THE FELLER AND ŠILOV BOUNDARIES OF A VECTOR LATTICE

BY

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1. Introduction

Let E denote a discrete countable set and, for each $x \in E$, let $P(x, \cdot)$ denote a probability measure on E. Define the operator P by setting

$$Ph(x) = \int P(x, dy)h(y).$$

Denote by H the vector space of bounded functions h for which $Ph = h$. In [3], under the assumption that P defines a transient Markov process, Feller adjoined to the space E a compact space B with the following property. Every function h in H has a continuous extension to $E \cup B$ and every continuous real-valued function on B is the restriction to B of some extended function. The boundary B is the Stone space of the Boolean algebra $\mathbf{S}$ of extremal points of the convex set $\mathbf{I} = \{h \in H \mid 0 \leq h \leq 1\}$.

Feller noted that if E is the open unit disc $|z| < 1$ there was a similar operator P for which the eigenspace H consisted of the bounded harmonic functions. For this case, he pointed out that the ideal boundary B could be constructed and adjoined in a similar way.

Since the vector lattice H of bounded harmonic functions on $|z| < 1$ is order isomorphic to the vector lattice L^∞ of bounded Lebesgue measurable real-valued functions on the unit circle $|z| = 1$, it follows that the set $\mathbf{S}$ of extremals corresponds to the set of equivalences classes of the characteristic functions 1_A, where A is a measurable set of positive measure. Consequently, the Stone space of $\mathbf{S}$ can be identified with the Stone space of the measure algebra. This space can also be looked at as the maximal ideal space of the Banach algebra L^∞.

It is known (c.f. [5]) that the maximal ideal space of L^∞ can be embedded in the maximal ideal space of the Banach algebra H^∞ of bounded analytic functions on $|z| < 1$. Furthermore, with this identification, the maximal ideal space of L^∞ is the Šilov boundary of H^∞.

While it is not true that a bounded harmonic function h on $|z| < 1$ is the real part of some function in H^∞, it is true that every such h has a continuous extension to the maximal ideal space of H^∞. Consequently, the vector lattice H can be viewed as a vector lattice on that compact extension K of $|z| < 1$ which is the maximal ideal space of H^∞. It follows (c.f. Bauer [1]) that there is a smallest closed subset of K on which each function in H attains its maximum. This set is called the Šilov boundary of H. It is easily seen that it coincides with the Šilov boundary of H^∞. As a result, the Feller and Šilov boundaries of H coincide.

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Let E be a locally compact space and let H denote a uniformly closed vector lattice of bounded continuous real-valued functions on E. Assume that H contains the constant functions. One of the purposes of this note is to discuss the relationship between the Feller and Šilov boundaries of H. It turns out that the Feller boundary is the space of connected components of the Šilov boundary. Consequently, they coincide if and only if the Šilov boundary is totally disconnected.

The remainder of this note is devoted to a discussion of the total boundary and of the problem of adjoining these boundaries to the original space E. In [3], Feller defined a space that he called the total boundary. It is obtained from $H = H^+ - H^+$, where $H^+ = \{h \geq 0 \mid Ph = h\}$, as a space of maximal ideals. We show that these ideals are the kernels of certain functionals $\Phi : H \rightarrow \bar{\mathbf{R}}$, where $\bar{\mathbf{R}}$ is the two-point compactification of $\mathbf{R}$. As an application, we give a proof of the assertion in [4] that the total boundary is, in a suitable sense, expressible as a union of Feller boundaries.

The Šilov boundary of H, when H consists of bounded functions, can always be adjoined to E so that the functions in H have continuous extensions. In general E is not dense in the resulting space. In case E is not compact, we give necessary and sufficient conditions for E to be dense. The adjunction of the total boundary is carried out and a corresponding condition given for E to be dense in the resulting space.

2. The Šilov boundary of H

Let H denote a vector space of bounded continuous real-valued functions on a locally compact space E. Assume that H contains the constant functions and that with respect to the partial order $\leq$, where $f \leq g$ if $f(x) \leq g(x)$ for all x, it is a lattice.

The following proposition, due to C. Constantinescu and A. Cornea, is proved in [2].

PROPOSITION 1. *There is a unique compactification K of E such that:*
(1) *each function in H has a unique continuous extension to K; and*
(2) *the extensions to K of the functions in H separate the points of $K - E$.*

Remark. In case E is $|z| < 1$ and H consists of all the bounded harmonic functions on E, the space K is the maximal ideal space of the Banach algebra H^∞ of bounded analytic functions on E. If $u \in H$, let v denote its harmonic conjugate. Define $\bar{u}$ on the maximal ideal space of H^∞ by setting

$$\bar{u}(\varphi) = \log|\varphi(\exp(u + iv))|.$$

When E is embedded in the maximal ideal space of H^∞, the functions $\bar{u}$ are continuous extensions of the functions u. The validity of the corona conjecture [5] then shows that the maximal ideal space of H^∞ is the space K.

The vector lattice of continuous functions on K obtained by extending the functions of H to K will also be denoted by H. If H does not separate the points of K, let K' be the compact space obtained from K by identifying points which are not separated by the functions in H. Then H can be viewed as a vector lattice of continuous functions on the compact space K'. Again, it is convenient to let the new vector lattice also be denoted by H.

Bauer [1] showed (Satz 2) that K' has a smallest closed subset on which each function of H attains its maximum. This set is called the Šilov boundary of H and will be denoted by $\partial(H)$. Further, in case H is uniformly closed, he showed (Korollar 2 von Satz 10) that each continuous function on $\partial(H)$ is the restriction to $\partial(H)$ of a unique function in H. Hence, H is isomorphic to $C(\partial(H))$ under a lattice preserving linear isomorphism.

