WEAK MINIMAL GENERATING SET REDUCTION THEOREMS FOR ASSOCIATIVE AND LIE ALGEBRAS

BY

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It is often difficult to obtain results for Lie algebras over arbitrary fields because the study of Lie algebras over fields of characteristic two and three as well as finite fields usually poses special problems. An attempt was made to develope methods using minimal generators which would be as independent as possible of the nature of the ground field. This led to the author's thesis[1] from which the present paper has been prepared.

Our theorems on Lie algebras essentially use the Jacobi identity only to show that certain subalgebras of Lie algebras are ideals. Hence these results for Lie algebras do not depend on the ground field. This fact also explains why analogous theorems hold for associative algebras. Several propositions determining the structure of certain quotient Lie algebras required a slightly more explicit use of the Jacobi identity. These results were not obtained for fields of characteristic two.

To derive the results of this paper only certain properties of minimal generating sets were used. We single these out by the following definition. A set S of elements of an algebra A weak minimally generates A, abbreviated S w.m.g. A, if

(1) S generates A as an algebra
(2) S consists of linearly independent elements of A
(3) No proper subspace of the vector space spanned by S generates A.

It is now possible to summarize the main results we obtain. Suppose S w.m.g. A and T is a non-empty subset of S. Let B be the subalgebra generated by $S - T$ and C the vector space spanned by T. Now, assume A is the direct sum as vector spaces of B and C and denote the projection of A onto C with respect to this decomposition by P. Then C becomes an algebra with multiplication $*$ defined by $c * c' = P(cc')$ for all $c, c' \in C$. The structure of C with this multiplication is determined. Next for each $c \in C$, $P(bc) = \beta_l(b)c$ and $P(cb) = \beta_r(b)c$, $b \in B$, where β_l and β_r are linear functionals from B into F. If A is either an associative or Lie algebra with dim $C \geqq 2$ then the kernel $\beta_l \cap$ kernel β_r is an ideal in A. In a slightly different direction, if B is an ideal in A and the base field is infinite then A is the direct sum as vector spaces of the subalgebra generated by all products of elements of B

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[1] *The structure of Lie algebras with large minimal generating sets*, Mathematics Ph.D. thesis, University of Notre Dame, Indiana, June 1964, written under the direction of Hans Zassenhaus.

and the vector space spanned by S. The paper concludes with a short appendix on weak minimal generating sets.

Let V be a vector space, not necessarily finite-dimensional, over a field F. Suppose $(a, b) \to ab$ is a map from $V \times V$ into V. For each $a \in V$ define R_a and L_a by $R_a(b) = ba$ and $L_a(b) = ab$, $b \in V$. If R_a and L_a for each $a \in V$ are linear transformations we say A is an algebra over F with multiplication denoted by ab. If it is not necessary to name the ground field F we will simply say A is an algebra.

Suppose A is an algebra having a decomposition $A = B \dotplus C$, where B is a subalgebra of A and C a vector subspace of A, and where $\dotplus$ denotes the direct sum of B and C as vector spaces. Let P be the projection of A onto C determined by this decomposition. Let $(C, *)$ denote the vector space C with multiplication $*$ defined by

$$c_1 * c_2 = P(c_1 c_2) \qquad \text{for } c_1, c_2 \in C.$$

If we let R_C^* and L_C^* denote right and left multiplication respectively in $(C, *)$ then $R_C^* = P \circ R_C$ and $L_C^* = P \circ L_C$. Therefore $(C,*)$ is an algebra. Observe if A is a square nil algebra, i.e. $a^2 = 0$ for all $a \in A$, then $(C, *)$ is also a square nil algebra. It does not follow if A is associative then $(C, *)$ is associative or if A is a Lie algebra then $(C, *)$ is a Lie algebra.

For any set S of elements of an algebra A let $V(S)$ denote the minimal vector subspace of A containing S and $\langle S\rangle$ the minimal subalgebra of A containing S. Suppose $A = \langle S - T\rangle + V(T)$ where S w.m.g. A and $\Phi \neq T \subset S$. That this sum must be direct (as vector spaces) follows immediately from the definition of S w.m.g. A when we observe $\langle W \cup (S - T)\rangle = A$, where W is any vector space complement of $\langle S - T\rangle \cap V(T)$ in $V(T)$.

Next, observe $A = V(S)$, S w.m.g. A, if and only if $\langle a, b\rangle = V(a, b)$ for all $a, b \in A$. Necessity is clear, while sufficiency would follow if

$$\langle a_1, \cdots, a_n\rangle = V(a_1, \cdots, a_n)$$

for $a_1, \cdots, a_n \in A$, n an arbitrary positive integer. However, it is immediate that the set $\{m \in A \mid m$ a product of elements of A not contained in the vector space spanned by these elements$\}$ has no element involving a least number of factors. Then $A = \langle S - T\rangle + V(T)$, where S w.m.g. A and $\Phi \neq T \subset S$, implies T w.m.g. $(V(T), *)$. For $s, t \in V(T)$ implies $st \in V(s, t) + \langle S - T\rangle$ so that $s * t \in V(s, t)$.

