THE SPECTRA OF CERTAIN TOEPLITZ MATRICES

BY

I. I. HIRSCHMAN, JR.[1]

1. Introduction

Let $f(e^{i\theta}) \in L^1(T)$ where T is the reals modulo 2π. If

$$f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} f^\wedge(r) e^{ir\theta}$$

then the Toeplitz matrix of order $n + 1$ associated with f is

$$M_n[f] = [f^\wedge(r - s)]_{r,s=0,1,\cdots,n}\,.$$

Let $\{\lambda_{n,i}\}_{i=0}^{n}$ be the eigen values of $M_n[f]$, that is the zeros of

$$\det [M_n - \lambda I_n] = 0.$$

For each n we define a measure α_n on the Borel sets in the λ-plane by

$$\alpha_n(E) = (n + 1)^{-1} \sum_{\lambda_{n,k} \in E} 1.$$

An important topic in the theory of Toeplitz matrices is the study of the asymptotic behavior of the measures α_n as $n \to \infty$. If $f(e^{i\theta})$ is real, in which case the matrices $M_n[f]$ are Hermitian, there is a simple and elegant solution. The support of each α_n is contained in the interval of the real line whose end points are ess inf f and ess sup f, and as $n \to \infty$ the α_n converge weakly to the measure α defined by

$$\alpha(E) = (2\pi)^{-1} \int_{f(e^{i\theta}) \in E} d\theta,$$

see [1, §7.5].

When $f(e^{i\theta})$ is not real, however, the problem is very difficult and the only results are those obtained by P. Schmidt and F. Spitzer in [5]. They assumed that f is a Laurent polynomial,

$$(1) \qquad f(e^{i\theta}) = \sum_{r=-k}^{h} f^\wedge(r) e^{ir\theta},$$

where $h, k > 0$, (otherwise the problem in question is trivial) and showed that there then exists a compact set C in the λ-plane, which can be described precisely, such that if N is any neighborhood of C the support of α_n must be in N provided n is sufficiently large. Moreover, no smaller closed set C has this property.

In this paper we will complete the investigation of Schmidt and Spitzer by showing that as $n \to \infty$ the α_n converge weakly to a measure α with support in C. This is rather easy. What is of greater interest is that we will obtain an explicit formula for α.

Received January 7, 1966.

[1] This research was supported by a National Science Foundation grant.

2. Preliminary results

We will need the following.

THEOREM 2a. *For* $g(e^{i\theta}) \in \text{Lip}\,\beta$ *for some* $\beta > 0$ *let* $M_n[g]$ *be the corresponding Toeplitz matrix. We define* $\sigma^+(g)$ *to be the set of complex values* λ *for which either*

$$g(e^{i\theta}) - \lambda = 0 \qquad \textit{for some} \quad \theta \in T,$$

or if this is not the case,

$$[\arg \{g(e^{i\theta}) - \lambda\}]_0^{2\pi} \neq 0.$$

Then if K *is any compact set such that* $K \cap \sigma^+(g) = 0$ *there is an integer* n_0 *such that* $\det M_n[g - \lambda] \neq 0$ *for* $\lambda \in K$, $n \geq n_0$, *and*

$$\lim_{n\to\infty} \frac{\det M_n[g - \lambda]}{\det M_{n-1}[g - \lambda]} = \exp\left\{ (2\pi)^{-1} \int_T \log\,[g(e^{i\theta}) - \lambda]\, d\theta \right\}$$

uniformly for $\lambda \in K$.

Proof. In the case $K = \{\lambda\}$ this follows from Baxter's inequality for finite section Wiener-Hopf operators in the L^2 formulation given by Reich [4]. With only very minor changes Reich's arguments can be made to give the result stated above.

Let $f(z)$ be defined as in (1) of §1. For convenience we set

$$D_n(\lambda) = \det M_n[f(e^{i\theta}) - \lambda].$$

For any $\rho > 0$ let

$$f_\rho(r^{i\theta}) = \sum_{r=-k}^{h} f^\wedge(r)\rho^r e^{ir\theta}.$$

Following Schmidt and Spitzer we define

$$C = \bigcap_{\rho>0} \sigma^+(f_\rho).$$

We can now give a new and very simple proof of part of their results.

THEOREM 2b. *If* N *is any neighborhood of* C *then the support of* α_n *is contained in* N, *provided* n *is sufficiently large.*

Proof. If $\lambda \notin C$ then there exists a $\rho > 0$ such that $\lambda \notin \sigma^+(f_\rho)$; that is

$$f_\rho(e^{i\theta}) - \lambda \neq 0, \qquad \theta \in T,$$

$$[\arg \{f_\rho(e^{i\theta}) - \lambda\}]_0^{2\pi} = 0.$$

Since $\sigma^+(f_\rho)$ is closed, there is a compact neighborhood $K(\lambda)$ of λ such that $K(\lambda) \cap \sigma^+(f_\rho) = \emptyset$. By Theorem 2a there exists an integer n_0 such that

$$\det M_n[f_\rho - \lambda] \neq 0 \quad \text{for} \quad \lambda \in K(\lambda), \quad n \geq n_0.$$

We now note the basic identity

$$\det M_n[f_\rho - \lambda] = D_n(\lambda)$$

from which it follows that $\text{supp}\,\alpha_n \cap K(\lambda) = \emptyset$ for $n \geq n_0$, etc.

For λ any fixed complex number let $\{m_\nu(\lambda)\}_{\nu=1}^{k+h}$ be the absolute values of the zeros of

$$z^k\{f(z) - \lambda\} = 0$$

arranged in non-decreasing order. The following identification given in [5] plays a fundamental role in all that follows.