In the course of the proof of these last two statements, Bauer showed that for $x \in K'$ the functional $f \to f(x)$, $f \in H$, is a lattice preserving functional if and only if $x \in \partial(H)$. Conversely, if $\varphi : H \to \mathbf{R}$ is a lattice preserving linear functional with $\varphi(1) = 1$, then φ corresponds to a point of the Šilov boundary. As a result, the Šilov boundary of H can be identified with the set of lattice preserving linear functionals for which $\varphi(1) = 1$.

3. The Feller boundary of H

Consider H as the vector lattice of functions on the compact space K'. Assume H is uniformly closed.

PROPOSITION 2. *Let* $\mathbf{I} = \{f \in H \mid 0 \leq f \leq 1\}$. *Then f is an extremal point of this convex set if and only if $f \mid \partial(H)$ is the characteristic function of an open and closed subset of $\partial(H)$.*

Proof. In view of Bauer's results, it suffices to consider the case where $H = C(X)$, X a compact space.

If f is the characteristic function of an open and closed subset A of X, then $f \in \mathbf{I}$. Assume $f = tg + (1 - t)h$, with $g, h \in \mathbf{I}$ and $0 < t < 1$. Since $f \geq tg$ and $(1 - t)h$, it follows that g and h vanish off A. Clearly, on A both g and h have to assume the value one. Hence, $f = g = h$ and so f is extremal.

Let $f \in \mathbf{I}$ have 1 as its maximum value, and let λ, ε be >0 with $\lambda + \varepsilon \leq 1$. Denote by g the function $([\lambda/(\lambda + \varepsilon)]f) \wedge \lambda$. Then $g \leq \lambda$ and $g \leq f$. If $h = f - g$ the maximum of h is $1 - \lambda$. The functions

$$h_1 = [1/(1 - \lambda)]h \quad \text{and} \quad g_1 = (1/\lambda)g$$

are in $\mathbf{I}$ and $f = g + h = \lambda g_1 + (1 - \lambda)h_1$.

Assume that $f \in \mathbf{I}$ is an extremal. Then, the maximum value of f is 1 and its minimum is 0. Assume that, for some x_0, $0 < f(x_0) < 1$. Let $\lambda = f(x_0)$. Since f is extremal, $f = g_1 = (1/\lambda)g$. Now

$$g(x_0) = \min\{\lambda^2/(\lambda + \varepsilon), \lambda\}$$

and so $\lambda = f(x_0) = \min\{\lambda/(\lambda + \varepsilon), 1\} = \lambda/(\lambda + \varepsilon)$. As the only restriction on ε is that $\varepsilon > 0$ and $\lambda + \varepsilon \leq 1$, this leads to a contradiction. Therefore, f is two-valued.

Corollary. *Let* $\mathbf{S}$ *denote the set of extremals of* $\mathbf{I}$. *Then* $\mathbf{S}$ *is a Boolean algebra and a sublattice of* H. *Further,* $\mathbf{S}$ *is isomorphic to the Boolean algebra* $\mathbf{O}$ *of open and closed subsets of* $\partial(H)$.

Definition. The Feller boundary of H is the Stone space of the Boolean algebra $\mathbf{S}$.

Theorem 1. *The Feller boundary of* H *is homeomorphic with the space of connected components of the Šilov boundary of* H. *Hence, the two boundaries coincide if and only if the Šilov boundary is totally disconnected.*

Proof. The Stone space of $\mathbf{S}$ is homeomorphic to the Stone space of $\mathbf{O}$. The points of the Stone space of $\mathbf{O}$ can be taken to be the maximal dual-ideals (i.e. ultrafilters) $\mathbf{U}$ of the lattice $\mathbf{O}$. A base for the topology of the Stone space is given by the sets

$$A^* = \{\mathbf{U} \mid A \in \mathbf{U}\} \quad \text{where } A \in \mathbf{O}.$$

In a compact space the connected component of a point is the intersection of all the open and closed sets which contain it. Hence, if $\mathbf{U}$ is an ultrafilter in $\mathbf{O}$, the intersection of all the sets in $\mathbf{U}$ is a connected component of $\partial(H)$. This defines a 1-1 function θ from the Stone space of $\mathbf{O}$ onto the set of connected components of $\partial(H)$.

Let D be the set of connected components of $\partial(H)$ and let

$$\pi : \partial(H) \to D$$

associate with each point its connected component. Give D the quotient topology. It has as a base the sets $A_1 = \pi A$, where A is an open and closed subset of $\partial(H)$. Since $\theta^{-1}A_1 = A^*$, it follows that θ is continuous. Consequently, θ is a homeomorphism.

It is known (c.f. [6]), for a compact space X, that X is totally disconnected if and only if $C(X)$ has no proper closed subring A for which (i) $1 \in A$ and (ii) $f^2 \in A$ implies $f \in A$. A closed subring A of $C(X)$ which contains the constants is a closed vector lattice that contains 1 and, conversely, if V is a closed vector lattice that contains 1, then V is a closed subring.

Lemma. *Let* $A \subseteq C(X)$ *be a closed subring containing* 1. *The following statements are equivalent*:

(1) $f^2 \in A$ *implies* $f \in A$; *and*

(2) $|f| \in A$ *implies* $f \in A$.

Proof. If $|f| \in A$, then $f^2 = |f|^2 \in A$. Hence (1) implies (2). On the other hand, since A is closed, $f^2 \in A$ implies $|f| = \sqrt{f^2} \in A$. Consequently, (2) implies (1).

Since H is isomorphic as a vector lattice to $C(\partial(H))$, this completes the proof of the following proposition.

