THE SPACE OF HOMEOMORPHISMS OF A DISC WITH n HOLES

BY

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In [3] M-E. Hamstrom and E. Dyer showed that the identity component of the space of homeomorphisms of an annulus onto itself, keeping the boundary pointwise fixed, is contractible. In [4] Hamstrom showed that, denoting by H_n the identity component of the space of homeomorphisms of a disc with n holes onto itself, keeping the boundary pointwise fixed, we have $\pi_i(H_n) = 0$ for all $i > 0$.

This paper shows that H_n is contractible for all n.

Remark. On all function spaces we shall use the compact-open topology, and refer the reader to Hu [5] for results. In particular, for the space X^Y, where X and Y are both compact metric, the induced metric topology agrees with the compact-open topology, [5, p. 102].

The theorem is proved in two steps, which are summarized here.

Define H_n^* to be the identity component of the space of homeomorphisms of a disc with n holes onto itself, keeping the boundary, and also one interior point a, pointwise fixed.

LEMMA 1. *H_{n-1} deformation retracts onto H_{n-1}^*.*

(A sketch proof follows. The details are given later.)

To every homeomorphism h in H_{n-1} we assign continuously a point $s(h)$ in in the interior of the universal cover of the manifold which lies above the point $h(a)$.

To every point of the interior of the universal cover we assign a canonically defined path to a chosen base point lying above a, and also a canonical isotopy of the manifold, keeping the boundary fixed. This isotopy starts at the identity, and makes the projection of the point follow the projection of the canonical path to the point a. By following a homeomorphism h with this isotopy for the point $s(h)$ we get a canonical isotopy of h to a homeomorphism h^*, which keeps a fixed. This h^* lies in H_{n-1}^*.

LEMMA 2. *H_{n-1}^* is homotopy equivalent to H_n.*

(The details of this proof are given later.)

We define an inclusion $i : H_n \to H_{n-1}^*$ by filling in one hole of the disc with n holes and extending $h \in H_n$ over this by the identity. Taking a as the centre of this filled-in disc, the extended map is in H_{n-1}^*.

The reverse map, $r : H_{n-1}^* \to H_n$ is constructed using a technique of H.

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Kneser [6]. This relies on the existence and uniqueness of certain conformal maps of annuli to produce canonical homeomorphisms of a standard annulus onto any other, given the map on the boundary.

From these two lemmas the theorem will follow by induction on n if H_0 is contractible. This, however, is just Alexander's theorem for the deformations of a 2-cell, [1], and so the induction is complete.

Proof of Lemma 1. Take a core C consisting of lines on the unit lattice in the plane, and thicken it as shown to form A_{n-1}, a standard disc with $n - 1$ holes. Choose the point a on C and an arc J joining a to the point b on the boundary as shown. Choose a point $\tilde{b}$ above b in the universal cover of the manifold, $\tilde{A}_{n-1}$. Then lift the arc J to $\tilde{A}_{n-1}$ to define $\tilde{a}$ above a in $\tilde{C}$.

For any $h \in H_{n-1}$ the arc hJ joins b to $h(a)$. Lifting this gives an arc in $\tilde{A}_{n-1}$ joining $\tilde{b}$ to a point $s(h)$ above $h(a)$. Under the compact-open topology s gives a continuous map

$$s : H_{n-1} \to \operatorname{int} \tilde{A}_{n-1} .$$

Suppose $f : I \to A_{n-1}$ is a path. (See Figure 1.) Then we say that a continuous family of homeomorphisms h_t in H_{n-1} follows f from $f(0)$ to $f(1)$ if

$$h_0 \text{ is the identity}$$

$$h_t(f(0)) = f(t) \qquad \text{for all } t.$$

If two paths can be composed, then this defines uniquely a composition of the two families to give a family which follows the new path.

Note. If we regard the path f as an isotopy of the point $f(0)$, then the family h is an extension of this isotopy to the manifold.

