CO-EQUALIZERS AND FUNCTORS

BY
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0. Introduction

If X and Y are objects of a category $\mathbf{C}$, let $| X, Y |$ denote their associated morphism set. Similarly if S and T are functors let $| S, T |$ denote the class (not necessarily a set) of natural transformations from S to T. Unless otherwise stated all functors will be assumed to be covariant. Let $R : \mathbf{V} \to \mathbf{W}$ be a functor. Then $X \in \mathbf{V}$ is a (*left*) *R-object* if for every $Y \in \mathbf{V}$ the mapping function

$$R : | X, Y | \to | RX, RY |$$

is a bijection.[1] We shall find in various circumstances certain conditions some necessary others sufficient for X to be an R-object. It is clear that such information could be of interest, however our objective is to consider the case $\mathbf{V} = \mathbf{V}(\mathbf{C}, \mathbf{D})$ a subcategory of the functor category $(\mathbf{C}, \mathbf{D})$ and $\mathbf{W} = \mathbf{W}(\mathbf{A}, \mathbf{D})$ a subcategory of $(\mathbf{A}, \mathbf{D})$ in which $R : \mathbf{V} \to \mathbf{W}$ is induced by a functor $J : \mathbf{A} \to \mathbf{C}$. Then to say that $S \in \mathbf{V}$ is an R-object means that for every $T \in \mathbf{V}$ and every $u' \in | SJ, TJ |$ there exists a unique $u \in | S, T |$ such that $uJ = u'$. The situation described arises frequently in connection with "uniqueness theorems". Thus to cite one celebrated example, if $\mathbf{V}$ is the category of homology theories on the category $\mathbf{C}$ of triangulable pairs and pair maps and if J is the functor which injects the subcategory "generated by" a single point then Eilenberg and Steenrod proved [3] that each homology theory S is an R-object in $\mathbf{V}$.

In this paper we shall be chiefly concerned with the case $\mathbf{A} = \mathbf{X}$, the subcategory of $\mathbf{C}$ consisting of a single object X and its $\mathbf{C}$-endomorphisms, J being the injection functor and we shall describe an R-object $S \in \mathbf{V}$ as an *X-functor in* $\mathbf{V}$. It follows that the X-functors are determined (up to natural equivalence in $\mathbf{V}$) by their action on $\mathbf{X}$.

In general our basic assumption is that there exists a functor $L : \mathbf{W} \to \mathbf{V}$ and a natural transformation $\alpha : LR \to 1$. L is sometimes (but not always) a left adjoint of R and then we find:

Theorem 0.1. *If L is a left adjoint of R then X is an R-object if and only if $\alpha X \in | LRX, X |$ is an isomorphism.*

One case in which 0.1 is involved is the following. Let $\mathbf{M} = \mathbf{M}_\Lambda$ denote the

Received May 11, 1966.

[1] X is an R-object if and only if (X, i_{RX}) is free over RX with respect to R in the sense of A. Frei, *Freie Objekte und multiplikative structuren*, Math. Zeitschrift, vol. 93 (1966), pp. 109–141. There is some overlap in Section 1 with Frei's results. In particular Theorem 0.1 as stated is essentially not new. (See however Remark 1.1.) Our applications are quite different.

category of modules over a commutative ring Λ with unit. Let $\mathbf{V} = \mathbf{V}(\mathbf{M}, \mathbf{M})$ denote the subcategory of Λ-linear functors and for a given $X \in \mathbf{M}$ let the objects of $\mathbf{W}$ be the Λ-linear functors from $\mathbf{X}$ to $\mathbf{M}_\Lambda$. If $G \in \mathbf{W}$, set

$$LG = \mathrm{Hom}_\Lambda(X, -) \otimes_\Lambda GX.$$

Then

$$\alpha SY \in | \mathrm{Hom}_\Lambda(X, Y) \otimes_\Lambda SX, SY |$$

may be defined by lifting the evaluation of the mapping function of S. It turns out that L is a left adjoint of R and we shall prove

THEOREM 0.2. *S is an X-functor in $\mathbf{V}$ if and only if S is naturally equivalent to $\mathrm{Hom}_\Lambda(X, -) \otimes_\Lambda N$ for some Λ-module N.*

0.2. does not destroy the interest in X-functors: one would still wish to find a suitable X for a given S. For example we shall prove that $\mathrm{Ext}^n(C, -)$ is a K_n-functor if

$$0 \to K_n \to P_1 \to P_2 \to \cdots \to P_n \to C \to 0$$

is an exact sequence such that P_i is projective $(1 \leq i \leq n)$.