PROPOSITION 3. *The Šilov boundary of H is totally disconnected if and only if H has no proper closed vector sublattice V for which* (i) $1 \in V$ *and* (ii) $|f| \in V$ *implies* $f \in V$.

In [3], for the vector lattice H of bounded functions f such that $Pf = f$, Feller showed (Theorem 9.1) that a function f in $\mathbf{I}$ is in $\mathbf{S}$ if $f > \lambda s$, $\lambda > 0$ and s in $\mathbf{S}$, implies $f \geq s$. A vector lattice H satisfying this condition will be said to have property (S).

PROPOSITION 4. *If the Šilov boundary of H is totally disconnected, then H has property* (S). *The converse is false. Specifically, there exist compact spaces X which are not totally disconnected for which $C(X)$ has property* (S).

Proof. It suffices to show that, for a totally disconnected space X, $C(X)$ has property (S).

Assume that f is a non-extremal function in $\mathbf{I}$ for which $f > \lambda s$, $\lambda > 0$ and s in $\mathbf{S}$, implies $f \geq s$. Since f is not an extremal,

$$\{x \mid f(x) = 0\} = Z(f) \neq \emptyset$$

and, for some x_0, $0 < f(x_0) < 1$. Let A be an open and closed set containing x_0 and denote by s the characteristic function of A. If f fails to vanish on A, then, for some $\lambda > 0, f > \lambda s$ and so $f(x_0) = 1$. Therefore, $A \cap Z(f) \neq \emptyset$. This shows that the connected component of x_0 contains a point of $Z(f)$, and so X is not totally disconnected.

Let X be a compact space with an open dense discrete subspace. Then, $C(X)$ has property (S). However, this does not imply that X is totally disconnected. Let X be compact and connected and let Y denote a second copy of X. Define a topology on the disjoint sum $Y + X$ by taking as basic open sets $\{y\}$, $y \in Y$, and $(O - F) + O$, O open in X and $F \subseteq X$ finite. The space $Y + X$ is compact and has Y as an open dense subset. It is not totally disconnected (this example was pointed out to me by B. A. Rattray).

4. An extension theorem for linear functionals

Let H denote an arbitrary vector lattice. A function $\Phi : H \to \bar{\mathbf{R}}$, where $\bar{\mathbf{R}}$ denotes the two-point compactification of $\mathbf{R}$, will be called a *linear functional* if (1) the image ΦH of H contains $\mathbf{R}$, (2) $\Phi(f + g) = \Phi(f) + \Phi(g)$, and (3) $\Phi(\lambda f) = \lambda\Phi(f)$, for λ real, whenever $\Phi(f) + \Phi(g)$ and $\lambda\Phi(f)$ are defined.

THEOREM 2. *Let H be a vector lattice and let 1 be a positive element of H. Assume that $\{0\}$ is the only subspace N of H for which*

(1) $f \in N$ *and* $|g| \leq |f|$ *implies* $g \in N$;

(2) $1 \notin N$.

Then there is a unique lattice preserving linear functional $\Phi : H \to \bar{\mathbf{R}}$ *with* $\ker \Phi = \{0\}$ *and* $\Phi(1) = 1$.

Proof. To simplify notations, for any real λ, let λ also denote $\lambda \cdot 1$. Let

$$H^* = \{f \in H \mid \exists\, n > 0, n \geq |f|\}.$$

Then, $\{0\}$ and H^* are the only subspaces of H^* which satisfy condition (1). This, as is well known (c.f. [1]), implies that H^* is the set of real multiples of 1.

LEMMA A. *Let $f > 0$ be in $H - H^*$. Then $f \wedge \lambda = \lambda$; for all real $\lambda \geq 0$.*

Proof. If $f > 0$ is in $H - H^*$ and $\lambda \geq 0$, then $f \wedge \lambda = \alpha \cdot 1 = \alpha$ for some $\alpha \geq 0$. This follows from the observation preceding the lemma.

Let $I = \{\lambda \geq 0 \mid f \wedge \lambda = \lambda\}$. Clearly $0 \in I$. The fact that

$$f \wedge (\lambda \wedge \mu) = (f \wedge \lambda) \wedge \mu$$

implies that $\mu \in I$ if $\mu \leq \lambda$ and $\lambda \in I$. Hence I is an interval. The interval I is closed. Let $\lambda_0 \notin I$. Then if $\alpha = f \wedge \lambda_0$, $\lambda > \alpha$ implies $\lambda \notin I$. Since I is an interval, it is clear when $\lambda > \lambda_0$. Assume $\alpha < \lambda \leq \lambda_0$. Then

$$f \wedge \lambda = f \wedge (\lambda_0 \wedge \lambda) = (f \wedge \lambda_0) \wedge \lambda = \alpha \wedge \lambda = \alpha < \lambda.$$

Assume that $\{\lambda \geq 0 \mid f \wedge \lambda = \lambda\}$ has an upper bound and let λ_1 denote the least upper bound of this interval. Consider $f - \lambda_1$. This function is positive and different from zero. Consequently, the hypotheses on H imply that, for some $n > 0$, $n \cdot (f - \lambda_1) \geq 1$. This implies $f \geq \lambda_1 + 1/n$ and so

$$f \wedge (\lambda_1 + 1/n) = \lambda_1 + 1/n,$$

which is a contradiction.

