ON THE ORDER ISOMORPHISM OF A PARTIALLY ORDERED LINEAR SPACE AND ITS ORDER DUAL

BY
RALPH DEMARR[1]

By considering various examples, the author has come to the conclusion that the following conjecture is probably true.

CONJECTURE. *If a partially ordered linear space X is order isomorphic to its order dual X', then it is possible to define an inner product* $(\cdot, \cdot)$ *on* $X \times X$ *in such a way that X becomes a real Hilbert space with this inner product. Furthermore, the inner product can be defined so that it has the following two properties:*

(a) if $x \in X$ *and* $x \geq 0$, *then* $(x, \cdot)$ *is a positive linear functional*; *hence*, $(x, \cdot) \in X'$;

(b) *if* $f \in X'$ *and* $f \geq 0$, *then there exists* $x \in X$ *with* $x \geq 0$ *such that* $(x, \cdot) = f$.

Although we are not able to prove or disprove this conjecture, we can prove it in a non-trivial special case. This is the main result of our paper, but we shall also discuss other results related to the order isomorphism of X and X'. Finally, we shall show that every real Hilbert space can be partially ordered so that it is order isomorphic to its order dual.

For the basic definitions of a partially ordered linear space and its order dual the reader may refer to Namioka [7, pp. 3–8]. Other references on this subject can be found in Birkhoff [1], Kantorovich [4], Nakano [6], and Vulikh [8].

DEFINITION 1. Let X be a partially ordered linear space and let X' be its order dual. Then X and X' are said to be order isomorphic to each other if there exist positive linear transformations $A : X \to X'$ and $B : X' \to X$ such that $B(A(x)) = x$ for all $x \in X$ and $A(B(f)) = f$ for all $f \in X'$.

ASSUMPTION 2. In this paper we assume that X, X', A, and B have the meanings given in Definition 1 and that X and X' are order isomorphic to each other. We use K and K' to denote the positive cones in X and X', respectively.

LEMMA 3. $X = K - K$.

Proof. By definition $X' = K' - K'$. Since X and X' are order isomorphic to each other, it follows that $X = K - K$.

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Since Namioka does not assume that a partial ordering is anti-symmetric (i.e., that $x \leqq y$ and $y \leqq x$ imply $x = y$), we shall prove the following lemma.

LEMMA 4. *The partial orderings in* X *and* X' *are anti-symmetric.*

Proof. We need only prove that if $0 \leq x$ and $x \leq 0$, where $x \in X$, then $x = 0$. Define $f = A(x)$; hence, $0 \leq f$ and $f \leq 0$, where the symbol 0 here refers to the zero functional in X'. By definition $0 \leq f(y)$ and $f(y) \leq 0$ for all $y \in K$. Since $X = K - K$, it follows that $f(z) = 0$ for all $z \in X$; hence, $f = 0$. Therefore, $x = B(f) = 0$. In the course of the preceding remarks it was shown that if $0 \leq f$ and $f \leq 0$, where $f \in X'$, then $f = 0$. Thus, the lemma is proved.

DEFINITION 5. For each $x, y \in X$ we define $E(x, y) = A(x)(y)$, where the right-hand expression denotes the value of the functional $A(x)$ at y. It is clear that E is a bilinear functional defined on $X \times X$ such that $E(x, y) \geq 0$ for all $x, y \in K$. For each $x \in X$ we define $F(x) = E(\cdot, x)$. It is easy to verify that F is a positive linear transformation mapping X into X'.

LEMMA 6. *If* $x \in X$ *and* $x \neq 0$, *then there exists* $f \in K'$ *such that* $f(x) \neq 0$.

Proof. Define $g = A(x)$. Since $x \neq 0$, it follows that $g \neq 0$, which means there exists $y \in K$ such that $g(y) \neq 0$. Define $f = F(y) \in K'$. Now

$$f(x) = F(y)(x) = E(x, y) = A(x)(y) = g(y) \neq 0.$$

LEMMA 7. *The mapping* F *is one-to-one.*

Proof. If $x \in X$ and $x \neq 0$, then by Lemma 6 there exists $f \in K'$ such that $f(x) \neq 0$. If we define $y = B(f)$, then $F(x)(y) = A(y)(x) = f(x) \neq 0$. Hence, $F(x) \neq 0$.