Let $GF(p^n)$ denote the Galois field of p^n elements. As usual let V^* denote the dual space of a vector space V. We now state the first theorem which completely determines the structure of $(V(T), *)$.

THEOREM 1.[2] *Let A be a vector space V with a basis S over a field*

[2] G. Leger has proved in his paper, *A particular class of Lie algebras*, Proc. Amer. Math. Soc., vol. 16 (1965), pp. 293–296, a theorem which contains this theorem for Lie algebras as a special case.

F, $F \neq GF(2)$. Then A is an algebra over F with S w.m.g. A if and only if multiplication is specified by

(1) $ab = \alpha(b)a + (\varepsilon(a) - \alpha(a))b$, *for all $a, b \in A$, where $\varepsilon, \alpha \in A^*$.*

Proof. If (1) holds

$$R_a = \alpha(a)I + a(\varepsilon - \alpha) \quad \text{and} \quad L_a = a\alpha + (\varepsilon(a) - \alpha(a))I,$$

where I denotes the identity transformation of A onto A. Clearly R_a and L_a are linear transformations and $\langle a,b \rangle = V(a, b)$ so that A is an algebra weak minimally generated by S.

Suppose S w.m.g. A. We wish to determine $\varepsilon, \alpha \in A^*$ such that (1) holds. Since S w.m.g. A, $\langle a \rangle = V(a)$ and $\langle a, b \rangle = V(a, b)$ for all $a, b \in A$. Then $a^2 = \varepsilon(a)a$ for all $a \in A$, where $a \to \varepsilon(a)$ is a map of A into F. Observe

$$\varepsilon(\lambda a)(\lambda a) = \lambda a^2 = \lambda^2 a^2 = \lambda^2 \varepsilon(a)a$$

mplies $\varepsilon(\lambda a) = \lambda\varepsilon(a)$, $\lambda \in F$, $a \in A$. If dim $A = 1$, $\varepsilon \in A^*$, $\alpha = 0$, and we are done. If not, consider linearly independent elements $a, b \in A$. Let

$$S(a, b) = (a + b)^2 - a^2 - b^2 = ab + ba.$$

Then $S(\lambda a, b) - \lambda S(a, b) = 0$, $\lambda \in F$. Written in terms of ε this becomes

$$\varepsilon(\lambda a + b)(\lambda a + b) - \varepsilon(\lambda a)(\lambda a) - \varepsilon(b)b \\ - \lambda(\varepsilon(a + b)(a + b) - \varepsilon(a)a - \varepsilon(b)b) = 0.$$

Then

$$\lambda[\varepsilon(\lambda a + b) - \lambda\varepsilon(a) - \varepsilon(a + b) + \varepsilon(a)] = 0 \tag{2}$$

$$\varepsilon(\lambda a + b) - \varepsilon(b) - \lambda\varepsilon(a + b) + \lambda\varepsilon(b) = 0 \tag{3}$$

since a and b are linearly independent. Suppose $\lambda \neq 0$; cancel λ in (2) and subtract (2) from (3) obtaining

$$(\lambda - 1)(\varepsilon(a + b) - \varepsilon(a) - \varepsilon(b)) = 0.$$

Since $F \neq GF(2)$, $\varepsilon(a + b) = \varepsilon(a) + \varepsilon(b)$.

Since $\langle a, b \rangle = V(a, b)$ we may write

$$ab = \alpha(b, a)a + \beta(a, b)b,$$

where α, β are maps from $A \times A$ into F. Suppose a and b are linearly independent elements of A. Consider

$$\alpha(b, a)a + \beta(a, b)b + \alpha(a, b)b + \beta(b, a)a = ab + ba = (a + b)^2 - a^2 - b^2 \\ = \varepsilon(a + b)(a + b) - \varepsilon(a)a - \varepsilon(b)b = \varepsilon(b)a + \varepsilon(a)b.$$

$\beta(a, b) + \alpha(a, b) = \varepsilon(a)$ and we may now write

$$ab = \alpha(b, a)a + (\varepsilon(a) - \alpha(a, b))b. \tag{4}$$

We next show $\alpha(b, a)$ depends only on the first variable. From the coefficient of a in the identity $(a + b)b = ab + b^2$ we conclude

$$\alpha(b, a + b) = \alpha(b, a),$$

while if there exist a, b, c linearly independent elements of A from the coefficients of a and c in $(a + c)b = ab + cb$ we conclude

$$\alpha(b, a + c) = \alpha(b, a) \quad \text{and} \quad \alpha(b, a + c) = \alpha(b, c).$$

We remark $\alpha(b, b) = \alpha(b, a)$ is an allowable definition since (4) still reduces to $b^2 = \varepsilon(b)b$. Therefore α is independent of the second variable and we may set $\alpha(b) = \alpha(b, a)$. Then (4) becomes

$$ab = \alpha(b)a + (\varepsilon(a) - \alpha(a))b$$

for all $a, b \in A$. Rewriting the above equation

$$R_a = a\alpha + (\varepsilon(a) - \alpha(a))I$$

we conclude $b \to \alpha(b)a$ is linear and hence $\alpha(b)$ is linear.