LEMMA B. *If $h \in H$, then one of h^+ and h^- is in H^*.*

Proof. Let $\lambda \geq 0$. Then

$$(h^+ - h^-) \wedge \lambda = h^+ \wedge (\lambda + h^-) - h^- = h^+ \wedge \lambda - h^-,$$

since $h^+ \wedge h^- = 0$. From this it follows that

$$[(h^+ - h^-) \wedge \lambda] \vee (-\lambda) = h^+ \wedge \lambda - h^- \wedge \lambda.$$

Consequently, Lemma A shows that if h^+ and h^- are both in $H - H^*$, then

$$(h \wedge \lambda) \vee (-\lambda) = 0 \qquad \text{for all } \lambda \geq 0.$$

Assume h^- is in $H - H^*$. Consider

$$h \wedge 1 - 1 = h^+ \wedge 1 - 1 - h^-.$$

It is negative and does not lie in H^*. Hence, by Lemma A,

$$(-1) = (h \wedge 1 - 1) \vee (-1) = (h \wedge 1 - 1) \vee (0 - 1) = (h \wedge 1) \vee 0 - 1,$$

and so $(h \wedge 1) \vee 0 = h^+ \wedge 1 = 0$. Applying Lemma A once again to h^+, it follows that h^+ is in H^* and in fact that $h^+ = 0$.

This argument applied to $-h$ shows that if h^+ is in $H - H^*$, then $h^- = 0$.

Define $\Phi : H \to \bar{\mathbf{R}}$ by setting $\Phi(f)$ equal to λ if $f = \lambda$, to $+\infty$ if f^+ is in $H - H^*$, and to $-\infty$ if f^- is in $H - H^*$. It is clear that $\Phi(\lambda f) = \lambda\Phi(f)$ whenever $\lambda \cdot \Phi(f)$ is defined. A consideration of the possibilities for $\Phi(f)$ and $\Phi(g)$ shows that $\Phi(f + g) = \Phi(f) + \Phi(g)$, whenever $\Phi(f) + \Phi(g)$ is defined. The functional Φ is increasing, since $f \leq g$ implies $f^+ \leq g^+$ and $f^- \geq g^-$. Furthermore, it preserves finite unions, since $(f \vee g)^+ = f^+ \vee g^+$ and $(f \vee g)^- = f^- \wedge g^-$. Consequently, Φ is lattice preserving.

The kernel of Φ is $\{0\}$ and $\Phi(1) = 1$. Let Ψ be any linear lattice preserving functional with $\Psi(1) = 1$. Then Ψ and Φ agree on H^*. If f^+ is in $H - H^*$, then $f = f^+$ and, applying Lemma A,

$$\Psi(f) \wedge n = \Psi(f \wedge n) = n,$$

for all $n > 0$. Hence $\Psi(f) = +\infty$. Similarly, $\Psi(f) = -\infty$ if f^- is in $H - H^*$. This shows that $\Phi = \Psi$.

Corollary. *Let $p \neq 0$ be a positive element of H. Let N be a subspace of H for which*

(1) *$f \in N$ and $|g| \leq |f|$ implies $g \in N$.*

(2) *$p \notin N$.*

(3) *N is maximal with respect to* (1) *and* (2).

Then there is a unique lattice preserving linear functional $\Phi : H \to \bar{\mathbf{R}}$ with $N = \ker \Phi$ and $\Phi(p) = 1$.

Proof. Consider the vector lattice H/N. Let 1 denote the positive element corresponding to p. Then 1 and H/N satisfy the conditions of the theorem. Hence, there is a unique lattice preserving linear functional $\Psi : H/N \to \bar{\mathbf{R}}$ with $\Psi(1) = 1$ and $\ker \Psi = \{0\}$. Composition of Ψ with the quotient mapping of H onto H/N defines Φ.

Definition. A maximal ideal of H is the kernel of a lattice preserving linear functional $\Phi : H \to \mathbf{R}$.

This corollary has as a consequence the following extension theorem for linear functionals which will be of use in discussing the total boundary. Let $p \neq 0$ be a positive element of H and let H_p^* denote $\{f \in H \mid \exists\, n, |f| \leq n \cdot p\}$.

Theorem 3. *Let $\varphi : H_p^* \to \mathbf{R}$ be a lattice preserving linear functional for which $\varphi(p) = 1$. Then there is a unique lattice preserving linear functional $\Phi : H \to \bar{\mathbf{R}}$ which extends φ.*

Proof. If $f \in H$ then, for all $\lambda > 0$, $(f \wedge \lambda p) \vee (-\lambda p)$ is in H_p^*. Hence, if Φ extends φ,

$$\Phi(f) = \lim_{\lambda\to+\infty} [(\Phi(f) \wedge \lambda) \vee (-\lambda)] = \lim_{\lambda\to+\infty} \varphi(f \wedge \lambda p) \vee (-\lambda p)).$$

This shows that φ has at most one extension as a lattice preserving linear functional.

Consider $N^* = \ker \varphi$. It is a subspace of H which satisfies conditions (1) and (2) of the corollary. Let $N \supseteq N^*$ be a subspace of H satisfying the conditions of the corollary, and let Φ be the functional corresponding to N. If $f \in H^*$, then $f - \varphi(f)p \in N^*$, and so

$$0 = \Phi(f - \varphi(f)p) = \Phi(f) - \varphi(f),$$

since $\Phi(p) = 1$. Consequently, Φ extends φ.

Remark. W. A. J. Luxemburg has informed me that a similar extension theorem for real-valued functionals is to be found in [7].

5. The total boundary

In [3], Feller defined a space that he called the total boundary. Let H denote the vector lattice $H^+ - H^+$, where $H^+ = \{h \geq 0 \mid Ph = h\}$ and P is the operator defined in the introduction. The points of the total boundary are taken to be what Feller called the ideals M in H^+, the positive cone of H, which are maximal with respect to some positive p in H.

In [3] a subset I of H^+ is called an ideal if

(1) $f \vee g \in I$ when $f, g \in I$.
(2) $f \in I$ and $0 \leq h \leq f$ implies $h \in I$.
(3) $f \in I$ and $\lambda \geq 0$ implies $\lambda f \in I$.