DEFINITION 8. A non-empty subset M of any partially ordered set is said to be down-directed if for every $x, y \in M$ there exists $z \in M$ such that $z \leq x$ and $z \leq y$. The term up-directed is defined by reversing the inequalities in the preceding statement. A non-empty subset M of K is said to be directed to 0 if M is down-directed and if $\inf M = 0$. (More detail on these matters may be found in Definition 1 of [2] or [3].)

LEMMA 9. *X is Dedekind complete. This means that if M is a down-directed subset of K, then* inf M *exists.*

Proof. Since X' is always Dedekind complete and X and X' are order isomorphic to each other, X must also be Dedekind complete.

The term "Dedekind complete" is due to McShane [5, pp. 9–11].

LEMMA 10. *The mapping* A *is o-continuous. This means that if* M, *where* $M \subset K$, *is directed to* 0, *then* $\inf \{A(x) : x \in M\} = 0$. *The proof below can be modified to show that* B *is also o-continuous.*

Proof. Let M be any subset of K which is directed to 0. If we define $M' = \{A(x) : x \in M\}$, then $M' \subset K'$ and M' is down-directed. Since X' is Dedekind complete, $g = \inf M'$ exists. It is clear that $0 \leq g$. Now for every $x \in M$ we have $g \leq A(x)$ *and, hence,* $B(g) \leq x$ for all $x \in M$. Therefore, $B(g) \leq 0$ and, hence, $g \leq 0$. By Lemma 4, $g = 0$, which proves that A is o-continuous.

LEMMA 11. *For each $z \in K$ the positive linear functional $F(z)$ is o-continuous.*

Proof. Let M be any subset of K which is directed to 0. Putting $f = F(z)$, we want to show that

$$\inf \{f(x) : x \in M\} = 0.$$

Now for each $x \in M$ we have $f(x) = A(x)(z)$. Since $\inf \{A(x) : x \in M\} = 0$ and M is down-directed, it follows that

$$\inf \{A(x)(y) : x \in M\} = 0$$

for all $y \in K$. Since $z \in K$, it follows that

$$\inf \{f(x) : x \in M\} = 0,$$

which proves that $F(z)$ is o-continuous.

We now make a few comments in preparation for the main theorem. In the main theorem we will assume that X is a complete vector lattice (i.e., that every non-empty subset of K has an infimum). However, it is not necessary to make exactly this assumption about X. For example, we could assume that X has the decomposition property in which case X' is a complete vector lattice [7, p. 27]. Since X and X' are order isomorphic to each other, it follows that X is also a complete vector lattice.

If X is a vector lattice, then we write $x^+ = x \vee 0$, $x^- = (-x)^+$, $|x| = x^+ + x^-$. Note that $x = x^+ - x^-$; this differs from Birkhoff [1, p. 219].

We now point out by means of an example that even if X is a complete vector lattice, the bilinear functional E (see Definition 5) may not be an inner product. Let X be the real linear space of all triples of real numbers which is partially ordered componentwise; hence, X is a complete vector lattice. It is easy to show that X' is exactly the same as X in the sense that if

$$x = (\alpha_1, \alpha_2, \alpha_3) \in X \quad \text{and} \quad f = (\beta_1, \beta_2, \beta_3) \in X',$$

then

$$f(x) = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 .$$

Now let us define $A : X \to X'$ and $B : X' \to X$ as follows:

$$A(x) = (\alpha_3, \alpha_1, \alpha_2) \quad \text{for } x = (\alpha_1, \alpha_2, \alpha_3);$$

$$B(f) = (\beta_2, \beta_3, \beta_1) \quad \text{for } f = (\beta_1, \beta_2, \beta_3).$$

Putting $x = (1, 0, 0)$ and $y = (1, 0, 1)$, it is easily seen that $E(x, y) \neq E(y, x)$ and $E(x, x) = 0$.