A result similar to 0.2 is available in the category $\mathbf{T}$ of topological spaces and maps, but in the category $\mathbf{Tb}$ of based spaces and based maps the analogue of L is not a left adjoint of R. What *is* to hand is a natural transformation $e : RL \to 1$ such that

$$eR = R\alpha \in | RLR, R | \tag{0.3}$$

and we still have a commutative diagram

$$\begin{array}{ccc} LRLR & \xrightarrow{LR\alpha} & LR \\ \downarrow{\scriptstyle \alpha LR} & & \downarrow{\scriptstyle \alpha} \\ LR & \xrightarrow{\alpha} & 1. \end{array}$$

It now becomes important to consider co-equalizers of $LR\alpha X$ and αLRX. We shall prove (in general)

THEOREM 0.4. *If αX is a co-equalizer of $LR\alpha X$ and αLRX and if $R\alpha X$ is epic then X is an R-object. If X is an R-object, if $R\alpha X$ is a co-equalizer of $RLR\alpha X$ and $R\alpha LRX$, and if αLRX or $LR\alpha X$ is epic then αX is a co-equalizer of $LR\alpha X$ and αLRX.*

Section 2 introduces the concept of a valuable functor for categories with a suitably enriched structure and Theorems 0.1 and 0.4 are applied. In a final section we show that the based topological product, smash, join, wedge, suspension and cone functors are all P-functors, where P is a 0-sphere (or an n-tuple of 0-spheres). I hope to consider in a subsequent paper the homotopy theory of P-functors. I am grateful to the referee for making a number of

helpful suggestions and wish also to acknowledge several interesting conversations with Kenneth Hughes.

1. R[-objects

In this section will be proved Theorems 0.1 and 0.4. For details concerning co-equalizers the reader is referred to [5] and [2]. Recall L is a left adjoint of R if there exist $\alpha \in | LR, 1 |$, $\beta \in | 1, RL |$ such that the compositions

$$L \xrightarrow{L\beta} LRL \xrightarrow{\alpha L} L, \qquad R \xrightarrow{\beta R} RLR \xrightarrow{R\alpha} R$$

are the identies i_L and i_R respectively.

Proof of 0.1. Suppose that X is an R-object. Then there exists a unique $v \in | X, LRX |$ such that $Rv = \beta RX$. Then $R(\alpha X \,.\, v) = R\alpha X \,.\, \beta RX = i_{RX} = R(i_X)$ which implies that $\alpha X \,.\, v = i_X$ and we have

$$v \,.\, \alpha X = \alpha LRX \,.\, LRv = \alpha LRX \,.\, L\beta RX = i_{LRX}\,,$$

as required. Conversely, suppose that αX is an isomorphism and let $u \in | RX, RY |$. Then

$$v = \alpha Y \,.\, Lu \,.\, \alpha X^{-1} \in | X, Y |$$

is such that $Rv = R\alpha Y \,.\, RLu \,.\, R\alpha X^{-1} = u \,.\, R\alpha X \,.\, R\alpha X^{-1} = u$. Moreover for any $w \in | X, Y |$ such that $Rw = u$, we have $w \,.\, \alpha X = \alpha Y \,.\, LRw = \alpha Y \,.\, Lu$ so that $w = v$.

Remark 1.1. We have not used the full force of the equality $\alpha L \,.\, L\beta = i_L$. It would have sufficed to assume the existence of $\gamma \in | R, RLR |$ such that $R\alpha \,.\, \gamma = i_R$ and $\alpha LR \,.\, L\gamma = i_{LR}$.

Proof of 0.4. Suppose that $R\alpha X$ is epic and that αX is a co-equalizer of $LR\alpha X$ and αLRX and let $u \in | RX, RY |$. Then we have a doubly-commutative diagram

$$\begin{tikzcd}
LRLRX \arrow[r, "LRLu"] \arrow[d, "\substack{\alpha LRX \\ LeRX \,=\, LR\alpha X}"] & LRLRY \arrow[d, "\substack{\alpha LRY \\ LeRY \,=\, LR\alpha Y}"] \\
LRX \arrow[r, "Lu"] & LRY.
\end{tikzcd}$$

That is to say we have

$$LR\alpha Y \,.\, LRLu = Lu \,.\, LR\alpha X \quad \text{and} \quad \alpha LRY \,.\, LRLu = Lu \,.\, \alpha LRX.$$

Since $\alpha Y \,.\, \alpha LRY = \alpha Y \,.\, LR\alpha Y$ we find easily that $\alpha Y \,.\, Lu \,.\, \alpha LRX = \alpha Y \,.\, Lu \,.\, LR\alpha X$. Hence there exists a unique $w \in | X, Y |$ such that $w \,.\, \alpha X = \alpha Y \,.\, Lu$. Then we have

$$Rw \,.\, R\alpha X = R\alpha Y \,.\, RLu = u \,.\, R\alpha X$$

which implies $Rw = u$. Moreover if $Rv = u$ then

$$v \, . \, \alpha X = \alpha Y \, . \, LRv = \alpha Y \, . \, Lu$$

so that $v = w$. Conversely let X be an R-object and let $w \in | LRX, Y |$ be such that $w \, . \, \alpha LRX = w \, . \, LR\alpha X$. Then $Rw \, . \, R\alpha LRX = Rw \, . \, RLR\alpha X$ and if $R\alpha X$ is a co-equalizer of RLRαX and $R\alpha LRX$ there exists a unique $u \in | RX, RY |$ such that $u \, . \, R\alpha X = Rw$. Let $v \in | X, Y |$ be the unique morphism such that $Rv = u$. Then

$$\begin{aligned} w \, . \, \alpha LRX = \alpha Y \, . \, LRw = \alpha Y \, . \, LRv \, . \, LR\alpha X &= \alpha Y \, . \, LR\alpha Y \, . \, LRLRv \\ &= \alpha Y \, . \, \alpha LRY \, . \, LRLRv = \alpha Y \, . \, LRv \, . \, \alpha LRX. \end{aligned}$$

