F^*$ the dual projection map of $X^*$ onto $F^*$. We set

$$T_F = j_F^* \circ T \circ j_F : F \to F^*.$$

The hypothesis that $T$ is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$ implies that each $T_F$ is continuous. Moreover for $u$ and $v$ in $F$,

$$(T_F u, u) = (Tu, u)$$

and

$$(T_F u - T_F v, u - v) = (Tu - Tv, u - v).$$

Hence $T_F$ satisfies the hypotheses of Theorem 1 with $X$ replaced by $F$. Since $T_F$ is continuous, we may apply Theorem 1 of [5] to obtain the fact that $T_F$ maps $F$ one-to-one onto $F^*$.

To prove Theorem 1, it suffices to show that 0 lies in $R(T)$, the range of $T$, since for every $w$ in $X^*$, the mapping $T_W u = Tu - w$ will satisfy the hypotheses of Theorem 1 if $T$ does.

Let $u_F$ be the unique solution in $F$ of the equation $T_F u_F = 0$. We know that

$$0 = (T_F u_F, u_F) = (Tu_F, u_F) \geq c(\| u_F \|)\| u_F \|.$$

Since $c(r) \to +\infty$ as $r \to +\infty$, there exists a constant $M > 0$ independent of $F$ in $\Lambda$ such that $\| u_F \| \leq M$ for all $F$ in $\Lambda$.

Since $X$ is reflexive by hypothesis, the closed ball of radius $M$ about the origin in $X$ is weakly compact. Hence there exists $u_0$ in $X$ such that for every $F_0$ in $\Lambda$, $u_0$ lies in the weak closure of the set

$$V_{F_0} = \bigcup_{F_0 \subset F} \{u_F\}.$$

Let $F$, $F_1$ be elements of $\Lambda$ with $F \subset F_1$. Then

$$c_M(\| u_{F_1} - u_F \|) \leq | (Tu_{F_1} - Tu_F, u_{F_1} - u_F) |.$$

On the other hand,

$$(Tu_{F_1} - Tu_F, u_{F_1} - u_F) = (Tu_{F_1}, u_{F_1} - u_F) - (Tu_F, u_{F_1}) + (Tu_F, u_F),$$

while

$$(Tu_{F_1}, u_{F_1} - u_F) = (T_{F_1} u_{F_1}, u_{F_1} - u_F) = 0,$$

$$(Tu_F, u_F) = (T_F u_F, u_F) = 0.$$

Hence

$$c_M(\| u_F - u_{F_1} \|) \leq | (Tu_F, u_{F_1}) |.$$

Let $q_M(r)$ be the continuous strictly increasing function which is the inverse of $c_M(r)$. (We may assume without loss of generality that $c_M(r) \to +\infty$ as $r \to +\infty$ and that $c_M(r) \to -\infty$ as $r \to -\infty$.) Then

$$\| u_{F_1} - u_F \| \leq q_M(| (Tu_F, u_{F_1}) |).$$

Let $f(v)$ be defined for $v$ in $X$ as

$$f(v) = \| v - u_F \| - q_M(| (T_{u_F}, v) |.$$

Then $f$ is weakly lower semi-continuous in $v$. We know by the preceding argument that $f(v) \leq 0$ for $v$ in $V_F$. Hence $f(v) \leq 0$ on the weak closure of $V_F$, and in particular $f(u_0) \leq 0$. Thus

$$\| u_0 - u_F \| \leq q_M(| (Tu_F, u_0) |).$$

Suppose now that $F$ is an element of $\Lambda$ which contains $u_0$. Then

$$(Tu_F, u_0) = (T_F u_F, u_0) = 0,$$

and hence

$$\| u_F - u_0 \| \leq q_M(0) = 0,$$

i.e. $u_F = u_0$ for such $F$.

Finally, for any $v$ in $X$, let $F$ be an element of $\Lambda$ which contains both $u_0$ and $v$. Then

$$(Tu_0, v) = (T_F u_0, v) = (T_F u_F, v) = 0,$$

so that $(Tu_0, v) = 0$ for all $v$ in $X$. Hence $Tu_0 = 0$, Q.E.D.

