ON THE CONCORDANCE OF LOCAL RINGS AND UNIQUE FACTORIZATION

BY

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Introduction

Every ring in this paper is commutative with an identity and a subring must always have the same identity as the containing ring. A ring A which has only one maximal ideal m is a quasi-local ring and is denoted by (A, m); a quasi-local ring (A, m) is a local ring if $\bigcap_{n=1}^{\infty} m^n = (0)$. By the natural topology of a quasi-local ring (A, m), we mean the m-adic topology of A.

The purpose of this paper is to investigate unique factorization in the direct limits of directed systems $(R_p, f_p^q)_{p \in I}$ of unique factorization domains (or UFD's) R_p relative to any directed set I. In particular, we concentrate on the case that all the UFD's are local rings (R_p, m_p) and every homomorphism f_p^q, $q \geq p$, is a local homomorphism; namely $f_p^q(m_p) \subseteq m_q$ for $q \geq p$. Let R be the direct limit of such a directed system of UF local domains R_p, and g_p the natural map of R_p into R for each $p \in I$. We prove that R is a UF local domain, if $g_p(R_p)$ for every $p \in I$ is a local ring having R as a concordant extension; that is, $g_p(R_p)$ is a topological subspace of R for the natural topologies. Applying the result and the fact that every regular (Noetherian) local ring is a UFD, we also prove that if $(E_p, f_p^q)_{p \in I}$ is a directed system of regular local rings (E_p, m_p) such that each homomorphism f_p^q maps a minimal basis of m_p into that of m_q for $q \geq p$, then the direct limit is a UF local domain.

1. Concordant extensions of local rings

Throughout this paper we shall denote by N the set of all positive integers.

DEFINITION. Let R_p and R_q be two quasi-local rings such that $R_p \subseteq R_q$; then R_q is said to be a *concordant extension* of R_p, if R_p is a subspace of R_q for the natural topologies.

From the definition we know that (R_q, m_q) is a concordant extension of (R_p, m_p) if and only if for any two integers h and k in N, we can find r and s in N such that

$$m_p^r \subseteq m_q^h \cap R_p \quad \text{and} \quad m_q^s \cap R_p \subseteq m_p^k .$$

Thus $R_p \subseteq R_q$ and $m_q \cap R_p = m_p$, i.e., R_q dominates R_p, is a necessary condition for R_q to be a concordant extension of R_p (cf. [2]).

For any directed set I let $(R_p)_{p \in I}$ be a family of local rings (R_p, m_p) such that R_q dominates R_p if $q \geq p$. Then the set $(R_p)_{p \in I}$ is a directed system of local rings under set inclusion; the direct limit $R = \lim_{\rightarrow} R_p = U_{p \in I} R_p$ is a

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quasi-local ring having the maximal ideal $M = \lim_{\rightarrow} m_p = U_{p \in I} m_p$, which dominates every R_p.

PROPOSITION. *Let $(R_p)_{p \in I}$ be a directed system of local rings (R_p, m_p) under set inclusion, such that R_q dominates R_p if $q \geq p$. Then $R = \lim_{\rightarrow} R_p$ is a local ring if R is a concordant extension of R_p for every $p \in I$. Conversely if R is a local ring and an R_p is a complete Noetherian local ring for the natural topology, then R is a concordant extension of the local ring R_p.*

Proof. The proof of the first statement is straightforward. The second part is a direct consequence of Chevalley's theorem in [4, p. 270].

Let $(R_p)_{p \in I}$ be given as in the proposition. If $R = \lim_{\rightarrow} R_p$ is a concordant extension of an R_p, then, as we shall see in the next lemma, every R_q for $q \geq p$ is necessarily a concordant extension of the R_p. The converse of this, however, is not true in general. For instance let K be a field, x an indeterminate and consider the following sequence of local rings:

$$K[[x]] \subset K[[x^{1/2}]] \subset K[[x^{1/4}]] \subset \cdots \subset K[[x^{1/2^i}]] \subset \cdots,$$

where $R_i = K[[x^{1/2^i}]]$ for each integer $i \in N_0 = N \cup \{0\}$ is the ring of formal power series in $x^{1/2^i}$ over K. By [2, Theorem 6, p. 65] it is not difficult to see that R_j is a concordant extension of R_i if $j \geq i$. On the other hand

$$R = \lim_{\rightarrow} R_i = \bigcup_{i \in N_0} R_i$$

is a quasi-local ring having the maximal ideal $M = \bigcup_{i \in N_0} m_i$, where $m_i = x^{1/2^i} R_i$. Since $x \in \bigcap_{n \in N} M^n$, R is not a local ring; therefore R can not be a concordant extension of R_i for all $i \in N_0$ by the proposition.