THEOREM 2c. *We have*

$$C = \{\lambda \mid m_k(\lambda) = m_{k+1}(\lambda)\}.$$

Proof. It is apparent that $\lambda \notin \sigma^+(f_\rho)$ if and only if $m_k(\lambda) < \rho < m_{k+1}(\lambda)$. For λ in C', the complement of C, choose ρ such that

$$(1) \qquad m_k(\lambda) < \rho < m_{k+1}(\lambda).$$

Then the function $f(z) - \lambda$ does not vanish on $|z| = \rho$ and inside this circle it has k zeros plus a pole of order k at $z = 0$. Thus $\log [f(z) - \lambda]$ is analytic and single valued in the annulus $m_k(\lambda) < |z| < m_{k+1}(\lambda)$, the branches differing by integral multiples of $2\pi i$. For any ρ satisfying (1) let

$$(2) \qquad G(\lambda) \exp\left\{(2\pi i)^{-1} \int_{|z|=\rho} \log [f(z) - \lambda] \frac{dz}{z}\right\}.$$

Clearly $G(\lambda)$ is well defined and is independent of ρ. Since $G(\lambda)$ is locally analytic everywhere in C' it is analytic in C'. We now have as a corollary of Theorem 2a the following result.

THEOREM 2d. *If $\lambda \notin C$ then*

$$\lim_{n\to\infty} D_n(\lambda)/D_{n-1}(\lambda) = G(\lambda)$$

uniformly for λ in any compact set K such that $K \cap C = \emptyset$.

3. The structure of C

The following result is due to Schmidt and Spitzer, however, our demonstration differs from their in some details.

THEOREM 3a. *C can be represented as a finite union of closed analytic arcs, where either distinct arcs are disjoint or, if not, their intersection consists of one or both common end points.*

Proof. Let $S(\lambda_0, \delta)$ be the disk $\{\lambda : |\lambda - \lambda_0| \leq \delta\}$. Take $\lambda_0 \in C$. The discriminant of $z^k\{f(z) - \lambda\}$ considered as a polynomial in z is a polynomial in λ whose zeros are the winding points for the roots of $z^k\{f(z) - \lambda\} = 0$ considered as functions of λ. We first consider the case when λ_0 is not itself a winding point. Choose δ so small that $S(\delta) = S(\lambda_0, \delta)$ contains no winding point. Then there exist analytic functions $\{z_\nu(\lambda)\}_{\nu=1}^{k+h}$ such that

$$z^k\{f(z) - \lambda\} = f^\wedge(h) \prod_{\nu=1}^{k+h} [z - z_\nu(\lambda)] \qquad \lambda \in S(\delta).$$

Let

$$\gamma_{\mu,\nu}(\delta) = \{\lambda \in S(\delta) : | z_\mu(\lambda)/z_\nu(\lambda) | = 1\},$$
$$\Gamma(\delta) = \bigcup_{\mu \neq \nu} \gamma_{\mu,\nu}(\delta).$$

Since $z_\mu(\lambda)/z_\nu(\lambda)$ has no zeros or poles in $S(\delta)$, $\Gamma(\delta)$ is the union of a finite number of analytic arcs each beginning and ending on the boundary of $S(\delta)$. Thus by taking δ if necessary still smaller $\Gamma(\delta)$ will consist of a finite number of analytic arcs where one end point of each arc is λ_0 and the other is on the boundary of $S(\delta)$, and the intersection of two distinct arcs is λ_0. Each spoke of $\Gamma(\delta)$ is now the carrier of a set of relations such as: $| z_1(\lambda) | = | z_2(\lambda) |$ and $| z_3(\lambda) | = | z_6(\lambda) | = | z_7(\lambda) |$. It is apparent that if λ_0 is not an isolated point of C then $C \cap S(\delta)$ must consist of one or more spokes from $\Gamma(\delta)$. Thus we need only prove that λ_0 cannot be an isolated point of C. If it were, then after suitable relabelling we would have

$$| z_\nu(\lambda) | < | z_\mu(\lambda) |, \qquad \nu = 1, \cdots, k; \mu = k + 1, \cdots, k + h$$

for all $\lambda \in S(\delta) - \{\lambda_0\}$, and

$$| z_1(\lambda_0) |, \cdots, | z_{k-p}(\lambda_0) | < | z_{k-p+1}(\lambda_0) | = \cdots = | z_k(\lambda_0) |$$
$$= | z_{k+1}(\lambda_0) | = \cdots = | z_{k+q}(\lambda_0) | < | z_{k+q+1}(\lambda_0), \cdots, | z_{k+h}(\lambda_0) |,$$

for some $p, q \geq 1$. Consider $\varphi(\lambda) = z_\mu(\lambda)/z_\nu(\lambda)$ where

$$k - p + 1 \leq \mu \leq k, \quad \text{and} \quad k + 1 \leq \nu \leq k + q.$$

Then $\varphi(\lambda)$ is analytic for $\lambda \in S(\delta)$ and

$$| \varphi(\lambda) | < 1, \quad \lambda \in S(\delta) - \{\lambda_0\}, \qquad \varphi(\lambda_0) = 1.$$

This contradicts the maximum modulus principle. Thus λ_0 cannot be an isolated point of C.

If λ_0 is a winding point we introduce a uniformizing parameter $\lambda - \lambda_0 = \Lambda^m$. If δ is small enough there exist functions $\{Z_\nu(\Lambda)\}_1^{k+h}$ analytic in $| \Lambda | \leq \delta$ and such that

$$z^k\{f(z) - \lambda_0 - \Lambda^m\} = f^\wedge(h) \prod_{\nu=1}^{k+h} [z - Z_\nu(\Lambda)]$$

there. We can now carry out the same argument in the Λ-plane, which we made before in the λ-plane, and at the end transfer the result to the λ-plane.