COROLLARY 1.1. *If a square nil algebra A has a decomposition*

$$L = \langle S - T\rangle \dotplus V(T),$$

$\Phi \neq T \subset S$, S w.m.g. A, *then* $(V(T), *)$ *is a Lie algebra.*

Proof. $P(c^2) = 0$ implies $c * c = 0$. Then

$$c * d = \alpha(d)c - \alpha(c)\, d, \qquad c, d \in V(T), \alpha \in V(T)^*,$$

implies

$$(c * d) * e = \alpha(d)\alpha(e)c - \alpha(c)\alpha(e)\, d, \qquad c, d, e \in V(T),$$

from which it follows $(V(T), *)$ satisfies the Jacobi identity.

We remark for a Lie algebra $L = V(S)$, of dimension m, S w.m.g. L, $m \geq 2$, there are precisely two non-isomorphic algebras corresponding to $\alpha = 0$ and $\alpha \neq 0$. This follows immediately once we observe that the codimension of the kernel of α in L is either zero or one.

PROPOSITION 1. *Suppose an algebra A over a field F has a decomposition* $A = \langle S - T\rangle \dotplus V(T)$, $\Phi \neq T \subset S$, S w.m.g. A. *Denote the projection map of A onto $V(T)$ by P. Then*

$$P(st) = \beta_l(s)t \quad \textit{and} \quad P(ts) = \beta_r(s)t, s \in (S - T), t \in V(T), \beta_l, \beta_r \in \langle S - T\rangle^*.$$

Proof. We prove only $P(st) = \beta_l(s)t$, $\beta_l \in \langle S - T\rangle^*$, the proof of $P(ts) = \beta_r(s)t$ being similar.

We have $P(st) \in \langle S - T, t\rangle \cap V(T) = V(t)$ since S w.m.g. A. Therefore for each $t \in V(t)$ we have $P(st) = \beta_t(s)t$ where β_t is a map from $\langle S - T\rangle$ into F. Define the linear functional δ_t from $V(T)$ into F by $\delta_t(\lambda t) = \lambda$,

$\lambda \in F$. Then $\beta_t = \delta_t \circ P \circ R_s$ and therefore $\beta_t \in \langle S - T\rangle^*$. We now show β_t is independent of t. Observe $\beta_{\lambda t}(s)(\lambda t) = P(s(\lambda t)) = \lambda P(st) = \lambda\beta_t(s)t$ implies $\beta_{\lambda t} = \beta_t$ for all $\lambda \neq 0$, $\lambda \in F$. While $\beta_{t+v}(s)(t + v) = P(s(t + v)) = P(st) + P(sv) = \beta_t(s)t + \beta_v(s)v$ implies $\beta_t(s) = \beta_v(s)$ for t and v linearly independent elements of $V(T)$.

Suppose an algebra A has a decomposition $A = B \dotplus C$, where B is a subalgebra of A and C a vector subspace of A. Let P denote the projection of A onto C and suppose $P(bc) = \beta_l(b)c$ and $P(cb) = \beta_r(b)C$, $b \in B$, $c \in C$, β_l, $\beta_r \in B^*$. Denote the kernel of β_l by K_l, of β_r by K_r, and set $K = K_l \cap K_r$.

THEOREM 2. *If A is a Lie algebra then $B^2 \subset K$ and if* $\dim C \geq 2$, *K is an ideal in A.*

Proof. Let $\beta = \beta_l$. Since L is square nil $\beta_l = -\beta_r$, kernel $\beta = K_l = K_r = K$. For $b, b' \in B$, $c \in C$

$$\begin{aligned}\beta(bb')c &= P((bb')c) \\ &= P(b(b'c)) - P(b'(bc)) && \text{(by the Jacobi identity)} \\ &= P(bP(b'c)) - P(b'P(bc)) && \text{(since } B \text{ is a subalgebra)} \\ &= \beta(b)\beta(b')c - \beta(b')\beta(b)c = 0.\end{aligned}$$

Therefore $B^2 \subset K$. Hence $KB \subset K$ and it would suffice to show $KC \subset K$ to conclude K is an ideal in A. Assume $\dim C \geq 2$. Given $c \in C$ there exists $c' \in C$ such that c and c' are linearly independent. For $b \in K$, $bc \in B$ and $bc' \in B$ so that we may write $\beta(bc)$ and $\beta(bc')$. Then

$$\begin{aligned}0 = \beta(b)(cc') &= P(b(cc')) \\ &= P((bc)c') - P((bc')c) \\ &= \beta(bc)c' - \beta(bc')c.\end{aligned}$$

Therefore $\beta(bc) = 0$, hence $bc \in K$, completing the proof.