*Proof of Theorem* 2. In this case, we let $\Lambda$ be the directed set of finite-dimensional subspaces of $D(T) = D(L)$ ordered by inclusion. As in the proof of Theorem 1, we let $j_F$ be the inclusion map of $F$ into $X$, $j_F^*$ the dual projection map of $X^*$ onto $F^*$, and

$$T_F = j_F^* T j_F : F \to F^*$$

which is well defined for $F$ in $\Lambda$ since $F \subset D(T)$.

Since $T_0$ is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$ and since every linear map $L$ is always continuous on finite-dimensional subspaces of $D(L)$, $T_F$ is continuous from $F$ to $F^*$ for every $F$ in $\Lambda$. Moreover, it satisfies conditions (I) and (II) by the same argument as in the preceding proof. Hence $T_F$ maps $F$ one-to-one onto $F^*$.

It suffices as in the proof of Theorem 1 to prove that there exists $u_0$ in $D(T)$ such that $Tu_0 = 0$. For each $F$ in $\Lambda$, there exists a unique $u_F \in F$ such that $T_F u_F = 0$. As before, there exists a constant $M > 0$ independent of $F$ such that

$$\| u_F \| \leq M$$

for all $F$ in $\Lambda$. Hence by the weak compactness of closed balls in $X$, there exists $u_0$ in $X$ such that for each $F_0$ in $\Lambda$, $u_0$ lies in the weak closure of the set

$$V_{F_0} = \cup_{F_0 \subset F} \{u_F\}.$$

We shall show first that $u_0$ lies in $D(T) = D(L)$. Let $v$ be an arbitrary element of $D(L) \cap D(L^*) = D(L) \cap D(L^*)$, and let $F$ be an element of $\Lambda$ which contains $v$.

Then

$$0 = (T_F u_F, v) = (Tu_F, v) = (Lu_F, v) + (T_0 u_F, v).$$

Since $v \in D(L^*)$,

$$(Lu_F, v) = (u_F, L^* v).$$

Since $\| u_F \| \leq M$ while $T_0$ maps bounded sets of $X$ into bounded sets in $X^*$, there exists a constant $M_1$ independent of $F$ in $\Lambda$ such that for all $F$ in $\Lambda$,

$$| (T_0 u_F, v) | \leq M_1 \| v \|.$$

Hence

$$| (u_F, L^* v) | \leq M_1 \| v \|.$$

Let $F_0$ be an element of $\Lambda$ containing $v$. For $u$ in $V_{F_0}$, it follows by the preceding argument that

$$| (u, L^* v) \| - M_1 \| v \| \leq 0.$$

Since the term on the left of the inequality is weakly continuous in $u$, it follows that the inequality persists on the weak closure of $V_{F_0}$, and, in particular, that

$$| (u_0, L^* v) | \leq M_1 \| v \|$$

for all $v$ in $D(L) \cap D(L^*)$. Since $L^*$ is the closure of its restriction to

$D(L) \cap D(L^*)$, it follows that

$$|(u_0, L^*v)| \leq M_1 \|v\|, \qquad v \in D(L^*),$$

i.e. $u_0 \in D(L^{**}) = D(L) = D(T)$, (since $L$ being closed implies that $L^{**} = L$).

Let $F$ and $F_1$ be two elements of $\Lambda$ with $F \subset F_1$. Then as in the proof of Theorem 1, we have

$$\|u_F - u_{F_1}\| \leq q_M(|(Tu_F, u_{F_1})|)$$

and

$$\|u_0 - u_{F_1}\| \leq q_M(|(Tu_F, u_0)|).$$

Since $u_0$ has been shown to lie in $D(T)$, there exists an element $F$ of $\Lambda$ which contains $u_0$. For such $F$,

$$(Tu_F, u_0) = (T_F u_F, u_0) = 0$$

and it follows that $\|u_F - u_0\| = 0$, i.e. $u_F = u_0$. Finally, for an arbitrary $v$ in $D(T)$, let $F$ be an element of $\Lambda$ which contains both $u_0$ and $v$. Then

$$(Tu_0, v) = (T_F u_0, v) = (T_F u_F, v) = 0.$$

Since $D(T)$ is a dense linear subset of $X$ while $Tu_0$ annihilates every $v$ in $D(T)$, it follows that $Tu_0 = 0$, Q.E.D.