It is clear that the local structure of C obtained above implies the global description of our theorem.

Scholium 3b. *In the representation of C as a union of analytic arcs we may assume that no winding point is an interior point of any arc.*

Proof. This follows because an arc can be broken into two arcs at any interior point. The advantage of this convention is that instead of having two categories of exceptional points in C, end points and winding points, we now have only one—end points.

4. The limiting distribution

Let c be an analytic arc in the representation of C given by Theorem 3a and Scholium 3b. Locally c has two sides which we designate by side 1 and side 2. Let us denote $G(\lambda)$ on side 1 by $G_1(\lambda)$ and $G(\lambda)$ on side 2 by $G_2(\lambda)$. We will show that both $G_1(\lambda)$ and $G_2(\lambda)$ can be continued across the interior of c, that $|G_1(\lambda)| = |G_2(\lambda)|$ for all λ on c, and that $G_1(\lambda)$ continued into side 2 is never equal to $G_2(\lambda)$ there. Let us parameterize c by means of the arc length s measured from any convenient point on c. We set

$$\Phi(s) = \frac{1}{2\pi}\left|\frac{d}{d\lambda}\frac{G_2(\lambda)}{G_1(\lambda)}\right|_{\lambda=\lambda(s)}. \tag{1}$$

Doing this for each arc c of C we define a measure α on C by

$$\alpha(ds) = \Phi(s)\, ds.$$

Our principal result is that α so defined is the weak limit of the α_n's as $n \to \infty$.

In order to carry out the program outlined above we proceed as follows. Any point in the interior of c is the center of a closed disk S such that $S \cap C$ is a closed subarc of c cutting across S. By taking S small enough we can insure that it does not contain any winding points. Let S_1 be the component of $S - c$ on side 1 and S_2 the component on side 2. As in §3 we can find analytic functions $\{z_\nu(\lambda)\}_1^{k+h}$ such that

$$z^k\{f(z) - \lambda\} = f^\wedge(h)\prod_{\nu=1}^{k+h}[z - z_\nu(\lambda)], \qquad \lambda \in S.$$

Let N_1 be the (unique) subset of h integers drawn from $\{1, 2, \cdots, k+h\}$ such that

$$|z_\nu(\lambda)| < |z_\mu(\lambda)|, \qquad \lambda \in S_1,$$

whenever $\mu \in N_1$, $\nu \notin N_1$, and let N_2 be similarly defined relative to S_2. We set

$$G_1(\lambda) = f^\wedge(h)\prod_{\nu\in N_1} - z_\nu(\lambda), \qquad G_2(\lambda) = f^\wedge(h)\prod_{\nu\in N_2} - z_\nu(\lambda).$$

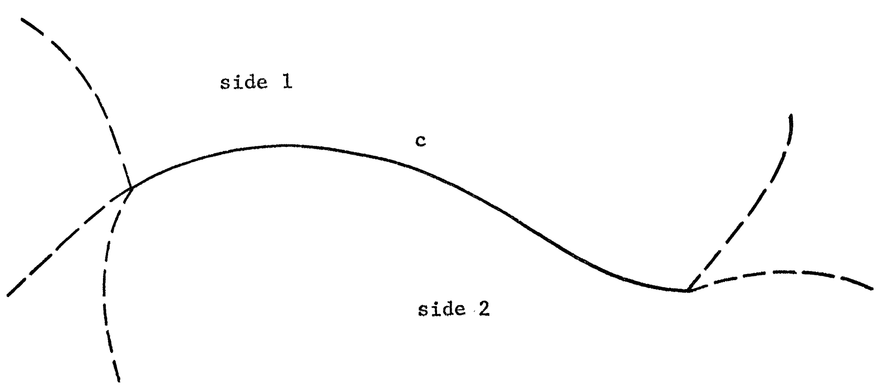

FIGURE 1

LEMMA 4a. *The functions* $G_1(\lambda)$ *and* $G_2(\lambda)$ *are analytic in* S *and*

$$G(\lambda) = G_1(\lambda), \qquad \lambda \in S_1,$$
$$= G_2(\lambda), \qquad \lambda \in S_2.$$

Proof. That both $G_1(\lambda)$ and $G_2(\lambda)$ are analytic in S is clear.
Let $\lambda \in S_1$. Then if $m_k(\lambda) < \rho < m_{k+1}(\lambda)$ we have

$$\log G(\lambda) = (2\pi)^{-1} \int_T \log [f(\rho e^{i\theta}) - \lambda]\, d\theta.$$

Now

$$\log [f(\rho e^{i\theta}) - \lambda]$$
$$= \log f^\wedge(h) + \textstyle\sum_{\nu \notin N_1} \log [1 - \rho^{-1} e^{-i\theta} z_\nu(\lambda)] + \textstyle\sum_{\nu \in N_1} \log [\rho e^{i\theta} - z_\nu(\lambda)].$$

If $\nu \notin N_1$ then

$$\log [1 - \rho e^{-i\theta} z_\nu(\lambda)] = -\textstyle\sum_{r=1}^{\infty} [\rho^{-1} e^{-i\theta} z_\nu(\lambda)]^r r^{-1},$$

while if $\nu \in N_1$ then

$$\log [\rho e^{i\theta} - z_\nu(\lambda)] = \log [-z_\nu(\lambda)] - \textstyle\sum_{r=1}^{\infty} [\rho e^{i\theta} z_\nu(\lambda)^{-1}]^r r^{-1}.$$

It follows that

$$\log G(\lambda) = \log f^\wedge(h) + \textstyle\sum_{\nu \in N_1} \log [-z_\nu(\lambda)],$$

etc.