The core C divides A_{n-1} into n annular regions, $B_1, \cdots, B_n$, each bounded by C and by one boundary curve $C_1, \cdots, C_n$ of A_{n-1}. Of these regions, $n - 1$ are square with C outside, and the other is rectangular with C inside. Extend each of these regions an equal distance on the other side of C as shown,

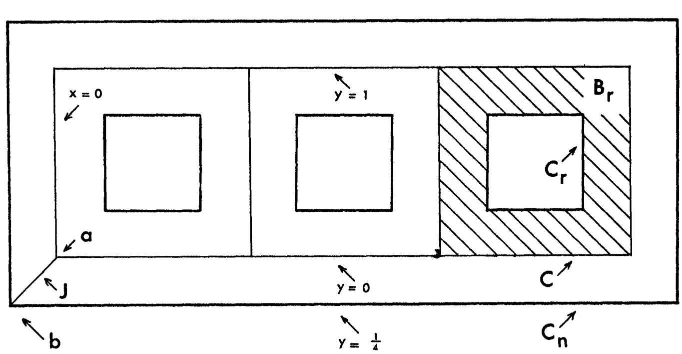

FIGURE 1

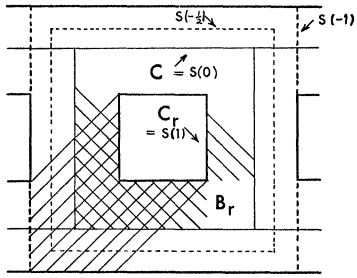

FIGURE 2

to form new square or rectangular annuli, with a part of C as a central core. (See Figure 2.) For each annulus define a family of parametrising rectangles $S(\lambda)$, each side of $S(\lambda)$ being at a distance λ from C normalised so that the boundary curve in A_{n-1} is $S(1)$, and the other boundary curve of the extended region is $S(-1)$.

Any point $x \in B_r - C_r$ lies on some $S(\lambda)$ for $0 \leq \lambda < 1$. For such x we define $f(x, t) \in H_{n-1}$, which maps the annulus

$$\{S(1), S(\lambda)\}$$

linearly onto the annulus

$$\{S(1), S((1-t)\lambda)\},$$

and

$$\{S(\lambda), S(-1)\}$$

linearly onto

$$\{S((1-t)\lambda), S(-1)\},$$

and is the identity outside the region. The maps $f(x, t)$ follow a straight line path, which we shall call $P(x)$, from x to a point of C. We have a continuous map

$$f : (B_r - C_r) \times I \to H_{n-1}$$

which sends $C \times I$ to the identity. Hence we have a continuous map

$$F : \operatorname{int} A_{n-1} \times I \to H_{n-1}.$$

Since the linear structure of A_{n-1} lifts to the universal cover, this defines an obvious map

$$\tilde{F} : \operatorname{int} \tilde{A}_{n-1} \times I \to H_{n-1}.$$

The paths P lift and define a continuous map $e : \operatorname{int} \tilde{A}_{n-1} \to \tilde{C}$ where $e(\tilde{x})$ is the end-point of the path $P(\tilde{x})$.

This produces and follows paths from the interior of the manifold to the core. We now show how to follow paths from points of C to the base-point a.

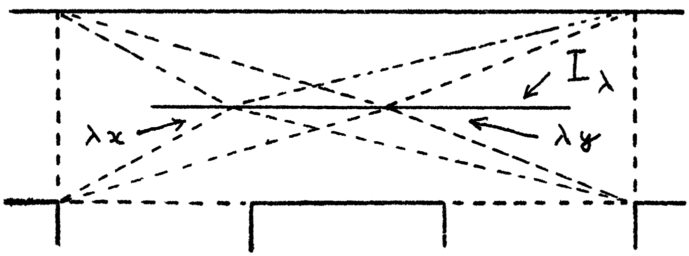

FIGURE 3
The rectangle around I_λ

For each side of a square in C we have two distance-preserving linear mappings $\lambda : I \to C$, depending on the orientation, so there are $6n + 2$ such λ. λI is then a directed straight-line segment of C. Take a rectangular area in A_{n-1} surrounding λI as shown. (See Figure 3.) For any two points λx, λy in λI, define a homeomorphism of the rectangular area by joining λx and λy to the four corners of the rectangle and mapping the four triangles formed from λx linearly to those formed from λy. Extend this to A_{n-1} by the identity, to give $h_\lambda(x, y)$ in H_{n-1}, which depends continuously on x and y. Define the continuous map