*Proof of Theorem* 3. It suffices as before to show that $0 \in R(T)$. We now let $\Lambda$ be the directed set of finite-dimensional subspaces $F$ of $X$ of dimension $\geq 2$. Let $j_F$, $j_F^*$, and $T_F = j_F^* T j_F : F \to F^*$ be as before. Then $T_F$ maps $F$ continuously into $F^*$, and for each $u$ in $F$,

$$|(T_F u, u)| = |(Tu, u)| \geq c(\|u\|)\|u\|$$

where $c(r) \to +\infty$ as $r \to +\infty$. Since $F$ is of dimension $\geq 2$, we may apply Theorem 1 of [6] to obtain the existence of at least one element $u_F$ of $F$ which is mapped by $T_F$ onto 0. For such $u_F$, we have as before $\|u_F\| \leq M$, where $M$ is a constant independent of $F$ in $\Lambda$.

By the reflexivity of $X$, there exists an element $u_0$ which lies in the weak closure of the set

$$V_{F_0} = \bigcup_{F_0 \subset F \in \Lambda} \{u_F\}$$

for every $F_0$ in $\Lambda$. Let $F$ and $F_1$ be two elements of $\Lambda$ with $F \subset F_1$. Then

$$c_M(\|u_{F_1} - u_F\|) \leq |(S(u_{F_1}, u_{F_1}) - S(u_F, u_{F_1}), u_{F_1} - u_F)|$$

while,

$$\begin{aligned}(S(u_{F_1}, u_{F_1}) &- S(u_F, u_{F_1}), u_{F_1} - u_F) \\ &= (Tu_{F_1} - Tu_F, u_{F_1} - u_F) + (S(u_F, u_{F_1}) - S(u_F, u_F), u_{F_1} - u_F).\end{aligned}$$

For the first summand on the right, we have as before

$$(Tu_{F_1} - Tu_F, u_{F_1} - u_F) = -(Tu_F, u_{F_1}).$$

Hence, if as in the proof of Theorem 1, $q_M(r)$ is the inverse function of $c_M(r)$, we have

$$\| u_{F_1} - u_F \| \leq q_M(| (Tu_F, u_{F_1}) | + | (S(u_F, u_{F_1}) - S(u_F, u_F), u_{F_1} - u_F)|).$$

Let $g$ be the function on $X$ given by

$$g(v) = \| v - u_F \| - q_M(| (Tu, v) | + | (S(u_F, v) - S(u_F, u_F), v - u_F) |).$$

Since $\| v - u_F \|$ is weakly lower semi-continuous in $v$, $q_M$ is continuous, and the argument of $q_M$ in the definition of $g$ is weakly continuous on bounded subsets of $v$ by hypothesis (b) of Theorem 3, it follows that $g(v)$ is weakly lower semi-continuous in $v$ on bounded subsets of $X$. Since $g(v) \leq 0$ on $V_F$, it follows that $g(u_0) \leq 0$, i.e.

$$\| u_0 - u_F \| \leq q_M(| (Tu_F, u_0) | + | (S(u_F, u_0) - S(u_F, u_F), u_0 - u_F) |).$$