LEMMA 4b. *We have*

$$(2) \qquad | G_1(\lambda) | = | G_2(\lambda) |, \qquad \lambda \in c \cap S,$$

and

$$(3) \qquad | G_1(\lambda) | > | G_2(\lambda) |, \ \lambda \in S_1, \qquad | G_2(\lambda) | > | G_1(\lambda) |, \ \lambda \in S_2.$$

Proof. The relation (2) is obvious.
Renumbering we may assume without loss of generality that for $\lambda \in S_1$

$$| z_\nu(\lambda) | < | z_\mu(\lambda) |, \qquad \nu = 1, \cdots, k;\ \mu = k+1, \cdots, k+h$$

while for $\lambda \in S \cap c$

$$| z_1(\lambda) |, \cdots, | z_{k-p}(\lambda) | < | z_{k-p+1}(\lambda) | = \cdots = | z_k(\lambda) |$$
$$= | z_{k+1}(\lambda) | = \cdots = | z_{k+q}(\lambda) | < | z_{k+q+1}(\lambda) |, \cdots, | z_{k+h}(\lambda) |.$$

Here $p, q \geq 1$. Choose a point λ_0 on $S \cap c$ such that

$$\frac{d}{d\lambda} \frac{z_\mu(\lambda)}{z_\nu(\lambda)} \bigg|_{\lambda=\lambda_0} \neq 0, \qquad \begin{array}{l} \nu = k-p+1, \cdots, k; \\ \mu = k+1, \cdots, k+q. \end{array}$$

Then if K_0 is a small circular neighborhood of λ_0, $z_\mu(\lambda)/z_\nu(\lambda) = w$ maps K_0 univalently. $c \cap K_0$ is mapped onto an arc of $| w | = 1$, while $S_1 \cap K_0$ is mapped into $| w | > 1$. Consequently $S_2 \cap K_0$ is mapped into $| w | < 1$. It follows that N_2 contains, in addition to $\{q + k + 1, \cdots, k + h\}$, min (p, q) integers from $\{k - p + 1, \cdots, k\}$. Thus $N_2 \neq N_1$, and this implies (3).

COROLLARY 4c. *If $\Phi(s)$ is defined as in (1) then $\Phi(s) \neq 0$ at any interior point of c.*

Let $\lambda(s)$ be a point in the interior of $c \cap S$. Construct a normal to c at $\lambda(s)$ into S_1. Let r be a small positive parameter. Let $\gamma(s, r)$ be the point on this normal at a distance $\frac{1}{2}r$ from $\lambda(s)$, and let $K(s, r)$ be the closed circular disk with center $\gamma(s, r)$ and radius r. If r is small enough $K(s, r) \subset S$. Note the angle $\omega_1(s, r)$ is negative.

LEMMA 4d. *Let $\alpha(d\lambda)$ be any weak limit point of $\{\alpha_n\}_1^\infty$. Then for r sufficiently small*

$$\int_{c \cap K(s,r)} \log \left[\frac{r}{d(s, r, t)} \right] \alpha(dt) = \frac{1}{2\pi} \int_{\omega_1(s,r)}^{\omega_2(s,r)} \log \left| \frac{G_2(\xi)}{G_1(\xi)} \right| d\theta. \tag{4}$$

We note that by Theorem 2b the support of the restriction of α to $K(s, r)$ is contained in $c \cap K(s, r)$ so that we may write $\alpha(dt)$ where dt is arc length on c. We also note that on the right hand side of (4) we have written ξ for $\xi(s, r, \theta)$.

Proof. Let us apply the Poisson-Jensen formula, see [6, §3.61], to the function $D_n(\lambda)/G_1(\lambda)^{n+1}$ and the circle $K(s, r)$. Dividing by $n + 1$ we obtain

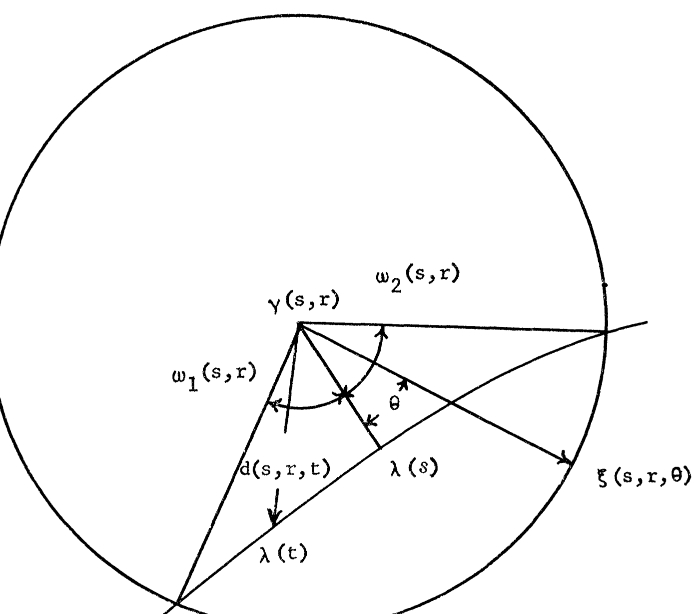

FIGURE 2

$$\frac{1}{n+1}\log\left|\frac{D_n(\gamma)}{G_1(\gamma)^{n+1}}\right| + \int_{K(s,r)}\log\left[\frac{r}{d(s,r,\xi)}\right]\alpha_n(d\xi)$$
$$= \frac{1}{2\pi(n+1)}\int_T \log\left|\frac{D_n(\xi)}{G_1(\xi)^{n+1}}\right| d\theta. \tag{5}$$