$$g_\lambda\colon \lambda I \times I \to H_{n-1}$$

by $g_\lambda(\lambda x, t) = h_\lambda(x, (1 - t)x)$. Then $g_\lambda(\lambda x, t)$ are homeomorphisms which follow the obviously parametrised path from λx to $\lambda 0$.

The linear and local metric structures of C lift to $\tilde{C}$ in an obvious way. For any two points $\tilde{x}$ and $\tilde{y}$ in $\tilde{C}$, there is a unique piecewise linear arc joining them, of length $\rho(\tilde{x}, \tilde{y})$. In particular, for each $\tilde{x} \in \tilde{C}$, we have the unique piecewise linear arc

$$\mu\tilde{x} : [0, \rho(\tilde{a}, \tilde{x})] \to \tilde{C}$$

such that $\mu\tilde{x}(0) = \tilde{a}$, $\mu\tilde{x}(\rho(\tilde{a}, \tilde{x})) = \tilde{x}$. Then $\mu\tilde{x}[i, i + 1]$ is a component of the inverse image of λI for some λ.

Let $\rho(\tilde{a}, \tilde{x}) = m + \nu(\tilde{x})$ where $0 \leq \nu(\tilde{x}) < 1$. Then the arc $\mu(\tilde{x})$ splits up into $m + 1$ linear segments, and hence can be represented as the sum of $m + 1$ linear segments each lying above some λI. By using the appropriate function g_λ, there is, for each segment, a family of homeomorphisms which follows the projection of the segment from its initial point to its final point. These segments combine to form the path $\mu(\tilde{x})$, with the parametrisation of arc length, normalised so that the total length of the path is 1. Similarly we combine the families of homeomorphisms to form one family following the projection of $\mu(\tilde{x})$ from x to a.

Since the function $h_\lambda(x, y)$ is continuous in x and y, and becomes the identity when x and y coincide, the definition of the family is the same when $\rho(\tilde{a}, \tilde{x}) = r + 1$, whether we take $m = r$ and $\nu = 1$, or $m = r + 1$ and $\nu = 0$. Hence the families above define globally a continuous function $G : \tilde{C} \times I \to H_{n-1}$, where $G(\tilde{x}, t)$ is the family defined by the path $\mu(\tilde{x})$.

For a point $\tilde{y}$ in int $\tilde{A}_{n-1}$ we have the paths $P(\tilde{y})$ from $\tilde{y}$ to $e(\tilde{y})$, and $\mu(e(\tilde{y}))$ from $e(\tilde{y})$ to $\tilde{a}$. Perform the first path in time $[0, \frac{1}{2}]$ and the second in $[\frac{1}{2}, 1]$ for each $\tilde{y}$. Then the homeomorphisms corresponding to each path combine to give the continuous map

$$K : \text{int } \tilde{A}_{n-1} \times I \to H_{n-1}$$

with $K(\tilde{y}, t)$ following the projection of the combined path from y to a.

Note that if $\tilde{y} = \tilde{a}$ then both paths become the constant path and $K(\tilde{y}, t) =$ identity, for all t.