It is easy now to determine the factor algebra structure when B is not an ideal.

PROPOSITION 2. *Suppose a Lie algebra L over a field F, characteristic of $F \neq 2$, contains an element a such that $ab = b + \alpha(b)a$ for all $b \in L$. Then $\alpha \in L^*$ and $c\,d = \alpha(d)c - \alpha(c)\,d$ for all c, $d \in L$, where $\alpha(a) = -1$.*

Proof. It is clear that $\alpha \in L^*$ with $\alpha(a) = -1$. Let D be any vector space complement of $V(a)$ in L. We have $cd = e + \beta(cd)a$, for some $\beta(cd) \in F$, $e \in D$, for c, $d \in D$. Then

$$\begin{aligned}e + \alpha(e)a = a(cd) &= (ac)\,d - (ad)c \\ &= 2(e + \beta(cd)a) + \alpha(c)(d + \alpha(d)a) - \alpha(d)(c + \alpha(c)a).\end{aligned}$$

By linear independence

(1) $e - \alpha(d)c + \alpha(c)\, d = 0$

(2) $\alpha(e) + 2\beta(cd) = 0.$

Because α is linear (1) implies $\alpha(e) = 0$, hence (2) implies $\beta(c\,d) = 0$.

Observe for $K \neq B$ the factor algebra L/K is weak minimally generated by a basis, while for $K = B$ it is necessary, of course, to suppose $B = \langle S - T\rangle$ and $C = V(T)$, where $\Phi \neq T \subset S$, S w.m.g. L. The next example shows all the possible factor algebras with a weak minimal generating set which is a basis occur. Let W be a vector space over F with basis $\{a, b, ab, c, d\}$. Then the following multiplication tables

	a	b	ab	c	d
a	0	ab	0	0	0
b	$-ab$	0	0	c	d
ab	0	0	0	0	0
c	0	$-c$	0	0	0
d	0	$-d$	0	0	0

	a	b	ab	c	d
a	0	ab	0	0	0
b	$-ab$	0	0	0	0
ab	0	0	0	0	0
c	0	0	0	0	0
d	0	0	0	0	0

	a	b	ab	c	d
a	0	ab	0	$-a$	0
b	$-ab$	0	0	$-b$	0
ab	0	0	0	$-2ab$	0
c	a	b	$2ab$	0	d
d	0	0	0	$-d$	0

turn W into Lie algebras weak minimally generated by $\{a, b, c, d\}$ with $B = \langle a, b\rangle$. In the first table $K = \langle a, ab\rangle \neq B$, in the second and third tables $K = B$ with L/K abelian and non-abelian, respectively.

The following lemma essentially settles the associative algebra case.

LEMMA. *Suppose A is an algebra over F. Then*

(i) $b(b'c) = (bb')c$, $b, b' \in B$, $c \in C$ *implies* K_l *an ideal in* B.

(ii) $(cb)b' = c(bb')$, $b, b' \in B$, $c \in C$ *implies* K_r *an ideal in* B.

(iii) $(bc)c = b(cc)$, $b \in B$, $c \in C$ *implies* $K_lC \subset K_l$

(iv) $c(cb) = (cc)b$, $b \in B$, $c \in C$ *implies* $CK_r \subset K_r$

(v) $c(bc') = (cb)c'$, $c, c' \in C$, $b \in B$, $\dim C \geq 2$ *implies* $KC \subset K_r$ *and* $CK \subset K_l$.

Proof. (i) $b(b'c) = (bb')c$ implies β_l is a homomorphism of B into F and hence its kernel is an ideal of B.

(ii) Similarly β_r a homomorphism implies K_r an ideal of B.

(iii) For $b \in K_l$, $bc \in B$, so that we may write $\beta_l(bc)$. Then $\beta_l(bc)c = P((bc)c) = P(b(cc)) = \beta_l(b)(cc) = 0$.

(iv) Similarly $b \in K_r$ implies $\beta_r(cb)c = \beta_r(b)(cc) = 0$.

(v) For $b \in K$, $bc' \in B$ and $cb \in B$, so that we may write $\beta_r(bc')$ and $\beta_l(cb)$. Then $\beta_r(bc')c = P(c(bc')) = P((cb)c') = \beta_l(cb)c'$. If c, c' are linearly independent we may conclude $\beta_r(bc') = 0$ and $\beta_l(cb) = 0$. Therefore $KC \subset K_r$ and $CK \subset K_l$ provided dim $C \geq 2$.

THEOREM 3. *If A is an associative algebra and dim $C \geq 2$ then K is an ideal* of A.