Here in the integral on the left hand side of (5) $d(s, r, \xi)$ is the distance from $\gamma(s, r)$ to a general point ξ in $K(s, r)$. It is apparent from Theorem 2d that

$$\lim_{n\to\infty}\frac{1}{n+1}\log\left|\frac{D_n(\gamma)}{G_1(\gamma)^{n+1}}\right| = 0.$$

Furthermore, the definition of weak limit point implies that for some subsequence $1 \leq n_1 < n_2 < \cdots$

$$\lim_{k\to\infty}\int_{K(s,r)}\log\left|\frac{r}{d(s,r,\xi)}\right|\alpha_{n_k}(d\xi) = \int_{C\cap K(s,r)}\log\left|\frac{r}{d(s,r,t)}\right|\alpha(dt).$$

Note that $d(s, r, t) = d(s, r, \lambda(t))$. Finally we assert that

$$\lim_{n\to\infty}\frac{1}{2\pi(n+1)}\int_T\log\left|\frac{D_n(\xi)}{G_1(\xi)^{n+1}}\right| d\theta = \frac{1}{2\pi}\int_{\omega_1(s,r)}^{\omega_2(s,r)}\log\left|\frac{G_2(\xi)}{G_1(\xi)}\right| d\theta.$$

To verify this let $\delta > 0$ be small and set

$$\frac{1}{2\pi(n+1)}\int_T\log\left|\frac{D_n(\xi)}{G_1(\xi)^{n+1}}\right| = I_1 + I_2 + I_3 + I_4$$

corresponding to the ranges of integration $(\omega_2 + \delta, \omega_1 - \delta)$, $(\omega_1 + \delta, \omega_2 - \delta)$, $(\omega_1 - \delta, \omega_1 + \delta)$, and $(\omega_2 - \delta, \omega_2 + \delta)$. By Theorem 2d

$$\lim_{n\to\infty} I_1 = 0, \qquad \lim_{n\to\infty} I_2 = \frac{1}{2\pi}\int_{\omega_1+\delta}^{\omega_2-\delta}\log\left|\frac{G_2}{G_1}\right| d\theta.$$

We have $I_3 = I_3' + I_3''$ where

$$I_3' = -\frac{1}{2\pi}\int_{\omega_1-\delta}^{\omega_1+\delta}\log|G_1(\xi)|\,d\theta, \qquad I_3'' = \frac{1}{2\pi(n+1)}\int_{\omega_1-\delta}^{\omega_1+\delta}\log|D_n(\xi)|\,d\theta.$$

Since I_3' is independent of n we pass on to I_3''. It is easy to see that for any complex number λ

$$\int_{\omega_1-\delta}^{\omega_1+\delta}\log|\xi(s,r,\theta) - \lambda|\,d\theta \geq \int_{-\delta}^{\delta}\log|r\sin\theta|\,d\theta.$$

Since $D_n(\xi) = (-1)^n(\xi - \lambda_{n,0})\cdots(\xi - \lambda_{n,n})$ we have

$$I_3''(n,\delta) \geq \frac{1}{2\pi}\int_{-\delta}^{\delta}\log|r\sin\theta|\,d\theta.$$

Now both the $\{\lambda_{n,k}\}_0^n$ and C lie inside the circle $|\xi| \leq \|f\|_\infty$ and consequently

$$\int_{\omega_1-\delta}^{\omega_1+\delta}\log|\xi(s,r,\theta) - \lambda_{n,k}|\,d\theta \leq \int_{-\delta}^{\delta}\log(2\|f\|_\infty)\,d\theta,$$

which implies that

$$I_3''(n, \delta) \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} \log (2 \| f \|_\infty) \, d\theta.$$

It follows that, for $n = 0, 1, \cdots$,

$$| I_3 | \leq B(\delta)$$

and similarly

$$| I_4 | \leq B(\delta),$$

where $B(\delta) \to 0$ as $\delta \downarrow 0$. Putting these facts together we obtain our desired conclusion.

We define the constant p by

$$p = \sqrt{3} - \pi/3.$$

LEMMA 4e. *Choose $t_1 < t_2$ so that $\lambda(t_1), \lambda(t_2) \in c \cap S$. Then*

$$\lim_{r \to 0+} \frac{1}{2\pi r} \int_{\omega_1(s,r)}^{\omega_2(s,r)} \log \left| \frac{G_2}{G_1} \right| d\theta = p\Phi(s)$$

uniformly for $t_1 \leq s \leq t_2$.

Proof. Let $g(\xi) = G_2(\xi)/G_1(\xi)$ and let $\varphi(s) = \arg \{\lambda(s) - \gamma(s, r)\}$. Then

$$\xi(r, s, \theta) - \lambda(s) = re^{i[\varphi(s)+\theta]} - \tfrac{1}{2}re^{i\varphi(s)}.$$

Since $| g(\xi) | = 1$ on c and $| g(\xi) | > 1$ below c

$$g'[\lambda(s)] = 2\pi g[\lambda(s)]\Phi(s)e^{-i\varphi(s)}.$$

Using Taylor's formula we see that

$$g[\xi(r, s, \theta)] = g[\lambda(s)] + g'[\lambda(s)][\xi(\theta) - \lambda(s)] + O(r^2),$$

and thus

$$\log | g[\xi(r, s, \theta)] | = 2\pi r(\cos \theta - \tfrac{1}{2})\Phi(s) + O(r^2).$$

Moreover, this holds uniformly for $t_1 \leq s \leq t_2$, $\omega_1(s, r) \leq \theta \leq \omega_2(s, r)$, and $0 < r < r_0$, if r_0 is sufficiently small. Furthermore, it is evident that

$$\lim_{r \to 0+} \omega_1(s, r) = -\pi/3, \qquad \lim_{r \to 0+} \omega_2(s, r) = \pi/3,$$

uniformly for $t_1 \leq s \leq t_2$. Combining these estimates we obtain our desired result.