LEMMA. *Let $(R_p)_{p \in I}$ be a directed system of local rings (R_p, m_p) under set inclusion, such that R_q dominates R_p if $q \geq p$. Then $R = \lim_{\rightarrow} R_p$ is a concordant extension of an R_p, if and only if for every integer $t \in N$ there exists an integer $n(t, p) \in N$ such that $m_q^{n(t,p)} \cap R_p \subseteq m_p^t$ for all $q \geq p$.*

The proof is easy, hence we omit it.

Let $(R_p, f_p^q)_{p \in I}$ be a directed system of local rings (R_p, m_p) such that all the homomorphisms f_p^q are local homomorphisms, and denote the natural map of each R_p into the direct limit R by g_p. Then R and each $g_p(R_p)$ are quasi-local rings having, respectively, the maximal ideals $\lim_{\rightarrow} m_p$ and $g_p(m_p)$. Therefore the set $(g_p(R_p))_{p \in I}$ is a directed system of quasi-local rings under set inclusion, such that $g_q(R_q)$ dominates $g_p(R_p)$ whenever $q \geq p$; furthermore $R = \bigcup_{p \in I} g_p(R_p)$. We remark that for any R_p of the directed system of local rings, the quasi-local ring $g_p(R_p)$ is a local ring if and only if the ker $(g_p) = \bigcup_{q \geq p} \ker (f_p^q)$ is a closed ideal of the R_p relative to the natural topology; evidently this is the case when the R_p is a Noetherian local ring or every f_p^q for $q \geq p$ is a monomorphism.

The following theorem is due to the proposition.

THEOREM 1. *Let $(R_p, f_p^q)_{p \in I}$ be a directed system of local rings R_p such that all the homomorphisms f_p^q are local homomorphisms. Then $R = \lim_{\to} R_p$ is a local ring if every $g_p(R_p)$ is a local ring having R as a concordant extension; the converse of this is also true, if all the R_p are complete Noetherian local rings for the natural topologies.*

COROLLARY 1. *Let $(R_p, f_p^q)_{p \in I}$ be a directed system of local rings (R_p, m_p) under local homomorphisms f_p^q, such that every $g_p(R_p)$ is a local ring. Then $R = \lim_{\to} R_p$ is a local ring, if for each $p \in I$ the following condition is satisfied: for every integer $t \in N$ there exists an $n(t, p) \in N$ such that*

$$m_q^{n(t,p)} \cap f_p^q(R_p) \subseteq f_p^q(m_p^t) \qquad \textit{for all } q \geq p.$$

Proof. First we prove that if the condition is satisfied for each $p \in I$ and each $t \in N$, then

$$(g_q(m_q))^{n(t,p)} \cap g_p(R_p) \subseteq (g_p(m_p))^t$$

for all $q \geq p$. Let a^* be any element of $(g_q(m_q))^{n(t,p)} \cap g_p(R_p)$; then $a^* = g_q(a) = g_p(a')$ for some $a \in m_q^{n(t,p)}$ and some $a' \in R_p$, so that there exists an $r \geq q$ such that $f_q^r(a) = f_p^r(a')$. Since

$$m_r^{n(t,p)} \cap f_p^r(R_p) \subseteq f_p^r(m_p^t)$$

by hypothesis, $f_q^r(a) \in f_p^r(m_p^t)$; hence

$$a^* = g_q(a) = g_r f_q^r(a) \in (g_p(m_p))^t$$

as we proposed to prove. Consequently R is a concordant extension of each $g_p(R_p)$ by the lemma, therefore the corollary follows from Theorem 1.

As a consequence of the Corollary 1 to Theorem 1, we have

COROLLARY 2. *Let $(R_p, f_p^q)_{p \in I}$ be a directed system of local rings (R_p, m_p) under local homomorphisms f_p^q, such that every $g_p(R_p)$ is a local ring. Then $R = \lim_{\to} R_p$ is a local ring if any one of the following conditions is fulfilled:*

(i) *for every $t \in N$ and every $p \in I$, we have that*

$$m_q^t \cap f_p^q(R_p) = f_p^q(m_p^t) \qquad \textit{whenever } q \geq p;$$

(ii) *for each $p \in I$, there exists an integer $n(p) \in N$ such that*

$$m_q^{n(p)} \subseteq f_p^q(R_p) \qquad \textit{for all } q \geq p.$$

The following corollary is a part of the statement of Lemma (10.3.1.3) in [3, p. 21].