Let $F$ be an element of $\Lambda$ which contains $u_0$. Then $(Tu_F, u_0) = 0$. Since $\| u_F \| \leq M$, there exists a weak neighborhood $V$ of $u_0$ in $X$ such that for all $w$ in $V$ and all $F$ in $\Lambda$, we have

$$\| S(u_F, u_0) - S(u_F, v) \| < \varepsilon$$

for a prescribed $\varepsilon > 0$. We may find $F_1$ in $\Lambda$ which contains $F$ and such that $u_{F_1} \in V$. Hence

$$\| u_0 - u_{F_1} \| \leq q_M(M\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence $u_0$ lies in the strong closure of the set $V_F$. Since $T$ is demicontinuous, $Tu_0$ lies in the weak closure of the set $T(V_F)$ for each $F$ in $\Lambda$ which contains $u_0$. However, $(Tu_F, v) \to 0$ on $\Lambda$ for each $v$ in $X$. Hence the intersection over $F$ in $\Lambda$ of the weak closures of $T(V_F)$ consists only of the single element 0. Hence $Tu_0 = 0$, Q.E.D.

*Proof of Theorem* 4. We take $\Lambda$ to be the directed set of finite-dimensional subspaces of $D(T)$ of dimension $\geq 2$. We obtain $u_0$ as in the proof of Theorem 3 and show that it lies in $D(T)$ as in the proof of Theorem 2. By the same argument as in the proof of Theorem 3, we then show that for every $F$ in $\Lambda$ which contains $u_0$ and for each $\varepsilon > 0$, there exists $F_1$ in $\Lambda$ with $F \subset F_1$ such that $\| u_0 - u_{F_1} \| < \varepsilon$. Since $T_0$ is demicontinuous, this implies that for each $v$ in $X$ and each $\varepsilon > 0$, we may obtain $F_1$ as above so that

$$| (T_0 u_{F_1} - T_0 u_0, v) | < \varepsilon \| v \|.$$

Now let $v$ be any element of $D(L) \cap D(L^*)$, where by the hypothesis on $L$, $D(L) \cap D(L^*)$ is dense in $X$. Since $v \in D(L)$, $(Tu_F, v) \to 0$ on $\Lambda$. Hence

$$(u_F, L^*v) + (T_0 u_F, v) \to 0$$

on $\Lambda$. For $F_1$ as above, we have

$$| (u_{F_1}, L^*v) + (T_0 u_{F_1}, v) - (u_0, L^*v) - (T_0 u_0, v) | < \varepsilon_1,$$

where $\varepsilon_1$ can be made arbitrarily small by suitable choice of $F_1$. Hence

$$(Tu_0, v) = 0$$

for all $v$ in $D(L) \cap D(L^*)$, and since this latter set is dense, $Tu_0 = 0$, Q.E.D.

## Section 2

In the present section, we give the simple proof of Theorem 5, as stated in the introduction.

*Proof of Theorem* 5. Since $T = (T + U) - U$, and $U$ is assumed to be demicontinuous, it suffices to prove that $T + U$ is demicontinuous, i.e. to replace $T$ by $T + U$. Hence we may assume without loss of generality that $T$ satisfies conditions (I) and (II) of the introduction and is locally bounded. By Theorem 1, we know that $T$ is a one-to-one mapping of $X$ *onto* $X^*$.

Let $u_k \to u$ strongly in $X$. Since $\{Tu_k\}$ is a bounded sequence by the local boundedness assumption and since $X^*$ is reflexive, to prove that $Tu_k \to Tu$ weakly in $X^*$, it suffices to show that if $Tu_k$ converges to $w$ in $X^*$, then $w = Tu$. Since $T$ is onto, there exists $v$ in $X$ such that $Tv = w$. Let $M$ be an upper bound for $\| u_k \|$ and $\| v \|$. Then

$$c_M(\| v - u_k \|) \leq | (Tv - Tu_k, v - u_k) |.$$

Since $Tu_k \to w = Tv$ weakly in $X^*$ while $v - u_k \to v - u$ strongly in $X$, we know that

$$(Tv - Tu_k, v - u_k) \to (0, v - u) = 0.$$

Hence

$$\| v - u_k \| \to 0,$$

i.e., $u = v$. Finally $Tu = w$, Q.E.D.