LEMMA 4f. *There exists positive constants a and A such that*

$$\frac{1}{r} \log \left[\frac{r}{d(s, r, t)} \right] \begin{cases} \leq \dfrac{1}{r} \log^+ \left[\dfrac{1}{4} + \left(\dfrac{s-t}{r} \right)^2 \right]^{-1/2} + A\chi_a\left(\dfrac{s-t}{r} \right) \\ \geq \dfrac{1}{r} \log^+ \left[\dfrac{1}{4} + \left(\dfrac{s-t}{r} \right)^2 \right]^{-1/2} - A\chi_a\left(\dfrac{s-t}{r} \right) \end{cases}$$

if $t_1 \leq s \leq t_2$, $t \in c \cap K(s, r)$ *and* $0 \leq r \leq r_0$ *for some* $r_0 > 0$. *Here* χ_a *is the characteristic function of* $[-a, a]$.

Proof. One easily sees that

$$d(s, r, t)^2 = \tfrac{1}{4}r^2 + (s - t)^2 + O(r(s - t)^2)$$

uniformly for $t_1 \leq s \leq t_2$, $t \in c \cap K(s, r)$, $0 \leq r \leq r_0$, provided $r_0 > 0$ is small enough, etc.

LEMMA 4g. *Under the above assumptions if* $t_1 < s_1 < s_2 < t_2$ *then*

$$\int_{s_1-}^{s_2+} \alpha(ds) = \int_{s_1}^{s_2} \Phi(s)\, ds.$$

Proof. For $\varepsilon > 0$ small let

$$I(\pm\varepsilon, r) = \int_{s_1\mp\varepsilon}^{s_2\pm\varepsilon} ds \int r^{-1} \log \left[\frac{r}{d(s, r, t)}\right] \alpha(dt),$$

where the integration with respect to t is over $c \cap K(s, r)$. Then

$$I(\varepsilon, r) \geq \int_{s_1-\varepsilon}^{s_2+\varepsilon} ds \int \left\{\log^+ \left[\frac{1}{4} + \left(\frac{s - t}{r}\right)^2\right]^{-1/2} - A\chi_a\left(\frac{s - t}{r}\right)\right\} \alpha(dt)$$

$$\geq \int \alpha(dt) \int_{s_1-\varepsilon}^{s_2+\varepsilon} \left\{\log^+ \left[\frac{1}{4} + \left(\frac{s - t}{r}\right)^2\right]^{-1/2} - A\chi_a\left(\frac{s - t}{r}\right)\right\} ds.$$

Since $\alpha(C) = 1$ we see that for r sufficiently small

$$I(\varepsilon, r) \geq p\alpha([s_1, s_2]) - 2aAr.$$

Similarly

$$I(-\varepsilon, r) \leq p\alpha([s_1, s_2]) + 2aAr.$$

At the same time by Lemma 4d

$$I(\pm\varepsilon, r) = \frac{1}{2\pi} \int_{s_1\mp\varepsilon}^{s_2\pm\varepsilon} ds \int_{\omega_1(s,r)}^{\omega_2(s,r)} \log \mid G_2(\xi)/G_1(\xi) \mid d\theta,$$

and therefore by Lemma 4e

$$\lim_{r\to 0+} (I(\pm\varepsilon, r) = p \int_{s_1\mp\varepsilon}^{s_2\pm\varepsilon} \Phi(s)\, ds.$$

Thus

$$\int_{s_1+\varepsilon}^{s_2-\varepsilon} \Phi(s)\, ds \leq \alpha([s_1, s_2]) \leq \int_{s_1-\varepsilon}^{s_2+\varepsilon} \Phi(s)\, ds,$$

and since ε is arbitrary our desired conclusion follows.

THEOREM 4h. *The measures* $\alpha_n(d\xi)$ *converge weakly to a measure* α *with support* C. α *is absolutely continuous with respect to the measure induced on each of the arcs of* C *by the arc length* s *and is given by the formula*

$$\alpha(ds) = \Phi(s)\,ds,$$

where $\Phi(s)$ *is defined by* (1).

Proof. In view of the preceding lemma we need only show that if λ is an end point of one of the arcs in the representation of C given in Theorem 3a and Scholium 3b, then $\alpha(\{\lambda\}) = 0$. Let us consider a typical case as in Figure 3. Draw a line from λ in any direction distinct from the directions at λ of the arcs of C having λ as an end point. Let $\gamma(r)$ be the point on this line at a distance $r/2$ from λ and with $\gamma(r)$ as center construct a circle $K(r)$ with radius r. Arguing as in the proof of Lemma 4d we find that, with an evident notation,

$$\int_{C\cap K(r)} \log\left[\frac{r}{d(s, r, t)}\right]\alpha(dt)$$
$$= \frac{1}{2\pi}\int_{\omega_1(r)}^{\omega_2(r)} \log|G_2(\xi)/G_1(\xi)|\,d\theta + \frac{1}{2\pi}\int_{\omega_2(r)}^{\omega_3(r)} \log|G_3(\xi)/G_1(\xi)|\,d\theta.$$

Making use of the relations

$$\lim_{\xi\to\lambda}|G_2(\xi)/G_1(\xi)| = 1, \qquad \xi \text{ in region } 2,$$
$$\lim_{\xi\to\lambda}|G_3(\xi)/G_1(\xi)| = 1, \qquad \xi \text{ in region } 3,$$

together with the fact that $|G_1(\xi)|$, $|G_2(\xi)|$, and $|G_3(\xi)|$ are all bounded away from 0 and ∞ near λ, (because the $z_\nu(\lambda)$ are) we see that