If αLRX is epic it follows that $w = \alpha Y \, . \, Lu$ and a similar calculation yields the same result if $LR\alpha X$ is epic. Moreover if $u' \, . \, \alpha X = w$ then $Ru' \, . \, R\alpha X = Rw = Ru \, . \, R\alpha X$. Hence $Ru' = Ru$ which implies $u' = u$, completing the proof.

2. Valuable functors

Let **E** denote the category of sets and functions. We recall that a *concrete* category, in the sense of Kelly [4], is a category **D** and a faithful functor from **D** to **E** denoted $X \to | X |, f \to | f |$. "Faithful" means that $| f | = | g |$ implies that $f = g$. If **D** is concrete then $f \in | X, Y |$ is an *identification* if $| f | : | X | \to | Y |$ is onto and if, given any function $k : | Y | \to | Z |$ and any $h \in | X, Z |$ such that $| h | = k \, . \, | f |$, there exists $g \in | Y, Z |$ such that $| g | = k$ (and $g \, . \, f = h$). Note that for every object X of **D** the identity morphism i_X is an identification.

A *concrete product* $(\otimes, r)$ in a concrete category **D** is a bifunctor

$$(X, Y) \to X \otimes Y, \quad (f, g) \to f \otimes g$$

and a natural transformation

$$r : | X | \times | Y | \to | X \otimes Y |$$

satisfying the condition $| h | \, . \, r = | k | \, . \, r$ implies $h = k$. We also require that $(\otimes, r)$ should admit natural associativity and commutativity isomorphisms γ and τ compatible with the associativity and commutativity bijections c and t in **E**. That is to say the following diagrams are commutative:

(2.1)
$$\begin{array}{ccc} |X| \times (|Y| \times |Z|) & \xrightarrow{c} & (|X| \times |Y|) \times |Z| \\ \downarrow{\scriptstyle |i_X| \times r} & & \downarrow{\scriptstyle r \times |i_Z|} \\ |X| \times |Y \otimes Z| & & |X \otimes Y| \times |Z| \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ |X \otimes (Y \otimes Z)| & \xrightarrow{|\gamma|} & |(X \otimes Y) \otimes Z| \end{array}$$

$$\begin{array}{ccc} |X| \times |Y| & \xrightarrow{t} & |Y| \times |X| \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ |X \otimes Y| & \xrightarrow{|\tau|} & |Y \otimes X|. \end{array}$$

$(\otimes, r)$ *admits sections* if for all $X, Y \in \mathbf{D}$ and all $x \in |X|$ there exists $\theta_x \in |Y, X \otimes Y|$ such that $|\theta_x|(y) = r(x, y)$ $(y \in |Y|)$.

Let $\mathbf{C}$ be a $\mathbf{D}$-category in the sense of Kelly [4, p. 21]. We recall that this means that there is a functor

$$(-, -) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$$

with the property that $|(X, Y)| = |X, Y|$ for all $X, Y \in \mathbf{C}$. If $S : \mathbf{C} \to \mathbf{D}$ is a functor let

$$E_S XY : |X, Y| \times |SX| \to |SY|$$

be the function such that

$$E_S XY(f, x) = |Sf|(x) \qquad (x \in |SX|, f \in |X, Y|). \tag{2.2}$$

Let $\mathbf{A}$ be a sub-category of $\mathbf{C}$. S is $\mathbf{A}$-*valuable* if for every $X \in \mathbf{A}$ and every $Y \in \mathbf{C}$ there exists a (necessarily unique) morphism

$$e_S XY \in |(X, Y) \otimes SX, SY|$$

such that

$$|e_S XY| \,.\, r = E_S XY. \tag{2.3}$$

S is $\mathbf{X}$-*constructive* if $e_S XX$ is an identification. We denote by $\mathbf{V}$ the full full sub-category of $(\mathbf{C}, \mathbf{D})$ whose objects are $\mathbf{A}$-valuable functors.