The following composition r will give the required deformation retraction by $r_1 = r \mid H_{n-1} \times 1$:

$$\begin{array}{ccccccccc} H_{n-1} \times I & \to & H_{n-1} \times H_{n-1} \times I & \to & H_{n-1} \times \text{int } \tilde{A}_{n-1} \times I & \to & H_{n-1} \times H_{n-1} & \to & H_{n-1} \\ (h, t) & & (h, h, t) & & (h, s(h), t) & & (h, K(s(h), t)) & & \end{array}$$

the last map being composition $f \times g \to g \circ f$. Put $r_t = r \mid H_{n-1} \times t$. Then r_0 is the identity, for $r(h, 0) = K(s(h), 0) \circ h = h$, since $K(\tilde{y}, 0) =$ identity for all $\tilde{y}$. $K(s(h), t)$ was defined to follow a path from the projection of $s(h)$ to a, and since $s(h)$ projects to $h(a)$, then

$$K(s(h), 1)(h(a)) = a.$$

So $r(h, 1)(a) = a$, therefore $r_1 H_{n-1}$ are homeomorphisms which keep a fixed.

$s(h) = \tilde{a}$ for $h \in H^*_{n-1}$, since $hJ \simeq J$ for such h, and so $r(h, t) = h$ for all t, i.e. $r_t \mid H^*_{n-1}$ is the identity. Hence r_1 will provide the deformation retraction when we show that its image lies in H^*_{n-1}. This follows immediately from the fact that r_1 is continuous and H_{n-1} is connected. Then the image lies only in one component of homeomorphisms keeping the boundary of A_{n-1} and a fixed, but we know that some of it lies in H^*_{n-1}, and so it all does.

For the second lemma we require a theorem about uniqueness and convergence of certain maps of annuli, similar to one quoted in [3].

It is a standard result in complex variable theory, see R. Courant [2, p. 38], that, given two non-intersecting Jordan curves γ_1 and γ_2 in the plane, with γ_1 inside γ_2 then there is an annulus $B(r)$, $1 \leq |z| \leq r$, and a homeomorphism w taking the annulus $B(r)$ onto the annulus G defined by γ_1 and γ_2, which is conformal on the interior of $B(r)$, and is uniquely determined by the orientation of the boundary and the image of one boundary point.

An important extension of this result is the following continuity property. Suppose that G_n is a sequence of annuli, point sets bounded by two Jordan curves, whose boundaries, γ_n, converge to the boundary γ of an annulus G in the sense of Fréchet. This means that the boundaries converge to γ as sets, and if two points P_n and Q_n on γ_n tend to P and Q on γ, then the whole arc $P_n Q_n$ must tend to one of the two arcs PQ. Then the values of r_n for G_n converge to the value r for G, and if the image of a convergent sequence of boundary points, one from each $B(r_n)$, is prescribed as a convergent sequence of boundary points of the corresponding G_n, then the resulting homeomorphisms w_n will converge uniformly to w.

The Riemann mapping theorem, which gives the similar result for discs certainly has this continuity property [2, p. 191]. But we can use this theorem to make the boundaries of the annuli analytic. The harmonic functions used by Courant to produce the map to $B(r)$ can now be extended across the boundary, and an examination of these gives the continuity property.

To define a canonical map—no longer conformal—which takes prescribed boundary values on an annulus we proceed as follows.

Suppose R is the annulus $1 \leq |z| \leq 2$, and f_1, f_2 are homeomorphisms of the inner and outer boundary curves onto themselves, which have the same orientation. Then these extend to a homeomorphism of R, given by

$$(r, \theta) \to (r, (2 - r)(f_1(\theta) + 2n_1 \pi) + (r - 1)(f_2(\theta) + 2n_2 \pi)).$$

This is uniquely determined by f_1 and f_2, and by $n_1 - n_2$, or by the angle change along the image of an arc in R joining the two boundary components, for example that part of the real axis $1 \leq x \leq 2$.

To map the annulus R onto a given annulus G with prescribed similarly oriented homeomorphisms g_1 and g_2 on the boundary, we find r, and $w : B(r) \to G$ as above, which restricts to w_1 and w_2 on the boundary. Choose $f_i = w_i^{-1} \circ g_i$, $i = 1, 2$, and extend to the homeomorphism $F(n)$ of R to itself, where $n = n_1 - n_2$. Shrink R radially to $B(r)$ and follow this with w. The combined map is a homeomorphism of R to G which uniquely determined on choosing n. This can be done by choosing the angle change along, say, the image in G of the arc τ, $1 \leq x \leq 2$, in R.