Proof. From (i) and (ii) we conclude K is an ideal in B. It then suffices to show $KC \subset K$ which follows from (iii) and (v) and $CK \subset K$ which follows from (iv) and (v).

The analysis of the structure of A/K where A is an associative algebra is more involved than that of L/K where L is a Lie algebra. Indeed the analysis is not quite complete, as we shall see. The following proposition and examples settle the case when $K = B$ under the natural assumption $B = \langle S - T \rangle$ and $C = V(T)$, $\Phi \neq T \subset S$, S w.m.g. A.

PROPOSITION 3. *Suppose a right or left alternative algebra A is of the form $A = V(S)$, S w.m.g. A; then $ab = \eta(a)b$ or $ba = \eta(a)b$ for all $b \in A$, $\eta \in A^*$.*

Proof. By Theorem 1 we have

$$cd = \alpha(d)c + (\varepsilon(c) - \alpha(c))\, d$$

for all $c, d \in A$, $\alpha, \varepsilon \in A^*$. Then $(cd)\, d = c(dd)$ implies

(1) $\alpha(d)(\varepsilon(c) - \alpha(c)) = 0$
(2) $(\alpha(d))^2 = \varepsilon(d)\alpha(d)$ for c, d linearly independent.

If $\varepsilon = 0$ then by (2) $\alpha = 0$ and we are done.
If $\varepsilon \neq 0$, and $\alpha = 0$, set $\varepsilon = \eta$.
If $\varepsilon \neq 0$, $\alpha \neq 0$ let $\varepsilon(b) = 1$ and $K =$ kernel ε. Then $A = V(b) \dotplus K$. Since $\varepsilon(K) = 0$, $\alpha(K) = 0$ by (2), while by (1), $\alpha(b) = \varepsilon(b)$. Set $\varepsilon = \alpha = \eta$.
The conclusion is symmetric in a and b hence it would follow also from the identity $d(dc) = (dd)c$.
Let $\{a, b, ab, c, d\}$ be a basis for a vector space W over F. Then

	a	b	ab	c	d
a	0	ab	0	0	0
b	ab	0	0	0	0
ab	0	0	0	0	0
c	0	0	0	0	0
d	0	0	0	0	0

and

	a	b	ab	c	d
a	0	ab	0	0	0
b	ab	0	0	0	0
ab	0	0	0	0	0
c	a	b	ab	c	d
d	0	0	0	0	0

determine associative multiplications on W, with $B = \langle a, b \rangle$ an ideal, and L/B abelian, L/B prossessing a left identity, respectively; both algebras are clearly w.m.g. by $\{a, b, c, d\}$.

We now begin the analysis of the case $\text{codim}_K\, B = 1$. Suppose we have $bc = c + \alpha(c)b$ for some element $b \in A$ and all $c \in A$. From $(b^2)c = b(bc)$ we conclude $\alpha(b) = 0$ and $\alpha(c) = 0$ provided c is linearly independent of b. Let D be a complement of $V(b)$ in A and suppose $cb = \beta(c)b + \delta(c)c$ for all

$c \in D$. Then $(cb)b = c(b^2)$ implies $\beta(c)\delta(c) = 0$ and $\delta(c)^2 = \delta(c)$. The following proposition takes care of the subcase when $\delta(c) = 0$ for all c.

PROPOSITION 4. *Suppose in an associative algebra A there exists an element b such that*

$$bc = c, \quad cb = \beta(c)b, \quad \beta(c) \in F, \qquad \textit{for all } c \in A,$$

then $dc = \eta(d)c$, $\eta \in A^*$, $\eta(b) = 1$.

Proof. Let D be a complement of $V(b)$ in A. The proposition would follow from Proposition 3 if we could show $cd \in V(cd)$ for c, $d \in D$, for then A would be weak minimally generated by a basis of A. Suppose $c\,d = e + \beta(cd)b$, $e \in D$, $\beta(c, d) \in F$. An easy calculation of $(cb)d = c(bd)$ leads to $\beta(c, d) = 0$ and hence $cd = \beta(c) = d$.

It is easy to give an example of an associative algebra having this factor algebra structure.

The subcase remains when $\delta(c) = 1$. for some c. By considering $(c + d)b = cb + db$ for b, c, d linearly independent we see $\delta(c) = 1$ for all $c \in A$. We now have $bc = c$ and $cb = c$ for some fixed $b \in A$ and all $c \in A$. Under these conditions no information can be derived about the product cd from the associative identities involving b so that an analogue of the last proposition is impossible. We next attempt to derive additional information by supposing $B = \langle S - T\rangle$ and $C = V(T)$ where $\Phi \neq T \subset S$, S w.m.g. $B + C$ and $A = (B + C)/K$. In this situation we know in terms of A and a complement D of $V(b)$ in A by Theorem 1 that

$$cd = \alpha(c)\, d + (\varepsilon(d) - \alpha(d))c + \eta(c, d)b, \qquad \alpha, \varepsilon \in D^*,$$