An ideal I of H^+ is maximal with respect to $p \in H^+$ if $p \notin I$ and p belongs to every ideal that properly contains I. If I is an ideal and $p \in H^+ - I$, there exists an ideal $M \supseteq I$ which is maximal with respect to p.

The connection between Feller's concept of maximal ideal and the definition of maximal ideal in the previous section is provided by the following proposition.

PROPOSITION 5. *Let $M \subseteq H^+$ be a Feller ideal which is maximal with respec to p. Then there exists a unique maximal ideal N of H with $N \cap H^+ = M$.*

Conversely, let N be a maximal ideal of H. Then $N \cap H^+$ is a Feller ideal maximal with respect to some $p \geq 0$.

Proof. Let $N = \{f \mid |f| \in M\}$. Then N is a subspace of H which satisfies conditions (1) and (2) of the corollary to Theorem 2. Let $N_1 \supseteq N$ be a subspace of H satisfying (1) and (2). Since $N_1 \cap H^+$ is a Feller ideal containing M, it follows that $N_1 \cap H^+ = M$. Therefore, if $f \in N_1$, $f^+, f^- \in N$ and so $f \in N$.

Conversely, let N be a maximal ideal of H and let $p \geq 0$ be the element in $H - N$ of condition (2) in the corollary to Theorem 2. The intersection $N \cap H^+$ is a Feller ideal I which does not contain p. Let $M \supseteq I$ be a Feller ideal maximal with respect to p. The maximality of the ideal N implies that $N = \{f \mid |f| \in M\}$ and so $N \cap H^+ = M$.

The *total boundary* of H is the topological space B^∞ obtained by equipping the set of maximal ideals N of H with the hull-kernel topology. This is the topology having the sets $O_h = \{N \mid h \notin N\}$, $h \in H$, as a base.

The space B^∞ is Hausdorff. Let $N_1 \neq N_2$ be two maximal ideals of H and let h_1 and h_2 be the positive functions in condition (2) of the corollary to Theorem 2 with $h_i \notin N_i$ for $i = 1, 2$. Then $N_i \in O_{h_i}$ and, as is easily seen by viewing the maximal ideals as kernels, $O_{h_1} \cap O_{h_2} = O_{h_1 \wedge h_2}$. Denote by k_i the function $h_i - h_1 \wedge h_2$. Then, $k_1 \wedge k_2 = 0$ and so $O_{k_1} \cap O_{k_2} = \emptyset$. It remains to show that $k_i \notin N_i$ for $i = 1, 2$.

It can be assumed that $h_2 \in N_1$ and $h_1 \in N_2$. Hence,

$$h_1 \wedge h_2 \in N_1 \cap N_2 .$$

Therefore, $k_i \notin N_i$ since $h_i \notin N_i$ for $i = 1, 2$.

On the set L of lattice preserving linear functionals $\Phi : H \rightarrow \bar{\mathbf{R}}$, the elements h of H determine the functions $\bar{h}$ defined by setting $\bar{h}(\Phi) = \Phi(h)$. Let L also denote the topological space obtained by giving L the weak topology induced by $\{\bar{h} \mid h \in H\}$.

Lemma. *Let U be a neighbourhood of a point $\Phi_0 \in L$. Then there exists $\varepsilon > 0$ and $h \in H$ such that $\Phi_0(h) = 0$ and*

$$\{\Phi \in L \mid \Phi(h) < \varepsilon\} \subseteq U.$$

Proof. Since U is a neighbourhood of Φ_0, there exists $\varepsilon > 0$, and functions $g_1, \cdots, g_n, k_1, \cdots, k_m$ in H with $\Phi_0 \in U_1 \cap U_2$, where

$$U_1 = \bigcap_{i=1}^{n} \{\Phi \mid \mid \Phi(g_i) - \Phi_0(g_i) \mid < \varepsilon\}$$

and

$$U_2 = \bigcap_{j=1}^{m} \{\Phi \mid \Phi(k_j) > 1/\varepsilon\}.$$

We first show that U_2 can be replaced by a neighbourhood of the same type as U_1.

Let $\lambda = 1/\varepsilon$. Since $\Phi_0 \in L$, there exists $p \geq 0$ with $\Phi_0(p) = \lambda + 1$. Assume $\Phi_0(k) = +\infty$ and let $q = p \wedge k$. Then

$$\{\Phi \mid \Phi(q) > \lambda + \tfrac{1}{2}\} \subseteq \{\Phi \mid \Phi(k) > \lambda\}.$$

As $\Phi_0(q) = \lambda + 1$, it follows that

$$\Phi_0 \in \{\Phi \mid \mid \Phi(q) - \Phi_0(q) \mid < \tfrac{1}{2}\} \subseteq \{\Phi \mid \Phi(k) > \lambda\}.$$

This shows that U_2 can be replaced by a set of the same type as U_1.

Now assume $\Phi_0 \in U_1 \subseteq U$. Let $h_i = g_i - \Phi_0(g_i) \cdot 1$. Then $h_i \in H$ and $\Phi_0(h_i) = 0$. Let $h = \sum_{i=1}^{n} \mid h_i \mid$. Since Φ_0 is lattice preserving, it is clear that $\Phi_0(h) = 0$ and that $\{\Phi \mid \Phi(h) < \varepsilon\} \subseteq U_1$.

Making use of this lemma, we are able to show that the total boundary is a quotient space of L.