For the remainder of this section let X be a fixed object of $\mathbf{A}$ and let Ω denote the functor $(X, -) : \mathbf{C} \to \mathbf{D}$. Notice that

$$E_\Omega ZY : |Z, Y| \times |X, Z| \to |X, Y|$$

is simply the composition function. It will be assumed that Ω is $\mathbf{A}$-valuable. Now let $\mathbf{W}$ be the category of *valuable X-germs*: that is to say the full sub-category of $(\mathbf{X}, \mathbf{D})$ whose objects are $\mathbf{X}$-valuable functors. Let $R : \mathbf{V} \to \mathbf{W}$ be defined by restriction. If $G \in \mathbf{W}$, let

$$LG = (X, -) \otimes GX : \mathbf{C} \to \mathbf{D}$$

and set $\alpha SY = e_S XY$. We have

THEOREM 2.4. *If r is surjective or if $(\otimes, r)$ admits sections then L is a functor from $\mathbf{W}$ to $\mathbf{V}$ and $\alpha : LR \to 1$ is a natural transformation. Moreover if indeed $(\otimes, r)$ admits sections, L is a left adjoint of R.*

Proof. To see that $LG \in \mathbf{V}$, set

$$e_{LG} ZY = (e_\Omega ZY \otimes i_{GX}) \,.\, \gamma.$$

Then if $g \in |Z, Y|, f \in |X, Z|$ and $x \in |GX|$ we have

$$|e_{LG} ZY| \,.\, r(g, r(f, x)) = (|e_\Omega ZY| \,.\, r(g, f), x) = r(g \,.\, f, x),$$

while

$$E_{LG} ZY(g, r(f, x)) = |\Omega g \otimes i_{GX}| \,.\, r(f, x) = r(g \,.\, f, x).$$

Hence 2.3 is satisfied if r is surjective. On the other hand if $(\otimes, r)$ admits sections then the calculation shows that $| e_{LG} ZY \,.\, \theta_g | \,.\, r = | LGg | \,.\, r$ which implies that $e_{LG} ZY \,.\, \theta_g = LGg$. Hence for all $x' \in | (X, Z) \otimes GX |$ we have

$$E_{LG} ZY(g, x') = | LGg |(x') = | e_{LG} ZY \,.\, \theta_g |(x') = | e_{LG} ZY | \,.\, r(g, x'),$$

verifying 2.3. Given $u \in \mathbf{W}$, $u : G \to H$ we understand that $(Lu)Y = i_{(X,Y)} \otimes uX$ and the functorial relations for L clearly hold. Now we have $LRSY = (X, Y) \otimes SX$. Thus we must show that for every $S, T \in \mathbf{V}$, $u : S \to T$ and $y \in | Y, Z |$ the following diagrams are commutative:

$$\begin{array}{ccc} (X, Y) \otimes SX & \xrightarrow{e_S XY} & SY \\ \downarrow{\scriptstyle \Omega g \otimes i_{SX}} & & \downarrow{\scriptstyle Sg} \\ (X, Z) \otimes SX & \xrightarrow{e_S XZ} & SZ \end{array} \qquad \begin{array}{ccc} (X, Y) \otimes SX & \xrightarrow{e_S XY} & SY \\ \downarrow{\scriptstyle i_{(X,Y)} \otimes uX} & & \downarrow{\scriptstyle uY} \\ (X, Y) \otimes TX & \xrightarrow{e_T XY} & TY. \end{array}$$

It is sufficient to prove that

$$| e_S XZ | \,.\, | \Omega g \otimes i_{SX} | \,.\, r = | Sg | \,.\, | e_S XY | \,.\, r$$

and that

$$| uY | \,.\, | e_S XY | \,.\, r = | e_T XY | \,.\, | i_{(X,Y)} \otimes uX | \,.\, r,$$

however the first equality simply expresses the functorial property of S and the second the naturality of u.

Now suppose that $(\otimes, r)$ admits sections and let

$$\beta G = \beta GX \in | GX, (X, X) \otimes GX |$$

be the section such that $| \beta GX |(x) = r(i_X, x)$ $(x \in | GX |, G \in \mathbf{W})$. Then if $g \in | X, Y |$, $x \in | GX |$ we have

$$| \alpha LGY | \,.\, | L\beta GY | \,.\, r(g, x) = | e_{LG} XY | \,.\, r(g, r(i_X, x)) = r(g, x)$$

which implies that $\alpha L \,.\, L\beta = i_L$. Finally if $x \in | SX |$ we have

$$| R\alpha S | \,.\, | \beta RS |(x) = | \alpha SX | \,.\, r(i_X, x) = | Si_X |(x) = x$$

which implies $R\alpha \,.\, \beta R = i_R$, completing the proof.

Combining 2.4 and 0.1 we have

THEOREM 2.5. *If $(\otimes, r)$ admits sections then S is an X-functor in $\mathbf{V}$ if and only if $\alpha S \in | LRS, S |$ is a natural equivalence.*

If $G \in \mathbf{W}$, let $eG = e_{LG} XX \in | RLGX, GX |$. G is a *constructive* X-germ if eG is an identification.

LEMMA 2.6. *$e : RL \to 1$ is a natural transformation,*

$$eR = R\alpha : RLR \to R,$$

eGX and $LeGX$ are epic. If $(\otimes, r)$ admits sections, or if r is surjective and G is constructive, then eG is a co-equalizer of $RLeG$ and $eRLG$.