The continuity property above now shows that if f_i and g_i are sequences of similarly oriented disjoint homeomorphisms of S^1 into the plane which converge uniformly to f and g, and if F_i, F, are homeomorphisms of R extending $f_i, g_i; f, g$, as above, then if the angle change along the arcs $F_i(\tau)$ converges to that along $F(\tau)$, the sequence F_i will converge uniformly to F.

Since, for metric spaces, sequential continuity implies continuity, we have a continuous map from the (space of similarly oriented embeddings of a pair of disjoint circles into the plane, with one inside the other) $\times$ (real line), representing the angle change, into the space of embeddings of an annulus in the plane, if we regard such embeddings as having the metric topology.

Proof of Lemma 2. Let A_n be a disc with n holes embedded in the plane so that one of the holes has $|z| = \frac{1}{2}$ as its boundary curve, and $\frac{1}{2} \leq |z| \leq 2$ lies in A_n, with the point $(2, 0)$ lying on a boundary curve. This implies that $n > 0$. Let A_{n-1} be A_n with the disc $|z| \leq \frac{1}{2}$ filled in, and choose the origin as the point a. Then the inclusion $i : H_n \to H_{n-1}^*$ is defined simply by extending $g \in H_n$ by the identity over $|z| \leq \frac{1}{2}$.

Let $h \in H_{n-1}^*$. Then h maps the circle $|z| = 1$ into a Jordan curve $\alpha(h)$

around a. Let $\rho(h)$ be the distance from a to this curve, and choose $\varepsilon(h) = \min(\frac{1}{2}, \frac{1}{2}\rho(h))$. Then the circle $\beta(h)$, $|z| = \varepsilon(h)$, lies inside the Jordan curve $\alpha(h)$. We have a continuous map from H^*_{n-1} to homeomorphisms of two disjoint circles into the plane taking $|z| = 1$ to $\alpha(h)$, and $|z| = \frac{1}{2}$ by radial contraction to $\beta(h)$.

Let the angle change under h along the arc $1 \leq x \leq 2$ be $\theta(h)$, and then specify that the angle change along $\frac{1}{2} \leq x \leq 1$ shall be $-\theta(h)$. This defines uniquely a homeomorphism from the annulus $\frac{1}{2} \leq |z| \leq 1$ to the plane, taking the boundary as prescribed. Since θ depends continuously on h then, by the continuity property above, so does this homeomorphism. We can now define a map on A_n using this one on $\frac{1}{2} \leq |z| \leq 1$, and h on the rest of A_n. Following this with a map shrinking the annulus $\varepsilon(h) \leq |z| \leq 1$ linearly onto $\frac{1}{2} \leq |z| \leq 1$, gives a homeomorphism $r(h)$ from A_n to itself which keeps the boundary pointwise fixed, and depends continuously on $h \in H^*_{n-1}$. Choosing h as the identity we have $\varepsilon(h) = \frac{1}{2}$, $\theta(h) = 0$ so that the map on the annulus, and hence the whole map $r(h)$, is the identity. Since H^*_{n-1} is connected, and r is continuous, $r(h)$ lies in the component of the identity, i.e. H_n, for all h.