Proposition 6. *Denote by $\pi : L \rightarrow B^\infty$ the function defined by setting $\pi(\phi) = \ker \Phi$. The topology of B^∞ is the quotient topology induced by π and the topology on L.*

Proof. It is clear that $\pi^{-1}O_h = \{\Phi \mid \Phi(h) \neq 0\}$. Hence, π is continuous.

Let now $U \subseteq L$ be open and saturated with respect to the equivalence relation defined by π. Pick $\Phi_0 \in U$, $\varepsilon > 0$, and $h \geq 0$ in H with $\Phi_0(h) = 0$ and $U \supseteq \{\Phi \mid \Phi(h) < \varepsilon\}$.

Let $p \in H$ be positive and such that $\Phi_0(p) = 1$. Denote by k the function $p + h$. Then $\Phi_0 \in \pi^{-1}O_k \subseteq U$.

Since $\Phi_0(k) = 1$, $\Phi_0 \in \pi^{-1}O_k$. Assume $\Phi \in \pi^{-1}O_k$, that is, $\Phi(k) > 0$. Then, $\Phi(h) > 0$ or $\Phi(h) = 0$. If $\Phi(h) > 0$, for $\lambda = \varepsilon/[2\Phi(h)]$, $\lambda\Phi \in U$. If $\Phi(h) = 0$, it is clear that $\Phi \in U$. In either case, for some $\lambda > 0$, $\lambda\Phi \in U$. Since $\pi(\Psi) = \pi(\Phi)$ if and only if, for some $\lambda > 0$, $\lambda\Phi = \Psi$, it follows that $\pi^{-1}O_k \subseteq U$.

In [4], Feller observes that the total boundary is obtained by taking, in some sense, a union of the boundaries defined by the vector lattices H_p^* as p runs through the positive cone of H, (in actual fact Feller uses the lattice of bounded functions of the form $f(x)/p(x)$, $f \in H$, which is isomorphic to H_p^*). His statement can be made explicit in the following way.

For each p, let B_p denote the set of lattice preserving linear functionals $\varphi : H_p^* \to \mathbf{R}$ for which $\varphi(p) = 1$. Equip each B_p with the weak topology. For each p, the extension theorem shows that there is an inclusion j_p of B_p in L. It follows from the compactness of B_p that each j_p is an embedding.

If B_p is now identified with the corresponding subspace of L, L itself is the union of the spaces B_p. Since B^∞ is Hausdorff and B_p is compact, the mapping π embeds each B_p in B^∞. Hence, B^∞ can then be thought of as a union of the spaces B_p, provided two functionals Φ and Ψ are identified when, for some $\lambda > 0$, $\lambda\Phi = \Psi$.

6. The boundaries as boundaries of E

Let H denote a vector lattice of bounded continuous real-valued functions on the locally compact space E. Assume that H contains the constants. Using the notation of Section 2, the Šilov boundary $\partial(H)$ of H is a closed subset of K', and if $\partial(H) \subseteq K' - \pi(E)$, where $\pi : K \to K'$ is the identification map, then $\partial(H)$ is homeomorphic with $B = \pi^{-1}(\partial(H))$. Hence, the Šilov boundary $\partial(H)$ of H can be adjoined to E by identifying it with the subspace B of the subspace $E \cup B$ of K.

The collection of open subsets O of E together with the collection of sets of the form $P - K$, where P is in the weak topology defined by the extensions to $E \cup B$ of the functions in H and $K \subseteq E$ is compact, form a base for the topology of the subspace $E \cup B$. In other words, the topology of $E \cup B$ is the coarsest Hausdorff topology for which E is an open subspace and the extensions to $E \cup B$ of the functions in H are all continuous.

In general, $\partial(H)$ does not lie in $K' - \pi(E)$. For example, if H consists only of constant functions, then $K' = \partial(H)$. However, it is always possible to adjoin $\partial(H)$ to E.

Denote by B any space homeomorphic to $\partial(H)$, for example, take B to be the subspace of the dual of H (equipped with the weak topology) consisting of the lattice preserving functionals φ for which $\varphi(1) = 1$. Let $E \cup B$ denote the following topological space. The underlying set is the disjoint sum of E and B. The topology is the coarsest Hausdorff topology for which E is an open subspace and each function in $\bar{H} = \{\bar{h} \mid h \in H\}$ is continuous, where $\bar{h}(y) = h(x)$ if $y = x \in E$ and $\bar{h}(y) = \varphi(h)$ if $y = \varphi \in B$. This topology has as a base $\{O \mid O \subseteq E \text{ is open}\} \cup \{P - K \mid P \text{ is in the weak topology determined by } \bar{H}, \text{ and } K \subseteq E \text{ is compact}\}$.

Remark. As applied to the situation studied by Feller in [3], this topology is coarser than the one he introduced into $E \cup B$. To show that Feller's topology coincides with the above topology it would be sufficient to show that for any sojourn set A there is an extremal s and $0 < \eta < 1$ with

$$A \supseteq \{i \mid s(i) > 1 - \eta\}.$$

When the Šilov boundary $\partial(H)$ lies in $K' - \pi(E)$ it is immediate that E is dense in $E \cup B$. In general, this is not the case. If E is itself compact then it is not dense in $E \cup B$, or if H consists of all the bounded continuous real-valued functions on E, then again E is not dense in $E \cup B$. This raises the question as to when E is dense in $E \cup B$.

Assume that E is not compact. The set E is dense in $E \cup B$ if and only if, for each $\varphi \in B$, every neighbourhood of φ intersects E. The following lemma describes the basic neighbourhoods of a point φ of B.

LEMMA. *Let $\varphi \in B$ and let U be a neighbourhood of φ in $E \cup B$. Then there exist $\varepsilon > 0$, $h > 0$ in H and $K \subseteq E$ compact with*

(i) $\varphi(h) = 0$ *and*

(ii) $U \supseteq \{y \in E \cup B \mid \bar{h}(y) < \varepsilon\} - K$.