The proof of Theorem 5 can obviously be combined with Theorem 2 to yield conclusions on the demicontinuity of $T$ in that theorem on $D(T)$. More generally, an examination of the argument yields the following conclusion:

THEOREM 6. *Let $T$ be a mapping defined on a subset $D(T)$ of $X$ and satisfying condition* (I) *of the introduction on $D(T)$. Suppose that $T$ maps onto $X$. Then $T$ is demicontinuous on $D(T)$.*

## Section 3

It is our purpose in the present section to present a simple result on the computability of the solutions $u$ of the equation $Tu = w$ under hypotheses like those of Theorem 1 in the case in which the Banach space $X$ is separable.

THEOREM 7. *Let $X$ be a reflexive, separable, complex Banach space, $T$ a mapping of $X$ into its dual space $X^*$ which is continuous from finite-dimensional subspaces of $X$ to the weak topology of $X^*$. Suppose that $T$ maps bounded* *subsets of* $X$ *into bounded subsets of* $X^*$. *Suppose also that* $T$ *is complex-monotone. Let* $\{\Psi_k\}$ *be a dense linearly independent set of elements of* $X$, $w$ *an element of* $X^*$. *For each integer* $j \geq 1$, *let* $F_j$ *be the subspace of* $X$ *spanned by* $\{\Psi_1, \cdots, \Psi_j\}$. *Then:*

(a) *For each* $j \geq 1$, *there is a unique solution* $u_j$ *in* $F_j$ *of the system of equations*

$$(Tu_j, \Psi_k) = (w, \Psi_k), \qquad 1 \leq k \leq j.$$

(b) *As* $j \to +\infty$, $\| u_j - u_0 \| \to 0$, *where* $u_0$ *is the unique solution of the equation* $Tu_0 = w$ *in* $X$.

(c) *If* $M = \sup_j \| u_j \|$, *then* $M$ *depends only upon* $\| w \|$ *and the function* $c(r)$ *of condition* (II), *and*

$$c_M(\| u_j - u_0 \|) \leq M \| Tu_j - w \|.$$

*If* $T$ *is also assumed to be continuous, then the right-hand side of the inequality will converge to zero, as well as the left.*

*Proof of Theorem* 7. Without loss of generality, we may assume that $w = 0$. The system of equations in part (a) is precisely equivalent to the equation $T_{F_j} u_j = j^*_{F_j} w = 0$, which we have already remarked to have a unique solution in $F_j$. Hence the conclusion of (a) follows.

For the proof of (b), we remark first that the existence and uniqueness of the solution $u_0$ of $Tu_0 = w$ is assured to us by Theorem 1. We next observe that $M = \max \{\| u_0 \|, \sup_j \| u_j \|\}$ depends only upon $\| w \|$ and the function $c(r)$ of condition (II) of the introduction which we hypothesize for $T$. Applying condition (I), we obtain

$$\begin{aligned} c_M(\| u_0 - u_j \|) &\leq | (Tu_0 - Tu_j, u_0 - u_j) | \\ &= | (w - Tu_j, u_0) - (w - Tu_j, u_j) | = | (w - Tu_j, u_0) | \end{aligned}$$

since $(w - Tu_j, u_j) = 0$. For every $v$ in the dense union of the $F_k$, $(w - Tu_j, v) = 0$ for $j \geq k$, where $v \in F_k$. Hence

$$(w - Tu_j, v) \to 0$$

for a dense set of elements $v$ in $X$. Since $w - Tu_j$ is uniformly bounded in norm for all $j$ because of $\| u_j \| \leq M$ and the assumption that $T$ maps bounded sets into bounded sets, it follows that $Tu_j - w$ converges weakly to zero. Hence $| (w - Tu_j, u_0) | \to 0$ as $j \to \infty$. Thus

$$c_M(\| u_0 - u_j \| \to 0$$

and hence

$$\| u_0 - u_j \| \to 0, \qquad j \to \infty.$$

For the proof of (c), we observe that

$$c_M(\| u_0 - u_j \|) \leq | (w - Tu_j, u_0) | \leq M \| w - Tu_j \|, \qquad \text{Q.E.D.}$$

## Section 4

In his forthcoming paper [14], Petryshyn has given an interesting application of the writer's results in [5] and [6] to the study of extensions of nonlinear operators satisfying generalized monotonicity conditions. It is the object of the present Section to reformulate the problem treated by Petryshyn in [14] and to establish results of the same type in a more general context.