COROLLARY 3. *Let $(R_p, f_p^q)_{p \in I}$ be a directed system of local rings (R_p, m_p) under local homomorphisms f_p^q. If $m_q = m_p R_q$ and R_q is a flat R_p-module whenever $q \geq p$, then $R = \lim_{\to} R_p$ is a local ring.*

Proof. This follows from [1, Proposition 9, p. 51] and (i) of Corollary 2 to Theorem 1.

2. Unique factorization in the direct limits of directed systems of UF local domains

THEOREM 2. *Let* $(R_p, f_p^q)_{p\in I}$ *be a directed system of UFD's* R_p *such that each homomorphism* f_p^q, $q \geq p$, *maps every non-unit of* R_p *into a non-unit of* R_q. *Then* $R = \lim_{\to} R_p$ *is a UFD if and only if every non-unit* ($\neq 0$) *of* R *is a finite product of irreducible elements.*

Proof. Evidently R is an integral domain, and the necessity is also obvious. To show the sufficiency, we let $\bar{x}$ be any non-unit ($\neq 0$) of R and set

$$\bar{x} = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_i \cdots \bar{x}_n = \bar{y}_1 \bar{y}_2 \cdots \bar{y}_j \cdots \bar{y}_k ,$$

where all $\bar{x}_i$'s and $\bar{y}_j$'s are irreducible elements of R. Since $R = \bigcup_{p\in I} g_p(R_p)$ and I is a directed set, there must exist some $g_p(R_p)$ which contains all the $\bar{x}$, $\bar{x}_i$ and $\bar{y}_j$. Therefore $\bar{x} = g_p(x)$, $\bar{x}_i = g_p(x_i)$ and $\bar{y}_j = g_p(y_j)$ for some non-units x, x_i and y_j of R_p. It follows that

$$g_p(x) = g_p(x_1) \cdots g_p(x_i) \cdots g_p(x_n) = g_p(y_1) \cdots g_p(y_j) \cdots g_p(y_k);$$

hence

$$(*) \quad f_p^q(x) = f_p^q(x_1) \cdots f_p^q(x_i) \cdots f_p^q(x_n) = f_p^q(y_1) \cdots f_p^q(y_j) \cdots f_p^q(y_k)$$

for some $q \geq p$. All the $f_p^q(x_i)$ and the $f_p^q(y_j)$ are non-units of R_q by the hypothesis on the f_p^q's, moreover they are irreducible in R_q; for if $f_p^q(x_i) = ab$ for some x_i (or y_j) and some non-units a, b of R_q, then clearly $g_q(a)$ and $g_q(b)$ are non-units of R and $g_p(x_i) = g_q(a)g_q(b)$ which contradicts the fact that $\bar{x}_i = g_p(x_i)$ is irreducible in R. Thus $(*)$ is a decomposition of $f_p^q(x)$ into prime factors in the UFD R_q; hence $n = k$, and we can rearrange the order of $f_p^q(x_i)$'s so that $f_p^q(x_i)$ and $f_p^q(y_i)$ are associates in R_q for each i. Accordingly $\bar{x}_i$ and $\bar{y}_i$ are associates in R for every i, which yields that R is a UFD.

COROLLARY. *Let* $(R_p, f_p^q)_{p\in I}$ *be a directed system of UF local domains* R_p *under local homomorphisms* f_p^q *such that* $R = \lim_{\to} R_p$ *is a local ring; then* R *is a UFD.*

The next theorem is by Theorem 1 and the corollary to Theorem 2.

THEOREM 3. *Let* $(R_p, f_p^q)_{p\in I}$ *be a directed system of UF local domains* R_p *such that every* f_p^q *is a local homomorphism. Then* $R = \lim_{\to} R_p$ *is a UF local domain if* $g_p(R_p)$ *is a local ring having* R *as a concordant extension for every* $p \in I$.

For any local ring (A, m) we define an order function v on A as follows:

$$v(a) = \min \{n \mid a \notin m^{n+1}\} \quad \text{if } 0 \neq a \in A$$
$$= \infty \quad \text{if and only if } a = 0.$$

THEOREM 4. *Let* $(E_p, f_p^q)_{p\in I}$ *be a directed system of regular local rings*

(E_p, m_p), such that every homomorphism f_p^q maps a minimal basis of m_p into that of m_q if $q \geq p$; then $E = \lim_{\to} E_p$ is a UF local domain.