LEMMA 12. *Let X be a complete vector lattice. For any $w \in K$ there exists a positive linear transformation $P : X \to X$ which has the property that*

$$P(x) = \sup\{(nw) \wedge x : n = 1, 2, 3, \cdots\} \qquad \textit{for all } x \in K.$$

Proof. See Nakano [6, pp. 41–43].

The positive linear transformation P defined above is called a projector by Nakano. The essential properties of projectors are intuitively obvious and, therefore, we will use them without comment. Detailed proofs of these properties can be found in Nakano [6, pp. 41–54].

THEOREM 13. *Let X be a complete vector lattice. If $E(x, y) = 0$ whenever $x \wedge y = 0$, then the above-stated conjecture is true with $E(\cdot, \cdot)$ as the inner product.*

Proof. First let us note that if $w \in K$ and P is the projector determined by w as in Lemma 12, then

$$E(P(x), w) = E(x, w) \quad \text{and} \quad E(w, P(x)) = E(w, x)$$

for all $x \in X$. This follows from the fact that $(x - P(x)) \wedge w = 0$ for all $x \in K$; hence,

$$E(x - P(x), w) = 0 = E(w, x - P(x))$$

for all $x \in K$. Since $X = K - K$, the desired equalities are obtained.

Let us now take any $w \in K$ with $w \neq 0$. Since $F : X \to X'$ (see Definition 5) is one-to-one by Lemma 7, $F(w) = f \in K'$ and $f \neq 0$. Consequently, there exists $z \geq w$ such that $f(z) > 0$. Now $f(z) = E(z, w) = E(P(z), w) = f(P(z))$, where P is the projector determined by w as in Lemma 12. Defining $z_n = (nw) \wedge z$ for all $n = 1, 2, 3, \cdots$, we see that $z_1 \leq z_2 \cdots$ and $\sup\{z_n\} = P(z)$ by Lemma 12. By Lemma 11 f is o-continuous so that $\lim_{n\to\infty} f(z_n) = f(P(z)) = f(z)$. However, $z_n \leqq nw$ for all n and since $f(z) > 0$, there must be an integer m such that $0 < f(z_m) \leq mf(w)$. Hence, $f(w) = E(w, w) > 0$. If we take any $x \in X$, then

$$E(x, x) = E(x^+, x^+) + E(x^-, x^-) \geqq 0;$$

if $x \neq 0$, then $x^+ \neq 0$ or $x^- \neq 0$ so that $E(x, x) > 0$. Note also that we have shown here that if $E(|x|, |y|) = 0$, then $|x| \wedge |y| = 0$.

Now take any $v \in K$ and define $u = B(F(v))$. Hence, $E(u, x) = E(x, v)$ for all $x \in X$. Now define $w = (u - v)^+$ and let P be the projector determined by w as in Lemma 12. Since $P((u - v)^-) = 0$, we have $w = P(w) = P(u - v) = u_1 - v_1$, where we put $u_1 = P(u)$ and $v_1 = P(v)$. Now we must have $E(u, u_1) = E(u_1, u_1)$ because $(u - u_1) \wedge u_1 = 0$ and $E(u_1, v) = E(u_1, v_1)$ because $u_1 \wedge (v - v_1) = 0$. Therefore, $E(u_1, u_1) = E(u_1, v_1)$ which means that $E(u_1, u_1 - v_1) = E(u_1, w) = 0$; hence, $u_1 \wedge w = 0$. Since $u \geq 0$ and $v \geq 0$, it follows that $u \geq (u - v)^+ = w$; hence, $u_1 = P(u) \geq w$ so that $0 = u_1 \wedge w = w$. Since $w = 0$, it follows that $u \leq v$. Now $E(u, v) = E(v, v)$ which means that $E(v - u, v) = 0$; hence, $0 = (v - u) \wedge v = v - u$.