η simply a mapping from $D \times D$ into F. If we suppose c, d are linearly independent the coefficients of c and d in the identity $(cc)\, d = c(cd)$ imply

$$\eta(c, d) = \alpha(c)(\alpha(d) - \varepsilon(d))$$

for c, $d \in D$. (The identity given by the coefficient of b in this calculation follows from the formula derived for $\eta(c, d)$.) Observe $\eta(c, d)$ is a bilinear functional from $D \times D$ into F. Furthermore, we must have $\eta(c, d) \neq 0$ for some c, $d \in D$, otherwise A would be weak minimally generated by a basis of $D \cup \{b\}$ contradicting Proposition 3. The author does not know whether such an algebra A can occur as a factor algebra $(B + C)/K$.

Finally, when the $\operatorname{codim}_B K$ is two, there is essentially one factor structure. There exists $b_l \in K_r$ and $\notin K_l$ and $b_r \in K_l$ and $\notin K_r$ which may be assumed normalized so that (when considered in the factor algebra with multiplication understood to be the natural factor algebra multiplication)

$$b_l\, c = c + \alpha(c)b_l + \beta(c)b_r \qquad b_r\, c = \delta(c)b_r\,,$$

$$b_l^2 = e_l\, b_l\,, \qquad cb_l = \gamma(c)b_l$$

$$cb_r = c + n(c)b_l + \xi(c)\, b_r \qquad b_r^2 = e_r\, b_r\,, \qquad e_l\,, e_r \in F,$$

for all $c \in D$, a complement of $V(b_l\,, b_r)$ in the factor algebra A/K.

(1) $b_r b_l = 0$ follows from (iii) and (iv) of Lemma 2.

Next

$$c(b_l b_r) = (cb_l)b_r = \gamma(c)b_l b_r \quad \text{implies} \quad b_l b_r \in V(b_l)$$

while

$$(b_l b_r)c = b_l(b_r c) = \delta(c)b_l b_r \quad \text{implies} \quad b_l b_r \in V(b_r),$$

hence

(2) $b_l b_r = 0.$

Now from $b_l^2 = b_l(b_l c)$ we conclude

(3) $\varepsilon_l = 1$
(4) $\alpha(c) = 0$

while from $cb_r^2 = (cb_r)b_r$ we conclude

(5) $\varepsilon_r = 1$
(6) $\xi(c) = 0.$

Finally, $0 = (b_r b_l)c = b_r(b_l c) = \delta(c)b_r + \beta(c)b_r$ so that

(7) $\delta(c) + \beta(c) = 0$, while

$$0 = c(b_r b_l) = (cb_r)b_l = \gamma(c)b_l + n(c)b_l$$

so that

(8) $\gamma(c) + n(c) = 0.$

For $c \in D$ let $c' = c + \beta(c)b_r + \eta(c)b_l$; then $b_l c' = c'$ and $c'b_r = c'$, hence $c'b_l = 0$, $b_r c' = 0$.

Let $D' = \{c' \mid c \in D\}$. Then D' is a vector space complement of $V(b_1, b_r)$ in the factor algebra. Then $0 = (c'b_1)d' = c'(b_1 d') = c' d'$ so that D' is abelian. Observe that the factor algebra is not weak minimally generated by a basis since $\langle b_l, b_r + c\rangle \in b_l, b_r, c$. Again the author has no example of an associative algebra $A = B + C$, $B = \langle S - T\rangle$, $C = V(T)$, $\Phi \neq T \subset S$, S w.m.g. A, which has A/K of the above structure.

Let S be a weak minimal generating set for an algebra A. Suppose $\tau : S \to V(S)$. The following construction gives a useful sufficient condition for $\tau(S)$ w.m.g. A. Suppose for the present that S is a linearly independent set of elements of A. A well defined product of the elements of S will be called a monomial in the elements of S. If $\langle S\rangle = A$ we may find a basis $B = S \cup M$, where M consists of monomials of the elements of S. We extend $\tau : S \to A$ to $\tau^B : A \to A$ by requiring

(1) τ^B restricted to S equals τ.
(2) If $\prod s_i \in M$, $s_i \in S$, then $\tau^B(\prod s_i) = \prod \tau(S_i)$, where parentheses in $\prod \tau(S_i)$ are inserted exactly as in $\prod s_i$.
(3) τ^B is linear

If S w.m.g. A and there exists a basis B as above such that $\tau^B(A) = A$ then $\tau(S)$ w.m.g. A.

THEOREM 4. *Let A be a finite-dimensional algebra over an infinite field F. Suppose $A = \langle S - T\rangle + V(T), \Phi \neq T \subset S$, S w.m.g. A, and $\langle S - T\rangle$ is an ideal of A. Then*

$$A = \langle S - T\rangle^2 \dotplus V(S).$$

Proof. Suppose $0 \neq \sum_{b \epsilon S} \lambda_b b \epsilon \langle S - T\rangle^2 \cap V(S)$. Then $\lambda_s \neq 0$ for some $s \epsilon S - T$ since $\langle S - T\rangle \cap V(T) = 0$.