Proof. Since U is a neighbourhood of φ there exist $\varepsilon > 0$, $g_1, \cdots, g_n$ in H, and $K \subseteq E$ compact with

$$[\bigcap_{i=1}^n \{y \in E \cup B \mid |\bar{g}_i(y) - \varphi(g_i)| < \varepsilon\}] - K \subseteq U.$$

Let $h_i = g_i - \varphi(g_i) \cdot 1$ and let $h = \sum_{i=1}^n |h_i|$. Then, since φ is lattice preserving, $\varphi(h) = 0$. Consequently,

$$\varphi \in \{y \in E \cup B \mid \bar{h}(y) < \varepsilon\} \subseteq \bigcap_{i=1}^n \{y \in E \cup B \mid |\bar{g}_i(y) - \varphi(g_i)| < \varepsilon\}.$$

With the aid of this lemma we give a proof of the following proposition

PROPOSITION 7. *Assume that E is not compact. Then E is dense in $E \cup B$ if and only if for any compact $K \subseteq E$ and h in H, $\inf_E h = 0$ implies $\inf_{E-K} h = 0$.*

Proof. Let $\varphi \in B$ and let $h > 0$ in H be such that $\varphi(h) = 0$. Then $\inf_E h = 0$ as $\varphi(1) = 1$. Since, for any compact $K \subseteq E$, $\inf_{E-K} h = 0$ it follows from the lemma that each neighbourhood of φ intersects E.

Conversely, let h be a function in H with $\inf_E h = 0$ and $\inf_{E-K} h = \varepsilon > 0$, for some compact $K \subseteq E$. Since $\inf_E h = 0$, h vanishes on B. Assume $\varphi(h) = 0$. Then

$$\{y \in E \cup B \mid \bar{h}(y) < \varepsilon\} - K$$

is a neighbourhood of φ which does not intersect E.

In case E is an open bounded region in $\mathbf{R}^n$ and H is the set of bounded solutions of Laplace's equation, then it is well known that the condition of Proposition 7 is satisfied.

In the case considered by Feller where H is defined by a transient Markov process the condition is also satisfied. While this follows from results of Feller [3], we give a direct proof. Assume $h > 0$ satisfies $\sum p(i, j)h(j) = h(i)$, $\inf_E h = 0$, but that for some finite set $K \subseteq E$, $\inf_{E-K} h = \varepsilon > 0$. Then h vanishes on K. Let

$$Z = \{i \mid h(i) = 0\}.$$

It is easy to see that, for $i \in Z$, $p(i, j) = 0$ if $j \in E - Z$. This implies, in the terminology of [3], that Z is a sojourn set. Since it is finite it contains a minimal sojourn set. This contradicts the assumption of transience.

The procedure used to adjoin B to E can be used to adjoin L to E. Let $h \in H$. Define $\bar{h}$ on the disjoint sum of E and L by setting $\bar{h}(y)$ equal to $h(y)$ if $y \in E$ and equal to $y(h)$ if $y \in L$. Using the set $\bar{H}$ of functions $\bar{h}$, $h \in H$, L may be adjoined to E in the same way that B was adjoined to E. The resulting topology on the disjoint union $E \cup L$ of E and L is Hausdorff. The subspace E is never dense in $E \cup L$, since $y \in \bar{E} - E$ implies that $y(1) = 1$.

The topological space $E \cup L$ can be used to adjoin the space B^∞ to E. Define $\pi_1 : E \cup L \to E \cup B^\infty$, by setting $\pi_1(y)$ equal to y if $y \in E$ and equal to ker y if $y \in L$. Then π_1 determines a quotient topology for $E \cup B^\infty$. If $h \in H$, let

$$O_h^* = \{y \in E \mid h(y) \neq 0\} \cup \{y \in B^\infty \mid h \notin y\}.$$

These sets are open since

$$\pi_1^{-1}(O_h^*) = \{y \in E \cup L \mid \bar{h}(y) \neq 0\}$$

and hence B^∞ is a subspace of $E \cup B^\infty$.

The quotient topology is Hausdorff. Instead of proving this directly, we introduce a coarser topology into $E \cup B^\infty$ which is Hausdorff and for which B^∞ is still a subspace. This topology is similar to one used by Feller in [3].

If $p > 0$ is in H, let $E_p = \{x \in E \mid p(x) > 0\}$. The boundary B_p can be adjoined to E_p by considering it as the Šilov boundary of the vector lattice of functions on E_p of the form $f = h/p$, where $h \in H_p^*$. This means that, for each function $f = h/p$, the function $\bar{f}$ is continuous, where $\bar{f}(y) = h(x)/p(x)$ if $y = x \in E_p$ and $\bar{f}(y) = \varphi(h)$ if $y = \varphi \in B_p$.

Denote by $j_p : E_p \cup B_p \to E \cup L$ the inclusion defined by setting $j_p(y)$ equal to y if $y \in E_p$ and equal to the extension of y to H if $y \in B_p$. Let i_p be the function $\pi_1 \circ j_p$. Then i_p is an inclusion of $E_p \cup B_p$ in $E \cup B^\infty$.

PROPOSITION 8. *There is a unique topology for $E \cup B^\infty$ for which each i_p embeds $E_p \cup B_p$ as an open subspace. This topology is Hausdorff. The subset E is dense in $E \cup B^\infty$ if and only if for each $p > 0$, E_p is dense in $E_p \cup B_p$.*

Proof. To simplify the notation, we identify $E_p \cup B_p$ with its image under i_p. With this identification, the first statement of the proposition holds if, for p and $q > 0$,

$$(E_p \cup B_p) \cap (E_q \cup B_q) = (E_p \cap E_q) \cup (B_p \cap B_q)$$

is open in $E_p \cup B_p$.