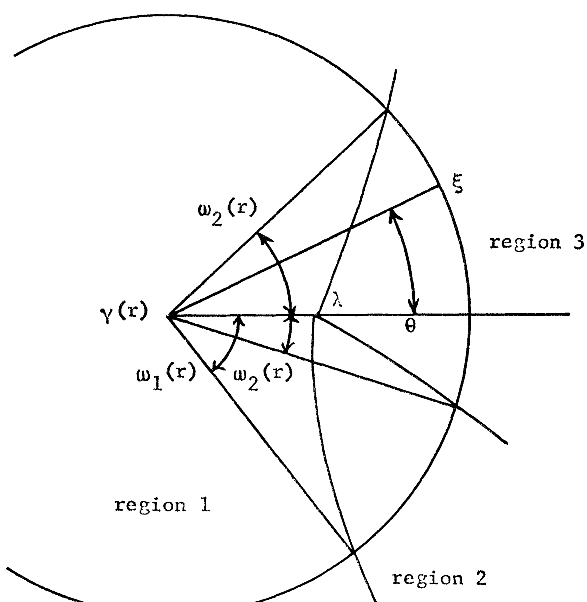

FIGURE 3

$$\lim_{r\to 0+} \int_{C\cap K(r)} \log\left[\frac{r}{d(s, r, t)}\right] \alpha(dt) = 0.$$

Since

$$\alpha(\{\lambda\}) \log 2 \leq \int_{C\cap K(r)} \log\left[\frac{r}{d(s, r, t)}\right] \alpha(dt)$$

it follows that $\alpha(\{\lambda\}) = 0$.

5. An example

Following Schmidt and Spitzer we consider the special case

$$f(z) = bz^{-1} + az$$

where $a > b > 0$. The range of $f_\rho(e^{i\theta})$ is the ellipse whose parametric equation is

$$x = (a\rho + b/\rho) \cos\theta, \qquad y = (a\rho - b/\rho) \sin\theta,$$

or in non-parametric form

$$x^2/(a\rho + b\rho^{-1})^2 + y^2/(a\rho - b\rho^{-1})^2 = 1,$$

and $\sigma^+(\rho)$ consists of this ellipse together with its interior. It is easily seen that $C = \bigcap_{\rho>0} \sigma^+(\rho)$ consists of the segment from $-\sqrt{4ab}$ to $+\sqrt{4ab}$ on the real axis. The roots of $z\{f(z) - \lambda\} = 0$ are $(\lambda \pm i\sqrt{4ab - \lambda^2})/2a$. It is apparent that in the situation pictured above

$$G_1(\xi) = -[\xi + i\sqrt{4ab - \xi^2}]/2, \qquad \xi \in S_1,$$

$$G_2(\xi) = -[\xi - i\sqrt{4ab - \xi^2}]/2, \qquad \xi \in S_2.$$

Thus, since

$$\frac{1}{2\pi}\left|\frac{d}{d\xi}\frac{G_1(\xi)}{G_2(\xi)}\right|_{\xi=x} = \frac{1}{\pi}(4ab - x^2)^{-1/2},$$

we have

$$\alpha(dx) = (1/\pi)(4ab - x^2)^{-1/2}\, dx.$$

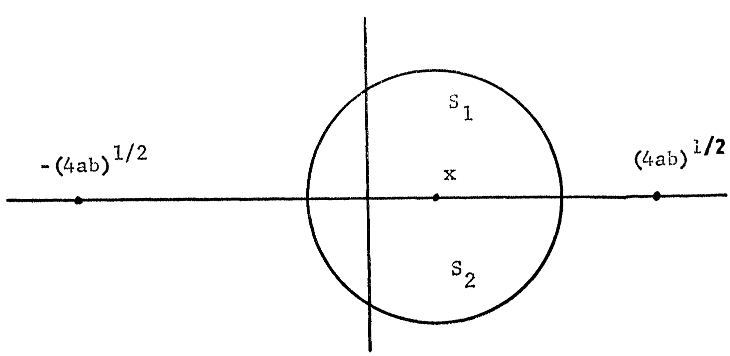

FIGURE 4

6. The case $\sigma^+(f) = R(f)$

We remarked in §1 if $f(\theta) \in L^1(T)$ is real, then the limit measure α is given by the formula

$$\alpha(E) = \frac{1}{2\pi} \int_{f(e^{i\theta}) \in E} d\theta.$$

In the present section we will show that this formula is valid whenever $\sigma^+(f) = R(f)$, $R(f)$ being the range of f, if f is at all well behaved.

LEMMA 6a. *If C is a compact set in the λ-plane which has two dimensional Lebesgue measure* 0, *then the restrictions to C of the functions* $\log |\lambda - a|$ *where $a \notin C$ are fundamental in* $\mathbf{C}(C)$, *the space of complex continuous functions on C with the uniform norm.*

Proof. Suppose that $a = a_1 + ia_2 \in C'$, the (open) complement of C; then, since for h real

$$\lim_{h \to 0} h^{-1}\{\log |\lambda - a + h| - \log |\lambda - a|\} = \mathrm{Rl}\left(\frac{1}{\lambda - a}\right),$$

$$\lim_{h \to 0} h^{-1}\{\log |\lambda - a + ih| - \log |\lambda - a|\} = -\mathrm{Im}\left(\frac{1}{\lambda - a}\right),$$

where the limits on the left are uniform for $\lambda \in C$, it is sufficient to know that the restrictions to C of $(\lambda - a)^{-1}$ where $a \notin C$ are fundamental in $\mathbf{C}(C)$. That this is so is a theorem of Hartogs and Rosenthal, see [3, p. 20].