Proof. The naturality of e follows by a special case of an argument already given and clearly $eR = R\alpha$. Suppose that $u, v \in | GX, W |$ are such that $u \,.\, eG = v \,.\, eG$. Then if $x \in | GX |$,

$$| u |(x) = | u | \,.\, | eG | \,.\, r(i_X, x) = | v | \,.\, | eG | \,.\, r(i_X, x) = | v |(x),$$

so that $u = v$. Thus eGX is epic. Now suppose that

$$u, v \in | (X, X) \otimes GX, W |$$

are such that $u \,.\, LeGX = v \,.\, LeGX$. Then if $g \in | X, X |$ and $x \in | GX |$ we have

$$| u | \,.\, r(g, x) = | u | \,.\, r(g, | eGX | \,.\, r(i_X, x)) = | u | \,.\, | LeGX | \,.\, r(g, r(i_X, x))$$
$$= | v | \,.\, | LeGX | \,.\, r(g, r(i_X, x)) = | v | \,.\, r(g, x),$$

which implies that $u = v$ and hence that $LeGX$ is epic. Let

$$w \in | (X, X) \otimes GX, W |$$

be such that $w \,.\, RLeG = w \,.\, eRLG$. If $(\otimes, r)$ admits sections then $w \,.\, \beta GX$ is the necessarily unique morphism k such that $k \,.\, eG = w$. Alternatively if r is surjective and G is constructive, let

$$k' : | GX | \to | W |$$

be such that $k'(x) = | w | \,.\, r(i_X, x)$ $(x \in | GX |)$. Then by a calculation similar to one already performed we find that $k' \,.\, | eG | \,.\, r = | w | \,.\, r$ and hence $k' \,.\, | eG | = | w |$. Since eG is an identification there exists k with $| k | = k'$ and having the desired property.

Combining 2.6 and 0.4 we obtain

THEOREM 2.7. *If $S \in \mathbf{V}$ and αS is a co-equalizer of $LR\alpha S$ and αLRS then S is an X-functor in $\mathbf{V}$. If S is an X-functor in $\mathbf{V}$, if αSX is an identification and if r is surjective then αS is a co-equalizer of $LR\alpha S$ and αLRS.*

As an application of 2.7 we have

THEOREM 2.8. *If (r, θ) admits sections then Ω is an X-functor in $\mathbf{V}$.*

For it suffices to show that $\alpha\Omega Y = e_\Omega XY$ is a co-equalizer of $LR\alpha\Omega Y$ and $\alpha LR\Omega Y$ $(Y \in \mathbf{C})$. Accordingly, suppose that

$$w \in | (X, Y) \otimes (X, X), Z |$$

is such that $w \,.\, LR\alpha\Omega Y = w \,.\, \alpha LR\Omega Y$ and let

$$\theta \in | (X, Y), (X, Y) \otimes (X, X) |$$

be the section such that $| \theta |(g) = r(g, i_X)$. Then if $g \in | X, Y |, f \in | X, X |$ we have

$$| w \,.\, \theta \,.\, e_\Omega XY | \,.\, r(g, f) = | w \,.\, \theta |(g \,.\, \mathrm{f})$$

$$= |w| \cdot r(g \cdot f, i_X) = |w| \cdot |\alpha LR\Omega Y| \cdot r(g, r(f, i_X))$$
$$= |w| \cdot |LR\alpha\Omega Y| \cdot r(g, r(f, i_X)) = |w| \cdot r(g, f)$$

which implies that $w \cdot \theta \cdot e_\Omega XY = w$. On the other hand if $k \in |(X, Y), Z|$ is such that $k \cdot e_\Omega XY = w$ we have $|k|(g) = |k \cdot e_\Omega XY| \cdot r(g, i_X) = |w| \cdot r(g, i_X) = |w \cdot \theta|(g)$, so that $k = w \cdot \theta$, which completes the proof.

If $G \in \mathbf{W}$, one may well ask: under what circumstances will there exist an X-functor S such that $RS = G$? Suppose that $\mathbf{D}$ is right-complete, and let $wY \in |LY, TY|$ be a co-equalizer of $LeGY, \alpha LGY \in |LRLGY, LGY|$ for each $Y \in \mathbf{C}$. It is easy to see that a mapping function can be chosen in a unique way to make T a functor from $\mathbf{C}$ to $\mathbf{D}$ and w a natural transformation. If $\otimes$ preserves co-equalizers (and $\mathbf{D}$ is right-complete) a standard argument shows that $\mathbf{V}$ is right-complete, so that we have $T \in \mathbf{V}$. Suppose that $(\otimes, r)$ admits sections, or that r is surjective and G is constructive. We have

THEOREM 2.9. *If $\mathbf{A} = \mathbf{X}$ or if $\otimes$ preserves co-equalizers then T is an X-functor in $\mathbf{V}$ and there exists a natural equivalence $v : RT \to G$.*

Proof. By 2.6, both eGX and wX are co-equalizers of $RLeGX = LeGX$ and $eRLGX = \alpha LGX$. Hence there is an equivalence $v \in |TX, GX|$ such that $eGX = v \cdot wX$. T is certainly $\mathbf{X}$-valuable for we may set

$$e_T XY = wY \cdot (i_{(X,Y)} \otimes v).$$

Then if $x \in |(X, X) \otimes GX|$, $g \in |X, Y|$, we have

$$|e_T XY| \cdot r(g, |wX|(x))$$
$$= |wY| \cdot r(g, |v \cdot wX|(x))$$
$$= |wY| \cdot r(g, |eGX|(x)) = |wY| \cdot |LeGY| \cdot r(g, x)$$
$$= |wY| \cdot |\alpha LGY| \cdot r(g, x) = |wY| \cdot |LGg|(x) = |Tg| \cdot |wX|(x).$$