Since $u = v$, we have $E(v, x) = E(x, v)$ for all $x \in X$. Since v is any element of K and $X = K - K$, it follows that $E(x, y) = E(y, x)$ for all $x, y \in X$.

Thus, we have shown that $E(\cdot, \cdot)$ is an inner product for X. It is obvious that $E(\cdot, \cdot)$ has property (a) of the conjecture. If $f \in K'$, define $x = B(f) \in K$. Hence, $f = A(x) = E(x, \cdot)$, so that $E(\cdot, \cdot)$ has property (b).

We must now show that X is norm complete with respect to the norm $\| \cdot \|$ determined by the inner product $E(x, y)$. Let $\{x_n\}$ be a Cauchy sequence of elements from X. Using the Cauchy-Schwarz inequality, we see that $\lim_{n\to\infty} E(x_n, y) = f(y)$ for all $y \in X$. By Nakano's theorem [6, p. 251] it follows that $f \in X'$. Let us define $x = B(f)$ so that $f = E(x, \cdot)$. Putting $\alpha = \sup\{\| x_n - x \| : n = 1, 2, 3, \cdots\}$ and $\beta_m = \sup\{\| x_n - x_m \| : n \geq m\}$, we obtain by elementary computations that

$$\| x_n - x \|^2 \leqq \alpha\beta_m + | E(x_n - x, x_m - x) | \qquad \text{for } n \geq m.$$

Therefore, for each fixed m we have $\limsup_{n\to\infty} \| x_n - x \|^2 \leqq \alpha\beta_m$. Since $\{x_n\}$ is a Cauchy sequence, it follows that $\lim_{m\to\infty} \beta_m = 0$; hence, $\lim_{n\to\infty} \| x_n - x \| = 0$. This proves that X is norm complete and completes the proof of the theorem.

There are at least two essentially different ways of partially ordering a real Hilbert space Y so that Y and Y' are order isomorphic to each other. We assume that $Y \neq \{0\}$.

I. If Y is a real Hilbert space, then Y is isomorphic to the space Y_0 of all real-valued functions $x(\cdot)$ defined on some set Ω such that

$$\sum \{| x(\sigma) |^2 : \sigma \in \Omega\} < \infty.$$

Taking the pointwise partial ordering of functions in Y_0 and carrying this back to Y by the isomorphism, it is easily shown that Y is a complete vector lattice. (We are implicitly assuming that Y_0 has the usual Hilbert space norm and that the mappings effecting the isomorphism are actually isometries). Since Y is a Banach lattice, we must have $Y' = Y^*$; see [7, p. 44]. But Y is naturally isomorphic to Y^* with one mapping $A : Y \to Y^*$ effecting the isomorphism being defined as follows: $A(x) = (x, \cdot)$ for all $x \in Y$. It then follows by routine calculations that Y and Y' are order isomorphic to each other.

II. Let Y be a real Hilbert space and take any $u \in Y$ with $\| u \| = 1$. Now define $K = \{x : \sqrt{2}\,(x, u) \geq \| x \|\}$. It is easily shown that K is a closed, generating cone with u as an interior point. We may then partially order Y by defining $x \leq y$ to mean that $y - x \in K$. We will show later that $\| x \| + \| y \| \leqq 2\| x + y \|$ for all $x, y \in K$; hence, K is normal [7, p. 30]. Referring to Corollary 5.5, p. 24, and Theorem 6.7, p. 31, of [7], we see that $Y' = Y^*$. As in the preceding example Y is naturally isomorphic to Y^* with the mapping A defined as it is there. To show that Y and Y' are order isomorphic to each other, we need only show that $A(x) \geq 0$ if and only if $x \geq 0$.