THEOREM 6b. *Let $f(e^{i\theta})$, $\theta \in T$, belong to the class* Lip η *for some $\eta > 0$, let $f(e^{i\theta})$ have the property that $\sigma^+(f) = R(f)$, and let $R(f)$ have two dimensional Lebesgue measure* 0. *Then if N is any neighborhood of $R(f)$, the support of α_n lies in N for all sufficiently large n, and as $n \to \infty$ the α_n converge weakly to the measure α defined by*

$$\alpha(E) = \frac{1}{2\pi} \int_{f(e^{i\theta}) \in E} d\theta.$$

Proof. The assertion concerning the supports of the α_n's follows from $\sigma^+(f) = R(f)$ and Theorem 2a. It also follows from Theorem 2a that if $a \notin R(f)$ then

$$\lim_{n \to \infty} \frac{1}{n+1} \log |D_n(a)| = \frac{1}{2\pi} \int_T \log |f(e^{i\theta}) - a| \, d\theta,$$

which we rewrite in the form

$$\lim_{n \to \infty} \int \log |\lambda - a| \, \alpha_n(d\lambda) = \int \log |\lambda - a| \, \alpha(d\lambda).$$

Let $\varphi(\lambda)$ be any continuous function defined in the complex plane. If $\varphi_0(\lambda)$ is the restriction of $\varphi(\lambda)$ to $R(f)$ then it follows from Lemma 6a that given $\varepsilon > 0$ there exist $\{a_k\}_1^K \in R(f)'$ and $\{b_k\}_1^K$ such that

$$|\varphi_0(\lambda) - \sum_{k=1}^{K} b_k \log|\lambda - a_k|| < \varepsilon \qquad \text{for all } \lambda \in R(f).$$

Then if

$$N = \{\lambda : |\varphi(\lambda) - \sum_{k=1}^{K} b_k \log|\lambda - a_k|| < 2\varepsilon\}$$

N is a neighborhood of $R(f)$. Consequently, since the α_n and α have mass 1,

$$\left| \int \varphi(\lambda)\alpha_n(d\lambda) - \int \varphi(\lambda)\alpha(d\lambda) \right| < 4\varepsilon$$

for all n sufficiently large; that is

$$\lim_{n\to\infty} \int \varphi(\lambda)\alpha_n(d\lambda) = \int \varphi(\lambda)\alpha(d\lambda).$$

But this is the definition of weak convergence of α_n to α.

The property that $\sigma^+(f) = R(f)$ is insured by either of the conditions:

1. $f(e^{i\theta})$ is even;
2. $R(f)$ does not separate the plane.

The property that $R(f)$ has two dimensional Lebesgue measure 0 is insured by the condition

$$f \in \text{lip } \tfrac{1}{2}.$$

Let $\Omega(dx)$ be a finite non-negative measure on the Borel sets of $[-1 \leq x \leq 1]$. $\Omega(dx)$ is said to belong to the class **S** if

$$\int_{-1}^{1} (1 - x^2)^{-1/2} \log \Omega_a(x)\, dx > -\infty,$$

where $\Omega(dx) = \Omega_s(dx) + \Omega_a(x)\,dx$ is the decomposition of $\Omega(dx)$ into its singular and absolutely continuous parts. Let $\{p(k, x)\}_{k=0}^{\infty}$ be the orthonormal polynomials corresponding to $\Omega(dx)$, normalized by the condition that the coefficient of x^k in $p(k, x)$ is positive.

For $c(x) \in L^1(\Omega)$ let

$$c^\wedge(r, s) = \int_{-1}^{1} p(r, x)p(s, x)c(x)\Omega(dx),$$

and let

$$M_n[c] = [c^\wedge(r, s)]_{r,s=0,\cdots,n}$$

be the corresponding Toeplitz matrix. Let $\{\lambda_{n,k}\}_0^n$ be the eigen values of $M_n[c]$ and let, as before, $\alpha_n = (n + 1)^{-1}\sum_{\lambda_{n,k}\in E} 1$. The methods of this section in conjunction with those of §6 of [2] suffice to prove the following.

THEOREM 6c. *Let $c(x)$, $-1 \leq x \leq 1$ belong to the class* Lip η, *for some $\eta > 0$, and let $R(c)$, the range of c, have two dimensional Lebesgue measure* 0. *Then if N is any neighborhood of $R(c)$ the support of α_n lies in N for all sufficiently large n, and as $n \to \infty$ the α_n converge weakly to the measure α defined by*

$$\alpha(E) = \frac{1}{\pi} \int_{c(\cos\theta)\in E} d\theta \qquad (0 \leq \theta \leq \pi).$$

References

1. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Berkeley and Los Angeles, Univ. of California Press, 1958.
2. I. I. Hirschman, Jr., *The strong Szegö limit theorem for Toeplitz determinants*, Amer. J. Math., vol. 88 (1966), pp. 577–614.
3. S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Amer. Math. Soc. Translations, no. 101, 1954.
4. E. Reich, *On non-Hermitian Toeplitz matrices*, Math. Scand., vol. 10 (1962), pp. 145–152.
5. P. Schmidt and F. Spitzer, *The Toeplitz matrices of an arbitrary Laurent polynomial*, Math. Scand., vol. 8 (1960), pp. 15–38.
6. E. C. Titchmarsh, *The theory of functions*, Oxford, Oxford Univ. Press, 1932.

Washington University
St. Louis, Missouri