PERMUTABLE CONGRUENCES IN A LATTICE[1]

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The purpose of this paper is to give certain conditions which are equivalent to the permutability of congruences in lattices and to reveal their utility by proving certain results as direct consequences of them. Thus we formulate the conditions which are equivalent to the permutability of two congruence relations on a lattice in Theorem 1. Making use of these conditions we prove in Theorem 2 that any two p-neutral congruences on a lattice permute. (A congruence of the form $x \, \theta \, y$ if and only if $x + a = y + a$ for some $a \, \theta \, 0$ is said to be a p-neutral congruence on the lattice (cf. [5]).) As corollaries to Theorem 2 we get the known results—any two standard congruences on a lattice permute, any two congruences on a weakly complemented lattice permute and any two congruences on a relatively complemented lattice permute. Further these conditions enable us to give a proof of the result—any two congruences on a distributive lattice permute if and only if L is relatively complemented.

In Theorem 3 we prove: any two congruences on a discrete modular lattice are permutable if and only if L satisfies condition (α); where (α) says that for all a, b, c in L with $a \succ b \succ c$ either (a, b) is projective with (b, c) or there exists a complement d of b in (a, c).($a \succ b$ means a covers b).

Theorem 4 is a generalization of Theorem 3 to weakly modular lattices and proves that any two congruences on a semi-discrete, weakly modular lattice L are permutable if and only if L satisfies condition (β); for all $a > b > c$ either (b, c) is a lattice translate (cf. [8]) of (a, b) or there exists a complement d of b in (a, c).

It is well known that two congruence relations θ and ϕ on a lattice L are said to be permutable if and only if $a \, \theta \, b$; $b \, \phi \, c$ implies the existence of a d in L such that $a \, \phi \, d$; $d \, \theta \, c$.

Next we give the conditions which are necessary and sufficient for the permutability of two congruence relations on a lattice.

THEOREM 1. *The following conditions on a lattice L are equivalent.*

(i) *The two congruences θ and ϕ on L permute.*

(ii) *For every comparable pair of elements (a, c), $a \, \theta \, b$; $b \, \phi \, c$ imply the existence of a d in L with $a \, \phi \, d$; $d \, \theta \, c$.*

(iii) *For all triples (a, b, c) forming a chain in that order $a \, \theta \, b$; $b \, \phi \, c$ imply the existence of a d in L with $a \, \phi \, d$; $d \, \theta \, c$.*

Received January 5, 1965.

[1] Forms a part of the Doctoral Thesis submitted to the University of Madras in January 1964.

(iv) *For all triples* (a, b, c) *forming a chain in that order,* $a\,\theta\, b$; $b\,\phi\, c$ *imply the existence of a* d *in* L *forming a chain* (a, d, c) *in that order with* $a\,\phi\, d$, $d\,\theta\, c$.

Proof. (i) implies (ii) implies (iii) obviously.

(iii) implies (iv). Let $a < b < c$ with $a\,\theta\, b$; $b\,\phi\, c$. Then there exists a d' in L such that $a\,\phi\, d'$; $d'\,\theta\, c$. Consider $d = (a + d')c$. Now $a = (a + a)c$ and $c = (a + c)c$. Hence $a\,\phi\, d$; $d\,\theta\, c$ and $a \leq d \leq c$. The other part when $a > b > c$ can be proved similarly.

(iv) implies (iii) trivially.

(iii) implies (ii). Let $a < c$; $a\,\theta\, b$; $b\,\phi\, c$. This implies $a = ac\,\theta\,(a + b)c$; $(a + b)c\,\phi\, c$. Now $a < (a + b)c < c$. Hence by (iii) there exists a d in L with $a\,\phi\, d$; $d\,\theta\, c$. Thus condition (ii) holds.