Let us take any x_0, $y_0 \in Y$ such that $x_0 > 0$ and $y_0 > 0$. Since

$$0 < \| x_0 \| \leq \sqrt{2}\,(x_0, u),$$

there must exist a real number $\alpha > 0$ such that $\alpha(x_0, u) = 1$. Similarly, there exists a real number $\beta > 0$ such that $\beta(y_0, u) = 1$. Define $x_1 = \alpha x_0$ and $y_1 = \beta y_0$. Since x_1, $y_1 \in K$, we must have

$$\| x_1 \| \leq \sqrt{2}\,(x_1, u) = \sqrt{2} \quad \text{and} \quad \| y_1 \| \leq \sqrt{2}.$$

Since $(x_1 - u, u) = (y_1 - u, u) = 0$, it follows that

$$\| u \|^2 + \| x_1 - u \|^2 = \| x_1 \|^2 \quad \text{and} \quad \| u \|^2 + \| y_1 - u \|^2 = \| y_1 \|^2.$$

Therefore, $\| x_1 - u \| \leq 1$ and $\| y_1 - u \| \leq 1$. Consequently,

$$\begin{aligned} -1 \leq (x_1 - u, y_1 - u) &= (x_1, y_1) - (x_1, u) - (y_1, u) + (u, u) \\ &= (x_1, y_1) - 1; \end{aligned}$$

hence, $0 \leq (x_1, y_1) = \alpha\beta(x_0, y_0)$. Since $\alpha\beta > 0$, we have $0 \leq (x_0, y_0)$. Thus, we have $0 \leq (x, y)$ for all $x, y \in K$. From this it follows that if $x \geq 0$, then $A(x) = (x, \cdot) \geq 0$. For any $x, y \in K$ we have

$$(x + y, x + y) = (x, x) + 2(x, y) + (y, y) \geq (x, x);$$

hence, $\| x + y \| \geq \| x \|$. From this it follows that

$$\| x \| + \| y \| \leq 2\| x + y \| \qquad \text{for all } x, y \in K.$$

Let us now take any $x \in X$ such that $x \neq 0$ and $(x, y) \geq 0$ for all $y \in K$. We first show that $(x, u) > 0$. Assume the contrary; i.e., $(x, u) = 0$. Now define $\alpha = \| x \|^{-1}$ and put $x_1 = \alpha x$. If we define $y_1 = u - x_1$, then $\| y_1 \| = \sqrt{2}$ and $(y_1, u) = 1$, which means that $y_1 \in K$. But

$$(x, y_1) = (x, u) - \alpha(x, x) = -\| x \| < 0,$$

which contradicts the fact that $(x, y_1) \geq 0$. Hence, we must have $(x, u) > 0$. Now take $\beta > 0$ so that $\beta(x, u) = 1$ and then put $z = \beta x$. We will now show that $\| z - u \| \leq 1$. Assume the contrary; i.e., $\| z - u \| > 1$. Putting

$$\gamma = \| z - u \|^{-1} \quad \text{and} \quad w = u + \gamma(u - z),$$

we have

$$\| w \|^2 = \| u \|^2 + \gamma^2 \| u - z \|^2 = 2$$

and

$$(w, u) = (1 + \gamma)(u, u) - \gamma(z, u) = 1,$$

which means that $w \in K$. But

$$\begin{aligned} \beta(x, w) = (z, w) &= (1 + \gamma)(z, u) - \gamma(z, z) \\ &= 1 + \gamma - \gamma[\| u \|^2 + \| z - u \|^2] \\ &= 1 + \gamma - \gamma(1 + \gamma^{-2}) \end{aligned}$$

$$= 1 - \gamma^{-1} < 0,$$

which contradicts the fact that $(x, w) \geq 0$. Hence, we must have

$$\| z - u \| \leq 1.$$

Since $\| z \|^2 = \| u \|^2 + \| z - u \|^2 \leq 2$ and $(z, u) = 1$, it follows that $\beta x = z \geq 0$; hence, $x \geq 0$. This means that if $A(x) = (x, \cdot) \geq 0$, then $x \geq 0$.

The second example does not give a partial ordering equivalent to that in the first example except in the case that Y is one- or two-dimensional. If Y is at least three-dimensional and is partially ordered as in the second example, then Y is not a vector lattice. We leave it to the reader to verify this.

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University of Washington
Seattle, Washington