We want to prove

$$r \circ i \simeq 1 : H_n \to H_n \quad \text{and} \quad i \circ r \simeq 1 : H^*_{n-1} \to H^*_{n-1}.$$

The annulus $\frac{1}{2} \leq |z| \leq 1$ is a collar of a boundary component of A_n, and hence there is a continuous family of homeomorphisms h_t, $t \in [0, 1]$, shrinking A_n off this collar, with h_0 being the identity, and h_t mapping the boundary $|z| = \frac{1}{2}$ linearly to $|z| = \frac{1}{2} + \frac{1}{2}t$. This provides a homotopy

$$1 \simeq \alpha : H_n \to H_n$$

by taking a homeomorphism g on A_n into g on $h_t A_n \cong A_n$, extended by the identity over $\frac{1}{2} \leq |z| \leq \frac{1}{2} + \frac{1}{2}t$. Then α sends g to a homeomorphism which is the identity on $\frac{1}{2} \leq |z| \leq 1$, and this is unaltered by $r \circ i$. So

$$r \circ i \circ 1 \simeq r \circ i \circ \alpha = \alpha \simeq 1.$$

This proves the first part of the lemma.

Define a map f_1 from H^*_{n-1} to itself, so that $f_1(h)$ agrees with h outside $|z| = 1$, maps the disc $|z| \leq \frac{1}{2}$ by radial contraction to $|z| \leq \varepsilon(h)$, and fills the annulus in between similarly to the map $r(h)$. There is then an obvious homotopy $f_1 \simeq i \circ r$.

Now define a family f_t, $0 < t \leq 1$, of such maps, which have $f_t(h) = h$ outside $|z| = t$, and send $|z| \leq \frac{1}{2}t$ to $|z| \leq \varepsilon(h, t)$, filling the annulus between as above. (See Figure 4.) By the continuity property of the maps of annuli, this gives a continuous map $(0, 1] \times H^*_{n-1} \to H^*_{n-1}$. Taking $f_0 : H^*_{n-1} \to H^*_{n-1}$ to be the identity gives a map

$$f : I \times H^*_{n-1} \to H^*_{n-1}.$$

It remains to show that this is continuous at $(0, h)$.

The map $f_t(h)$

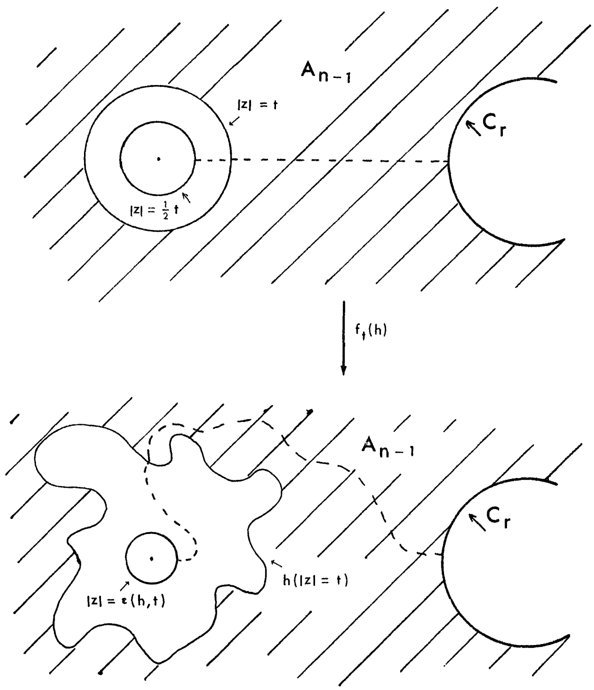

FIGURE 4

Given $\varepsilon > 0$, choose δ such that diam $h(|z| \leq \delta) < \varepsilon/4$. Choose a neighbourhood of h, radius $\varepsilon/4$, then diam $h'(|z| \leq \delta) < 3\varepsilon/4$ for h' in this neighbourhood. For x in A_{n-1} outside $|z| = \delta$,

$$\| f(0, h)(x) - f(t, h')(x) \| = \| h(x) - h'(x) \| < \varepsilon/4.$$

For x inside $|z| = \delta$,

$$\begin{aligned} \| f(0, h)(x) - f(t, h')(x) \| &\leq \| f(0, h)(x) - 0 \| + \| 0 - f(t, h')(x) \| \\ &< \varepsilon/4 + 3\varepsilon/4. \end{aligned}$$

So $\| f(0, h) - f(t, h') \| < \varepsilon$. Hence the map f is continuous and so the lemma is proved.

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