Adjoin a transcendental x to F and view A as $A \otimes_F F(x)$ over $F(x)$. Fix $t \epsilon T$ and consider $\tau : S \to V(S) \otimes_F F(x)$ defined by

$$\tau(s) = s + xt, \qquad \tau(b) = b \qquad \text{for } b \epsilon S - \{S\}.$$

Let $B = S \cup M$, M a set of monomials in $S - T$, be a basis for A. Then

$$\tau^B(m) = \sum_{b \epsilon M \cup S - T \cup \{t\}} (\delta_{mb} + xP_{mb})b, \qquad P_{mb} \epsilon F[x], \text{ for } m \epsilon M,$$

where

$$\delta_{mb} = 0 \quad \text{if } m \neq b$$
$$= 1 \quad \text{if } m = b.$$

Therefore

$$\det \tau^B = 1 + xP, \qquad P \epsilon F[x].$$

From the generalized form of Cramer's rule for vector-valued functions applied to $\tau^B(b)$ in terms of b for $b \epsilon M \cup S - T \cup \{t\}$ we conclude for $b \neq t$

$$b = \sum_{c \epsilon S - T \cup M} \frac{Q_{bc}}{1 + xP} \tau^B(c) + \frac{xQ_{bt}}{1 + xP} t, \qquad Q_{bc}, Q_{bt} \epsilon F[x]. \tag{1}$$

Then

$$\left(\sum \frac{Q_{bc}}{1 + xP} \tau^B(c) + \frac{xQ_{bt}t}{1 + xP}\right) t = bt \epsilon \langle S - T\rangle \tag{2}$$

since $\langle S - T\rangle$ is an ideal in A, while

$$t^2 - \varepsilon(t)t \epsilon \langle S - T\rangle, \quad \varepsilon(t) \epsilon F \tag{3}$$

by Theorem 2. Using (1), (2), and (3) we may calculate bc for $b, c \epsilon S - T \cup M$ concluding

$$bc - \frac{x^2 R_{bc}}{(1 + xP)^2} t \epsilon \langle \tau^B(S - T \cup M)\rangle \otimes_F F(x) \tag{4}$$

for some $R_{bc} \epsilon F[x]$. By supposition

$$\sum_{b \epsilon S} \lambda_b b = \sum_{c,d \epsilon S - T \cup M} \lambda_{cd}\, cd, \qquad \lambda_b, \lambda_{cd} \epsilon F.$$

Then

$$\sum_{b \epsilon S} \lambda_b b = \sum_{b \epsilon S - \{t\}} \lambda_b \tau^B(b) + (\lambda_t - x\lambda_s)t. \tag{5}$$

We conclude

$$\left((\lambda_t - x\lambda_s) + x^2\left(\frac{\sum_{c,d}\lambda_{cd}R_{cd}}{(1 + xP)^2}\right)t\right) \in \langle \tau^B(S - \{t\} \cup M)\rangle \otimes_F F(x), R_{cd} \in F[x].$$

Since $\lambda_s \neq 0$ the coefficient of t is a quotient of non-zero polynomials and since F is infinite there exists a specialization $\zeta : x \rightarrow \lambda$ such that these polynomials are non-zero. Then

$$t = (\zeta \circ \tau^B)(t) \in \langle(\zeta \circ \tau^B)(S - \{t\} \cup M)\rangle,$$

$\zeta \circ \tau^B$ is onto since $(1 + \lambda P)^2 \neq 0$, contradicting $\zeta \circ \tau^B(S)$ w.m.g. A.

It is clear from the last proof that "infinite field" could be replaced by "field with sufficiently many elements." The number of elements necessary could be calculated once a basis in terms of the generators was specified for a given algebra. It is interesting to rephrase the last theorem as an extension theorem.

COROLLARY. *Suppose A is an algebra over an infinite field and S* w.m.g. A. *Then there exists an algebra $\hat{A}$ over F* such that

(1) *A is an ideal in $\hat{A}$,*

(2) *$S \cup \{t\}$ w.m.g. $\hat{A}$, where $\hat{A} = A \dotplus V(t)$ if and only if*

$$A = A^2 \dotplus V(S).$$

Proof. The necessity follows immediately from the theorem. To show sufficiency we construct an $\hat{A}$ given $A = A^2 \dotplus V(S)$. Let $A = A + V(t)$, $ta = at = 0$ for all $a \in A$ and $t^2 = 0$. Then $\hat{A}^2 \subset A^2$ so that $S \cup \{t\}$ w.m.g. A.

Appendix

In this appendix we answer some of the natural questions which arise concerning the concept of weak minimal generating set.