Proof. Let $S_p = \{a_1^{(p)}, a_2^{(p)}, \cdots, a_{\lambda_p}^{(p)}\}$, $\lambda_p \in N$, be a minimal basis of m_p for each $p \in I$ such that $f_p^q(S_p) \subseteq S_q$ if $q \geq p$. Letting

$$\{x_1, x_2, \cdots, x_{\lambda_p}\} \quad \text{and} \quad \{y_1, y_2, \cdots, y_{\lambda_q}\}$$

be two sets of indeterminates, we put

$$E_p[x] = E_p[x_1, x_2, \cdots, x_{\lambda_p}] \quad \text{and} \quad E_q[y] = E_q[y_1, y_2, \cdots, y_{\lambda_q}]$$

for any pair p and q of I such that $q \geq p$. Applying the fact that $f_p^q(S_p) \subseteq S_q$, we define an extension $\bar{f}_p^q$ of f_p^q on $E_p[x]$ into $E_q[y]$ as follows: for each $\tau \in E_p[x]$, $\bar{f}_p^q(\tau) = \tau_0$ is to be an element of $E_q[y]$ which is obtained from τ by replacing each coefficient α of τ with $f_p^q(\alpha)$ and each x_i with some y_j provided that $f_p^q(a_i^{(p)}) = a_j^{(q)}$. We remark that each coefficient of $\bar{f}_p^q(\tau) = \tau_0$ is either an image of a coefficient of τ, or an image of a sum of some coefficients of τ under f_p^q respectively. Now we prove that E is a local ring by applying the first part of Corollary 2 to Theorem 1. Obviously every f_p^q is a local homomorphism and every $g_p(E_p)$ is a local ring; hence we only need to prove that the equality

$$m_q^t \cap f_p^q(E_p) = f_p^q(m_p^t)$$

is true for every integer $t \in N$. Let us assume that $m_q^t \cap f_p^q(E_p) \neq (0_q)$ where 0_q is the zero of E_q, for otherwise there is nothing to prove; and take any non-zero element b from $m_q^t \cap f_p^q(E_p)$. Since $v(b) < \infty$, every inverse image $f_p^{q^{-1}}(b)$ of b in E_p also has finite order which does not exceed $v(b)$; among all the inverse images of b in E_p we choose one, say c, which possesses the largest order. If $v(c) = n$, then we have

$$c = \psi(a_1^{(p)}, a_2^{(p)}, \cdots, a_{\lambda_p}^{(p)}) = \psi(a^{(p)})$$

for some form (a homogeneous polynomial) ψ of degree n in $E_p[x]$; since $c \notin m_p^{n+1}$, not all the coefficients of ψ are in m_p. Put $\bar{f}_q^p(\psi) = \psi_0$; then ψ_0 is a form of degree n in $E_q[y]$ and

$$b = f_p^q(c) = f_p^q(\psi(a^{(p)})) = \psi_0(a_1^{(q)}, a_2^{(q)}, \cdots, a_{\lambda_p}^{(q)}) = \psi_0(a^{(q)}).$$

We assert that not all the coefficients of ψ_0 are in m_q. By the previous remark we already know that each coefficient $\bar{u}$ of ψ_0 has an inverse image $u = f_p^{q^{-1}}(\bar{u})$ in E_p, which is either a coefficient of ψ or a sum of coefficients of ψ. Therefore if all the coefficients $\bar{u}$'s of ψ_0 are in m_q, then all those corresponding inverse image u's of $\bar{u}$'s are in m_p; this suggests to us that there must exist a form $\eta\ (\neq \psi)$ of degree n in $E_p[x]$ having all the coefficients u's in m_p, such that $\bar{f}_p^q(\eta) = \psi_0$.

Set

$$\eta(a_1^{(p)}, a_2^{(p)}, \cdots, a_{\lambda_p}^{(p)}) = \eta(a^{(p)}) = c'.$$

Then

$$b = \psi_0(a^{(q)}) = f_p^q(\eta(a^{(p)})) = f_p^q(c') \quad \text{and} \quad v(c') > n = v(c)$$

which contradicts the choice of c; hence not all the coefficients of ψ_0 are in m_q, as asserted. It follows that $v(b) = v(\psi_0(a^{(q)})) = n$, since E_q is a regular local ring (cf. [2, p. 85]); consequently $n \geq t$ by the fact that $b \in m_q^t$, and, therefore

$$b = f_p^q(c) \in f_p^q(m_p^n) \subseteq f_p^q(m_p^t)$$

which means that

$$m_q^t \cap f_p^q(E_p) \subseteq f_p^q(m_p^t).$$

Evidently $m_q^t \cap f_p^q(E_p) \supseteq f_p^q(m_p^t)$, so that

$$m_q^t \cap f_p^q(E_p) = f_p^q(m_p^t),$$

which we wish to show. Thus E is a local ring, and we can conclude that E is a UFD by the corollary to Theorem 2. This completes the proof of the theorem.

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