Next we prove (ii) implies (i).

$$a\,\theta\, b;\ b\,\phi\, c \Rightarrow abc\,\phi\, ab;\ ab\,\theta\,(a + b);\ (a + b)\,\phi\,(a + b + c)$$

$$\Rightarrow \exists\ d \text{ in } L \text{ with } abc\,\theta\, d;\ d\,\phi\,(a + b);\ (a + b)\,\phi\,(a + b + c) \quad (\text{by (ii) as } abc < (a + b))$$

$$\Rightarrow abc\,\theta\, d;\ d\,\phi\,(a + b + c).$$

Also

$$a\,\theta\, b;\ b\,\phi\, c \Rightarrow abc\,\theta\, ac;\ (a + c)\,\phi\,(a + b + c).$$

Therefore

$$a\,\theta\, b;\ b\,\phi\, c \Rightarrow ac\,\theta\, d;\ d\,\phi\,(a + c) \text{ for some } d \text{ in } L$$

$$\Rightarrow \exists\ e \text{ in } L \text{ such that } ac\,\phi\, e;\ e\,\theta\,(a + c) \quad (\text{by (ii) as } ac < (a + c)$$

$$\Rightarrow a = a + ac\,\phi\, a + e;\ a + e\,\theta\, a + c;\ a + c\,\theta\, e;\ e\,\phi\, ac;\ ac\,\phi\, ce;$$
$$ce\,\theta\, c(a + c) = c.$$

$$\Rightarrow a\,\phi\, a + e;\ a + e\,\theta\, e;\ e\,\phi\, ce;\ ce\,\theta\, c.$$

$$\Rightarrow \exists f \text{ in } L \text{ such that } a\,\phi\, a + e;\ a + e\,\phi\, f;\ f\,\theta\, ce;\ ce\,\theta\, c \quad (\text{by (ii) as } ce < (a + e)).$$

$$\Rightarrow a\,\phi\, f;\ f\,\theta\, c.$$

Therefore θ and ϕ permute, i.e., (i) holds.

As an important consequence of this we get,

THEOREM 2. *Any two p-neutral congruences of a lattice permute.*

Proof. Let θ and ϕ be two p-neutral congruences on L determined by the p-neutral ideals I and J respectively.

$$a < b < c;\ a\,\theta\, b;\ b\,\phi\, c$$

$$\Rightarrow a + t_1 = b + t_1;\ b + t_2 = c + t_2 \text{ for } t_1 \in I \text{ and } t_2 \in J$$

$$\Rightarrow a = ac\,\phi\,(a + t_2)c;\ (a + t_2)c\,\theta\,(a + t_1 + t_2)c = (c + t_1 + t_2)c = c$$

$$\Rightarrow \text{there exists a } d = (a + t_2)c \text{ in } L \text{ such that } a \leq d \leq c \text{ and } a\,\phi\, d;\ d\,\theta\, c.$$

Hence by condition (iv) the two congruences θ and ϕ permute.

As corollaries to Theorem 2 we get,

COROLLARY 1. *Any two standard congruences on a lattice permute.*

Proof follows as any standard congruence (cf. [4]) is a p-neutral congruence.

COROLLARY 2. *Any two congruences on a weakly complemented lattice permute.*

Proof follows as any congruence of a weakly complemented lattice is a standard congruence (cf. [4]).

COROLLARY 3. *Any two congruences of a relatively complemented lattice permute.*

COROLLARY 4. *Any two congruences on a distributive lattice L permute if and only if L is relatively complemented.*

Proof. The first part follows from Corollary 3. For the second part let L be a distributive lattice with permutable congruence relations. Let $a < b < c$. Now $a \,\theta\, b$ and $b \,\phi\, c$ for the congruences θ and ϕ on L generated by the intervals (a, b) and (b, c) respectively (cf. [8] for the congruence generated by an interval). Also $\theta \wedge \phi = 0$ (the null congruence on L (cf. [3]). As θ and ϕ permute on L by condition (iv) of Theorem 1 there exists a d in L with $a \leq d \leq c$ and $a \,\phi\, d$; $d \,\theta\, c$.

Now $bd = a$ and $b + d = c$. For if $bd \neq a$ then $a \;(\theta \wedge \phi)\; bd$ will imply $\theta \wedge \phi \neq 0$, a contradiction. Similarly $b + d = c$. Whence d is the complement of b in (a, c). Hence the lattice L is relatively complemented.