Since $|wX| = |v^{-1} \cdot eGX|$ is surjective, 2.3 is satisfied with $S = T$ and $Z = X$. It follows easily that v is a natural equivalence $RT \to G$, that $e_T XY = wY \cdot Lv$ and hence in view of the doubly-commutative diagram

$$\begin{array}{ccc}
LRLRT & \xrightarrow{LRLv} & LRLG \\
\Big\downarrow {\scriptstyle LR\alpha T \,=\, LeRT \atop \scriptstyle \alpha LRT} & & \Big\downarrow {\scriptstyle Leg \atop \scriptstyle \alpha LG} \\
LRT & \xrightarrow{Lv} & LG
\end{array}$$

that αT is a co-equalizer of $LR\alpha T$ and αLRT. 2.7 implies that T is an X-functor.

Suppose that $\mathbf{A} = \mathbf{X}$ or that $\otimes$ preserves co-equalizers. Combining 2.7 and 2.9 yields the following corollary.

COROLLARY 2.10. *If $(\otimes, r)$ admits sections there is a one-to-one correspond-*

ence between the family of natural equivalence classes of valuable **X**-*germs and the family of natural equivalence classes of X-functors in* **V**. *If r is surjective then there is a one-to-one correspondence between the family of natural equivalence classes of constructive* **X**-*germs and the family of natural equivalence classes of* **X**-*constructive X-functors in* **V**.

3. Λ-modules

In the category of modules $\mathbf{M} = \mathbf{M}_\Lambda$, the tensor product $(\otimes_\Lambda, r)$ is concrete: r denotes the function $(a, b) \to a \otimes b$. A functor $S : \mathbf{M} \to \mathbf{M}$ is Λ-*linear* if for all $A, B \in \mathbf{M}$, all $f_1, f_2 \in |A, B| = |\mathrm{Hom}_\Lambda(A, B)|$ and all $\lambda_1, \lambda_2 \in \Lambda$ we have

$$S(\lambda_1 . f_1 + \lambda_2 . f_2) = \lambda_1 . Sf_1 + \lambda_2 . Sf_2 \in |SA, SB|.$$

It follows easily that S is $\mathbf{M}$-valuable if and only if S is Λ-linear. Let $\mathbf{V}$ be the full subcategory of $\mathbf{M}$-valuable functors and let $X \in \mathbf{M}$. Then the bilinearity of composition implies that $\Omega = (X, -) = \mathrm{Hom}_\Lambda(X, -)$ belongs to $\mathbf{V}$. Since $(\otimes, r)$ admits sections, 2.5 states that S is an X-functor in $\mathbf{V}$ if and only if $\alpha S : (X, -) \otimes SX \to S$ is a natural equivalence. One of the assertions of Theorem 0.2 is thus proved. 2.8 implies that Ω is an X-functor in $\mathbf{V}$. If we now let $T = (X, -) \otimes N$ it is easily verified that αTY is equivalent to $\alpha\Omega Y \otimes i_N$ and that $LR\alpha TY$ and $\alpha LRTY$ are equivalent to $LR\alpha\Omega Y \otimes i_N$ and $\alpha LR\Omega Y \otimes i_N$ respectively. It follows that αTY is a coequalizer of $LR\alpha TY$ and $\alpha LRTY$ and hence that T is an X-functor, which completes the proof of 0.2.

Let $C \in \mathbf{M}$, $n \geq 1$ and let

$$0 \to K_n \to P_n \to \cdots \to P_2 \to P_1 \to C \to 0$$

be an exact sequence in which P_i is projective $(1 \leq i \leq n)$. We have

THEOREM 3.1. $\mathrm{Ext}^n(C, -)$ *is a K_n-functor in* **V**.

Proof. In view of 0.2 we need only establish a natural isomorphism

$$\mathrm{Ext}^n(C, Y) \approx \mathrm{Hom}_\Lambda(K_n, Y) \otimes_\Lambda \mathrm{Ext}^n(C, K_n). \tag{3.2}$$

There is certainly a natural isomorphism $\mathrm{Ext}^n(C, Y) \approx \mathrm{Ext}^1(K_{n-1}, Y)$ [6, p. 102] and so we need only consider the case $n = 1$. Accordingly let $0 \to K \to P \to C \to 0$ be exact with P projective and let S denote $\mathrm{Ext}^1(C, -)$. Then $LRSY = (K, Y) \otimes SK$ and we find easily that $\alpha SY(a \otimes E) = aE$ $(a \in (K, Y), E \in SK)$, where aE denotes the composite extension obtained by completing the diagram

$$\begin{tikzcd}
E: & 0 \arrow[r] & K \arrow[r] \arrow[d, "a"] & M \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\
aE: & 0 \arrow[r] & Y \arrow[r, dotted] & ? \arrow[r, dotted] & C \arrow[r] & 0.
\end{tikzcd}$$