To show this it suffices to show that each point $y_0 \in B_p \cap B_q$ has a neighbourhood in $E_p \cup B_p$ which lies in $(E_p \cup B_p) \cap (E_q \cup B_q)$. Consider the function $r = p \wedge q$. It is in H_p^* and so $f = r/p$ has a continuous extension $\bar{f}$ to B_p, given by

$$\bar{f}(y) = y(r) = y(p) \wedge y(q) = y(q),$$

where y denotes also the extension of y to H. Since $y_0 \in B_p \cap B_q$, $y_0(q) = \bar{f}(y_0) \neq 0$. There therefore exists a neighbourhood U of y_0 in $E_p \cup B_p$ on which $\bar{f}$ never vanishes.

The neighbourhood U lies in $(E_p \cup B_p) \cap (E_q \cup B_q)$. Let $y \in U \cap E_p$. Then, $0 \neq \bar{f}(y) = f(y)$ and so $0 \neq r(y) \leq p(y) \wedge q(y)$. This implies $q(y) \neq 0$ and so $y \in E_q$. If $y \in U \cap B_p$, then $0 \neq \bar{f}(y) = y(q)$ and so $q \notin \ker y$. Consequently, $y \in B_q$, when B_q is viewed as a subset of B^∞.

Since each subspace $E_p \cup B_p$ is Hausdorff, this topology for $E \cup B^\infty$ is Hausdorff provided any two points of B^∞ lie in one of the subspaces $E_p \cup B_p$. Let $y_1 \neq y_2$ be two points of B^∞ with $y_i = \ker \Phi_i$, for $i = 1, 2$. Assume $p_i > 0$ are such that $\Phi_i(p_i) = 1$. The argument used to show B^∞ is Hausdorff also shows that p_1 and p_2 can be chosen so that $p_1 \wedge p_2 = 0$. Let $p = p_1 \vee p_2$. Then $\Phi_i(p) = 1$, for $i = 1, 2$, and so y_1, y_2 lie in B_p.

If each E_p is dense in $E_p \cup B_p$, it is clear that E is dense in $E \cup B^\infty$. Since each $E_p \cup B_p$ is open, the converse holds.

In case E is an open region in $\mathbf{R}^n$ and H^+ is the set of positive solutions of Laplace's equation, each E_p coincides with E. Applying Proposition 7, $E = E_p$ is dense in $E_p \cup B_p$ if, for any compact K and h in H_p^*, $\inf_E h/p = 0$ implies $\inf_{E-K} h/p = 0$. Assume that $\inf_{E-K} f/p = \varepsilon > 0$. Since $\inf_E h/p = 0$ this implies that h/p vanishes on K. Consequently f vanishes on K and so $h = 0$. This is a contradiction.

In the case of a transient Markov process the condition is also satisfied. Assume $K \subseteq E_p$ is finite and that, for $h \in H_p^*$, $\inf_{E_p} h/p = 0$, but $\inf_{E_p - K} h/p = \varepsilon > 0$. Then $Z = \{i \mid h(i) = 0\}$ is a subset of K and hence finite. This leads to a contradiction by the argument used above for the bounded case.

PROPOSITION 9. *The topology for $E \cup B^\infty$ defined in Proposition 8 is coarser than the quotient topology induced by π_1. Furthermore, it induces the hull-kernel topology on B^∞.*

Proof. Let $O \subseteq E_p \cup B_p$ be open and let $y_0 \in O \cap B_p$. Then, there exist $\varepsilon > 0$, $h > 0$ in H_p^* and compact $K \subseteq E_p$ with $y_0(h) = 0$ and

$$O \supseteq \{y \in E_p \cup B_p \mid (h/p)^-(y) < \varepsilon\} - K.$$

The continuity of π_1 follows from the fact that

$$\pi_1^{-1}\{y \in E_p \cup B_p \mid (h/p)^{\sim}(y) < \varepsilon\} = \{y \in E \cup L \mid \bar{h}(y) < \varepsilon \cdot \bar{p}(y)\}.$$

Since π_1 is continuous, $\pi = \pi_1 \mid L$ is continuous. Consequently, the hull-kernel topology on B^{∞} contains the subspace topology induced by topology defined in Proposition 8. To show that this subspace topology is the hull-kernel topology, it suffices to show, for each h and $p > 0$, that $O_h \cap B_p$ is open in B_p. Let $q = p \wedge h$ and let $y_0 \in O_h \cap B_p$. Then

$$y_0 \in O_q \cap B_p = \{y \in B_p \mid y(q) \neq 0\}.$$

Since $q \in H_p^*$, this set is open.

Remark. It does not look as though the topology for $E \cup B^{\infty}$ for which all the $E_p \cup B_p$ are open subspaces coincides with the quotient topology induced by π_1. If, for example, each j_p embeds $E_p \cup B_p$ as an open subspace of $E \cup L$, then these two topologies coincide. However, in the case where E is a bounded region in $\mathbf{R}^n$ and H is the set of harmonic functions, it is not true that each $E_p \cup B_p$ is embedded by j_p into $E \cup L$. Let $p > 0$ be a minimal function. Then H_p^* has dimension one and as $E_p = E$, $E_p \cup B_p$ is the one-point compactification of E. If $E_p \cup B_p$ could be embedded in $E \cup L$, each function in H would extend continuously to the point at infinity. Since this clearly cannot happen, it follows that j_p does not embed $E_p \cup B_p$ in $E \cup L$.

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