In the case of general lattices relative complementation is not a necessary condition for the permutability of congruences in lattices, as any two congruences of a simple lattice permute and a simple lattice need not always be relatively complemented. However we have weaker conditions of such lattices. Regarding modular lattices we have,

THEOREM 3. *Any two congruences on a discrete, modular lattice L permute if and only if for all $a \succ b \succ c$ either b has a complement d in (a, c); or (a, b) and (b, c) are projective.*

Proof. It is well known that if a congruence θ permutes with all congruence relations ϕ of a family of congruences Φ then θ permutes with $\bigcup_{\phi \in \Phi} \phi$ (cf. [2]) Also any congruence on a discrete lattice can be expressed as a sum of congruences generated by prime intervals. Further one can easily see that any congruence on a discrete modular lattice L is a minimal congruence on L if and only if it is generated by a prime interval. Thus any two congruences on a discrete modular lattice L permute if and only if any two minimal congruences on L permute.

We shall show next that the condition stated is precisely the condition required for any two minimal congruence relations on L to permute.

Let L be a discrete modular lattice with permutable congruence relations; and let $a \succ b \succ c$. Let θ, ϕ be the congruences on L generated by (a, b) and (b, c) respectively Then θ and ϕ are minimal congruence relations on L. Hence either $\theta \wedge \phi = 0$ or $\theta = \phi$. If $\theta = \phi$ then (a, b) and (b, c) are projective (cf. [6].)

Next let $\theta \wedge \phi = 0$. As θ and ϕ permute on L, by condition (iv) of Theorem 1 there exists a d in L such that $a \geq d \geq c$ with $a \, \phi \, d$; $d \, \theta \, c$. Now $bd = c$ and $b + d = a$; otherwise it will contradict the fact $\theta \wedge \phi = 0$. Thus d is the complement of b in (a, c). L being modular, we have the further relation $a \succ d \succ c$ as well.

Conversely let L be a discrete modular lattice satisfying the condition of the theorem. Let θ and ϕ be two minimal congruences on L. If $\theta = \phi$ then θ and ϕ permute trivially. Let $\theta \neq \phi$; and let $a \, \theta \, b$; $b \, \phi \, c$ with $a > b > c$. Consider

$$a = a_1 \succ a_2 \succ \cdots \succ a_n = b \quad \text{and} \quad b = b_1 \succ b_2 \succ \cdots \succ b_m = c$$

two finite maximal chains connecting (a, b) and (b, c) respectively. Then we have the following: (i) any (a_{i-1}, a_i) is projective with (a_{j-1}, a_j); (ii) any (b_{k-1}, b_k) is projective with (b_{l-1}, b_l); (iii) No (a_{i-1}, a_i) is projective with (b_{j-1}, b_j).

Consider $a_{n-1} \succ b_1 \succ b_2$. Now (a_{n-1}, b_1) is not projective to (b_1, b_2); hence there exist a d_1 in L with $a_{n-1} \succ d_1 \succ b_2$ such that $a_{n-1} \, \phi \, d_1$; $d_1 \, \theta \, b_2$. Now $d_1 \succ b_2 \succ b_3$ and by similar argument we can get d_2 with $d_1 \succ d_2 \succ b_3$ such that $d_1 \, \phi \, d_2$; $d_2 \, \theta \, b_3$ etc $\cdots$. Proceeding thus we can get a d in L such that $a \, \phi \, d$ and $d \, \theta \, c$ with $a \geq d \geq c$. Thus θ and ϕ satisfy condition (iv) of Theorem 1 and hence are permutable. Thus the proof is complete.

Theorem 3 can be generalized to semi-discrete, weakly modular lattices thus:

THEOREM 4. *Any two congruences on a semi-discrete, weakly modular lattice L permute if and only if for all a, b, c in L with $a \succ b > c$ either (a, b) is a lattice translate of (b, c) or b has a complement d in (a, c).*

Proof. Note the difference in the above condition from that of the condition in Theorem 3 i.e., b does not cover c here. This difference is due to the fact that unlike modular lattices, weakly modular lattices do not satisfy the Jordan Dedekind chain condition. The rest of the proof otherwise follows on similar lines as in Theorem 3.

I thank Professor V. S. Krishnan for his constant help during the preparation of this paper.

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