(For details see [6, p. 66].) In view of 2.7 it will be sufficient to verify that SY is the difference cokernel of $LR\alpha SY$ and $\alpha LRSY$. Now αSY is certainly an epimorphism, for P is projective and hence the following diagram can always be completed:

$$\begin{tikzcd} F\colon & 0 \arrow[r] & K \arrow[r, "X"] \arrow[d, dotted] & P \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & Y \arrow[r] & N \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

Thus we have only to prove that if $\sum (1 \leq i \leq n) a_i E_i = 0$, $a_i \in (K, Y)$ then

$$\sum (1 \leq i \leq n) a_i \otimes E_i = (\alpha LRSY - LR\alpha SY)(t),$$

where $t \in (K, Y) \otimes ((K, K) \otimes SK)$. Let $b_i \in (K, K)$ be obtained by completing the following diagram $(1 \leq i \leq n)$:

$$\begin{tikzcd} F\colon & 0 \arrow[r] & K \arrow[r, "X"] \arrow[d, dotted, "b_i"] & P \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\ E_i\colon & 0 \arrow[r] & K \arrow[r] & M_i \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

Then $\sum a_i E_i = 0$ implies that there exists $h \in (P, Y)$ such that $h\mathrm{X} = \sum a_i \,.\, b_i$. Let $\oplus K$ denote the direct sum of n copies of K and $\pi_i \in (\oplus K, K)$ the projection onto the i^{th} summand $(1 \leq i \leq n)$. P being projective there exists $g \in (P, \oplus K)$ such that $(\sum a_i \,.\, \pi_i) \,.g = h$, for without loss of generality we may assume that $\sum a_i \,.\, \pi_i \in (\oplus K, Y)$ is an epimorphism. If we now let

$$t = \sum \{a_i \otimes ((b_i - \pi_i \,.\, g \,.\, \mathrm{X}) \otimes F\},$$

the desired equality is easily verified.

We conclude this section with the observation that multifunctors in $\mathbf{M}$ of whatever variance may be studied. Thus for example a functor contravariant in one argument and covariant in another may be regarded as a covariant functor from $\mathbf{M}^{\mathrm{op}} \times \mathbf{M}$ to $\mathbf{M}$ and if we set

$$((X_1, X_2), (Y_1, Y_2)) = \mathrm{Hom}_\Lambda (Y_1, X_1) \oplus \mathrm{Hom}_\Lambda (X_2, Y_2),$$

we convert $\mathbf{M}^{\mathrm{op}} \times \mathbf{M}$ into an $\mathbf{M}$-category.

4. Spaces

The categories $\mathbf{T}$ and $\mathbf{Tb}$ are certainly concrete. We have in fact a commutative diagram of forgetful functors:

$$\begin{tikzcd} \mathbf{Tb} \arrow[rr, "F"] \arrow[dr] & & \mathbf{T} \arrow[dl] \\ & \mathbf{E} & \end{tikzcd}$$

[1, 1.2] shows that a morphism of **T** is an identification if and only if it is an identification map in the usual sense and the following lemma is easily proved.

LEMMA 4.1. *$f \in$ **Tb** is an identification if and only if Ff is an identification in* **T**.

In **T** or **Tb** let (X, Y) denote $|X, Y|$ enriched with the compact-open topology. Specifically, when appropriate, the base point of (X, Y) is the constant map at the base point of Y. Many different concrete products offer themselves, however let $(\times, r)$ denote the topological product in **T** and the based topological product in **Tb**, r being the identity function. Then $(\times, r)$ admits sections in **T** but *not* in **Tb**. In **Tb** the smash product is concrete and it admits sections but it imposes severe restrictions on the valuable functors.

Let **A** denote the sub-category of **T** or of **Tb** whose spaces are locally-compact and Hausdorff and let $X \in$ **T** or **Tb**. We have

LEMMA 4.2. *$(X, -)$ is* **A***-valuable relative to $(\times, r)$.*

Proof. The composition function $\theta : (Z, Y) \times (X, Z) \to (X, Y)$ respects base points, thus we have only to prove that θ is continuous provided that Z is locally compact and Hausdorff. Given U (open) $\subseteq Y$ and C (compact) $\subseteq X$ it will be sufficient to show that $\theta^{-1}[C, U]$ is open, where

$$[C, U] = \{f \in |X, Y| \mid f(C) \subseteq U\}.$$

Suppose $(g, h) \in \theta^{-1}[C, U]$ so that $g \, . \, h(C) \subseteq U$. If $z \in h(C)$ let $V(z)$ be an open set such that $z \in V(z)$ and $\mathrm{Cl}\,(V(z))$ is compact (Cl = closure). Since Z is regular and $g(z) \in U$, we may also require that $\mathrm{Cl}\,(V(z)) \subseteq g^{-1}(U)$. But $h(C)$ is compact, hence there exist $z_i \in h(C)$ $(1 \leq i \leq n)$ such that $h(C) \subseteq \bigcup (1 \leq i \leq n) V(z_i) = V(\text{say})$. Then

$$\mathrm{Cl}\,(V) = \bigcup\,(1 \leq i \leq n)\ \mathrm{Cl}\,(V(z_i))$$

is compact and since

$$(g, h) \in [\mathrm{Cl}\,(V), U] \times [C, V] \subseteq \theta^{-1}[C, U],$$

θ is continuous.