It is clear that weak minimal generating sets exist for finite-dimensional algebras but need not exist for an infinite-dimensional algebra as the following example shows. Let Q denote the rational numbers. Let $k = Q\ (w_n \mid w_n$ a primitive 2^n root of unity, n a positive integer). Let $K = k(x)$ be a transcendental extension of K. Consider the algebra $A = K[z_n \mid (z_n)^{2^n} = x, n$ a positive integer]. Suppose $\langle S\rangle = A$. It is clear S is infinite. We show given $s, s' \in S$ either $\langle S - \{s\}\rangle = A$ or $\langle S - \{s'\}\rangle = A$, and therefore A would have no weak minimal generating sets. For $s \in S$, $K(s) \subseteq K(z_m)$, where m is the greatest integer n occurring in either a numerator or denominator term $\lambda_n z_n$ of s, $\lambda_n \neq 0$, and s assumed to be expressed with numerator and denominator relatively prime over K. Since $K(z_m)$ is a finite normal extension of K, hence has finite cyclic Galois group, every subgroup of which is determined by its order, every subfield of $K(z_m)$ is one of the $K(z_n)$, $1 \leq n \leq m$. Therefore $K(s) = K(z_n)$ for some n. For $s' \in S$, $K(s') = K(z_n')$. Therefore $K(s) \subseteq K(s')$ or $K(s') \subseteq K(s)$ when $n \leq n'$ or $n' \leq n$, respectively. Since $K(s) = K[s]$ and $K(s') = K[s']$ either $\langle S - \{s\}\rangle = A$ or $\langle S - \{s'\}\rangle = A$.[3]

[3] This example resulted from a conversation with Max Zorn.

The term weak minimal generating set is justified by the fact that weak minimal generating sets for finite-dimensional algebras may have distinct number of elements. Consider the vector space A with basis $\{a, b, c, d\}$ over a field F and multiplication specified by the following table:

	a	b	c	d
a	0	c	b	0
b	$-c$	0	0	0
c	$-b$	0	0	a
d	0	0	$-a$	0

Then $\{a, b, d\}$ and $\{c, d\}$ both w.m.g. A. Observe though, A is neither a Lie algebra nor an associative algebra. The author has no examples of a finite-dimensional Lie or associative algebra which have two weak minimal generating sets consisting of different numbers of elements.

We now indicate how an example of an infinite-dimensional Lie algebra could be constructed which is weak minimally generated by two elements and also by three elements.

Suppose S is a set of elements; then an element of $(\cdots (SS)S) \cdots S$ is called a standard monomial in S of length n. The standard monomials span the free Lie algebra generated by S, but are not linearly independent because of the square nil identity (e.g. $0 = (\mathrm{ab})(\mathrm{ab}) = ((ab)a)b - ((ab)b)a)$. If we inductively extend a basis consisting of standard monomials of length $\leq k - 1$ to a basis for those of length $\leq k$, then those monomials of length k in the basis will lead to linear relations of standard monomials of weight $2k$. With respect to this basis we can now see what happens if we factor a free Lie algebra α generated by $S = \{a, b\}$ by the relations

$$a = ((ab)a)(ab) = (((ab)a)a)b - (((ab)a)b)a$$

and

$$b = ((ab)b)ab) = (((ab)b)a)b) - (((ab)b)b)a.$$

If I is the minimal two sided ideal in L containing these relations, the linear dependence relations among standard monomials induced by factoring by I arise simply by multiplying the above relations successively by a and b. It follows that L/I is an infinite-dimensional algebra so constructed that both $\{a, b\}$ and $\{ab, (ab)a, (ab)b\}$ w.m.g. L/I.

Finally, if we cannot produce an example of a finite-dimensional Lie algebra with two distinct w.m.g. sets, can we prove that all w.m.g. sets have the same number of elements? We now sketch a proof under the hypothesis L is a nilpotent Lie algebra (probably too strong an assumption). It has been shown that a finite-dimensional Lie algebra L is nilpotent if and only if every maximal subalgebra of L is an ideal.[4] This implies L is nilpotent if and only if

[4] Donald Barnes, *Nilpotency of Lie algebras*, Math. Zeitschrift, vol. 79 (1962), pp. 237–238.

$L^2 = \Phi L$, where ΦL is the intersection of all maximal proper subalgebras of L, called the Frattini subalgebra of L. It follows if S w.m.g. a finite-dimensional nilpotent Lie algebra L then $L = \Phi L + V(S)$, i.e. $V(S)$ has codimension equal to the dimension of L^2 in L. Simply observe for $\sum_{s \epsilon S} \lambda_s s$, $\lambda_{s_0} \neq 0$, $\lambda_s \epsilon F$, there exists a maximal subalgebra M of L containing $\langle S - \{s_0\}\rangle$ and not containing $\sum_{s \epsilon S} \lambda_s s$.

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