Let $H$ be a complex Hilbert space, $T$ and $K$ two linear operators with a common dense domain $D(T)$ in $H$ and with range in $H$, with $K$ closeable and $R(K)$ dense in $H$. Then $T$ is said to be $K$-positive definite if there exist positive constants $\alpha_1$ and $\alpha_2$ such that

$$(Tu, Ku) \geq \alpha_1 \| u \|^2, \qquad \| Ku \|^2 \leq \alpha_2 (Tu, Ku)$$

for all $u$ in $D(T)$. On $D(T)$, we may define a Hermitian inner product $[u, v]$ by

$$[u, v] = (Tu, Kv)$$

since $(Tu, Kv) = (Ku, Tv)$ by polarization. The first inequality above tells us that the inner product $[u, v]$ defines a pre-Hilbert space structure on $D(T)$ with a bounded injection $J$ into $H$. Let $H_0$ be the completion of $D(T)$ with respect to the inner product $[\cdot, \cdot]$. Then $J$ may be extended by continuity to a bounded linear mapping of $H_0$ into $H$. Moreover, $J$ is one-to-one and identifies $H_0$ with a linear subset of $H$. (Indeed, suppose $\{u_k\}$ is a sequence from $D(T)$ such that $u_k \to u$ in $H_0$ while $u_k \to 0$ in $H$. Since $\| Ku_k \|$ is uniformly bounded, we may assume that $Ku_k$ converges weakly to an element $w$ of $H$. Since the weak and strong closures of the graph of $K$ coincide, it follows that $w = 0$. Hence for every $v$ in $D(T)$, $[u_k, v] = (Ku_k, Tv) \to 0$, which implies that $u = 0$.)

The second inequality $\| Ku \|^2 \leq \alpha_2 (Tu, Ku)$ implies that the mapping $u \to Ku$ of $D(T)$ into $H$ is bounded from the $H_0$-norm to $H$ and therefore can be extended uniquely by continuity to a bounded linear mapping $K_0$ of $H_0$ into $H$. Let $K_0^*$ be the adjoint mapping of $H$ into $H_0$. Since the range of $K$ is dense in $H$ by hypothesis, $K_0^*$ is one-to-one.

In [14], Petryshyn considers a nonlinear mapping $P$ from $D(T)$ into $H$ which satisfies the conditions:

$$|(Pu - Pv, K(u - v))| \geq \alpha_3 \| u - v \|_{H_0}^2, \qquad u, v \in D(T),$$

$$(Pu, K_0 v) \quad \text{is continuous in} \quad u \quad \text{on} \quad H_0 \quad \text{for each} \quad v \quad \text{in} \quad H_0 .$$

He shows the existence of an unique extension $P_0$ of $P$ whose domain lies in $H_0$ with range in $H$, whose range is all of $H$, and which also satisfies both of the above conditions. This result may be obtained as a special case of the following simple generalization:

THEOREM 8. *Let $X$ and $Y$ be two complex Banach spaces, with $X$ reflexive. Let $P$ be a (possibly) nonlinear operator with domain $D(P)$ a dense subset of $X$*

*and with range in* $Y^*$, $K_0$ *a bounded linear operator from* $X$ *to* $Y$ *whose range is dense in* $Y$. *Suppose that* $P$ *and* $K_0$ *satisfy the following conditions:*

(*1*) *For each integer* $N > 0$, *there exists a continuous strictly increasing function* $c_N$ *on* $R^1$ *with* $c_N(0) = 0$ *such that for all* $u$ *and* $v$ *in* $D(P)$ *with* $\| u \| \leq N$, $\| v \| \leq N$,

$$(4.1) \qquad |(Pu - Pv, K_0 u - K_0 v)| \geq c_N(\| u - v \|).$$

(*2*) *There exists a real function* $c$ *on* $R^1$ *with* $c(r) \to +\infty$ *as* $r \to +\infty$ *such that for all* $u$ *in* $D(P)$,

$$(4.2) \qquad |(Pu, K_0 u)| \geq c(\| u \|)\| u \|.$$

(*3*) *For each* $v$ *in* $X$, $(Pu, K_0 v)$ *is continuous in* $u$ *on* $X$.