Let $\mathbf{C}^n$ denote the category of n-tuples of members of the category **C** and let $X \in \mathbf{C}^n$ denote $(X_1, X_2, \cdots, X_n)$ where $X_i \in \mathbf{C}$, $1 \leq i \leq n$. Then $\mathbf{T}^n$ becomes a **T**-category if we define (X, Y) to be the topological product $\prod (1 \leq i \leq n)(X_i, Y_i)$. A similar definition provides $\mathbf{Tb}^n$ with the structure of a **Tb**-category. The product of continuous functions being continuous we have

LEMMA 4.3. *The functor $(X, -)$ from $\mathbf{T}^n$ to* **T** *or from $\mathbf{Tb}^n$ to* **Tb** *is $\mathbf{A}^n$-valuable.*

Let $P_i \in$ **Tb** be a discrete space with exactly two points: p_i and the base point $*$ $(1 \leq i \leq n)$, and let **V** be the category of **P**-valuable functors from $\mathbf{Tb}^n$ to **Tb**. The remainder of this paper is devoted to the study of the P-functors in **V**. In view of 2.10 the interest centers on the constructive **P**-germs. We have

LEMMA 4.4. *Every functor* $G : \mathbf{P} \to \mathbf{Tb}$ *is* **P**-*constructive.*

Proof. (P, P) is a discrete semi-group with identity element $*$ generated by the n commuting idempotents ϕ_i, where

$$\begin{aligned}(\phi_i(x))_j &= x_j \quad (j \neq i) \\ &= * \quad\; (j = i)\end{aligned} \qquad (x \in P).$$

$(P, P) \times GP$ consists of 2^n copies of GP each of which is mapped continuously by E_G, one copy being mapped identically. Thus

$$eG \in |\, (P, P) \times GP, GP \,|$$

is well defined and is moreover an identification.

A morphism $f \in \mathbf{Tb}$ is a *projection* if $f \,.\, f = f$. We remark that a functor $G : \mathbf{P} \to \mathbf{Tb}$ can be regarded simply as a pair (W, f) where $W = GP$ and f is an n-tuple of commuting projections $f_i = G\phi_i$ $(1 \leq i \leq n)$. The pairs (W, f) and (W', f') are *equivalent* if there exists an equivalence $h \in |\, W, W' \,|$ such that $h \,.\, f_i = f_i' \,.\, h$ $(1 \leq i \leq n)$. 2.10 and 4.4 now imply

THEOREM 4.5. *There is a one-to-one correspondence between the family of equivalence classes of pairs* (W, f) *and the family of natural equivalence classes of* **P**-*functors in* **V**.

If (W, f) is a pair, let us construct the P-functor T (determined up to natural equivalence) corresponding to (W, f). We first observe that the functor $\Omega = (P, -)$ can be replaced by the based topological product functor $\Pi : \mathbf{Tb}^n \to \mathbf{Tb}$, for there is a natural equivalence $\lambda : (P, -) \to \Pi$ given by the rule

$$\lambda Y(g) = (g_1(p_1), g_2(p_2), \cdots, g_n(p_n)) \qquad (g \in (P, Y)).$$

Then if G is the **P**-germ corresponding to (W, f), LGY is equivalent to $\Pi Y \times W$ and $LRLGY$ to $\Pi Y \times (P, P) \times W$. Moreover $LeGY$ and αLGY are described by the rules

$$LeGY(y, x, w) = (y, eG(x, w))$$

$$\alpha LGY(y, x, w) = (y \,.\, x, w).$$

Now $y \to y \,.\, \phi_i$ is simply the projection $\pi_i : \Pi Y \to \Pi Y$ which replaces the i^{th} coordinate by the basepoint. Since (P, P) is generated by ϕ_i $(1 \leq i \leq n)$, it follows that $vY \in |\, \Pi Y \times W, TY \,|$ will be a co-equalizer of $LeGY$ and αLGY if vY is the quotient map and TY the space obtained from $\Pi Y \times W$

by performing the identification

$$(4.6) \qquad (\pi_i(y), w) = (y, f_i(w)) \qquad (y \in \Pi Y, w \in W, i = 1, 2, \cdots, n).$$

It may now be verified that the based topological product Π, the smash $Y_1 \# Y_2$, the wedge functors (thin, fat or indifferent), various join, suspension and cone functors are all P-functors since they are (or are naturally equivalent to) functors obtained by the above construction. To test whether a given $S : \mathbf{Tb}^n \to \mathbf{Tb}$ is a P-functor one simply chooses $W = SP, f_i = S\phi_i$ $(1 \leq i \leq n)$ and examines whether S is naturally equivalent to the resulting T.

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