*Then there exists a unique extension* $P_0$ *of* $P$ *mapping from its domain in* $X$ *to* $Y^*$ *such that* $P_0$ *maps one-to-one onto* $Y^*$ *and for each* $v$ *in* $X$, $(P_0 u, K_0 v)$ *is continuous in the* $X$*-norm for* $u$ *running through* $D(P_0)$. *For this extension,* (4.1) *and* (4.2) *hold with* $P$ *replaced by* $P_0$.

*Proof of Theorem* 8. Since $K_0$ is a bounded linear map of $X$ into $Y$, $K_0^*$ is a bounded linear mapping of $Y^*$ into $X^*$. The function $u \to K_0^* Pu$ is demicontinuous from $X$ to $X^*$ since for each $v$ in $X$,

$$(K_0^* Pu, v) = (Pu, K_0 v)$$

is continuous in the $X$-norm for $u$ running through $D(P)$. Since $X$ is complete in the strong topology and $X^*$ is complete in the weak topology, the mapping $K_0^* P$ may be extended uniquely to a demicontinuous mapping $Q$ from the whole of $X$ into $X^*$.

For $u$ in $D(P)$, we have from (4.1)

$$(4.3) \quad |(Qu - Qv, u - v)| = |( Pu - Pv, K_0(u - v)| \geq c_N(\| u - v \|)$$

and from (4.2),

$$(4.4) \qquad |(Qu, u)| = |(Pu, K_0 u)| \geq c(\| u \|)\| u \|.$$

Extending $Q$ by demicontinuity, we observe that both sides of the inequalities (4.3) and (4.4) are continuous on the graph of $Q$ in $X_s \times X_w^*$, where $X_s$ is $X$ in the strong topology and $X_w^*$ is $X^*$ taken with the weak topology. Hence (4.3) and (4.4) hold for all $u$ and $v$ in $X$.

Applying Theorem 1, above, or the corresponding results of [5], we see that $Q$ maps $X$ onto $X^*$. Let $D_0 = Q^{-1}(R(K_0^*))$. Since $K_0$ has a dense range in $Y$, $K_0^*$ is an injective map of $Y^*$ into $X^*$ and has an inverse $(K_0^*)^{-1}$ mapping $R(K_0^*)$ onto $Y^*$. We define $P_0$ by

$$D(P_0) = D_0, \qquad P_0 u = (K_0^*)^{-1}Qu, \qquad u \in D(P_0).$$

Since $R(Q) = X^*$ and $R((K_0^*)^{-1}) = Y^*$, $R(P_0)$ must be all of $Y^*$. Since

$Q$ is one-to-one and $(K_0^*)^{-1}$ is one-to-one, $P_0$ is one-to-one. Since

$$(P_0 u, K_0 v) = (K_0^*(K_0^*)^{-1}Qu, v) = (Qu, v)$$

$(P_0 u, K_0 v)$ is continuous in $u$ for fixed $v$ by the demicontinuity of $Q$. Since the last equation would be true for any $P_0$ satisfying the conditions of the theorem, $P_0$ is uniquely determined by the conditions of the theorem. We observe that (4.1) and (4.2) go over to $P_0$ by the demicontinuity of $Q$. To complete the proof of Theorem 8, we need only show that $P_0$ is really an extension of $P$.

For $u$ in $D(P)$, however, $Qu = K_0^* Pu$ does lie in $R(K_0^*)$. Hence $u$ lies in $D(P_0)$ and

$$P_0 u = (K_0^*)^{-1}Qu = (K_0^*)^{-1}K_0^* Pu = Pu, \qquad \text{Q.E.D.}$$

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The University of Chicago
Chicago, Illinois