THE BAR CONSTRUCTION AND ABELIAN $H$-SPACES

BY

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If $X$ is an associative $H$-space with unit Dold and Lashof [2] have given a method for constructing a classifying space $B_X$ which generalizes the classifying spaces for topological groups. In this note we present a construction which has three advantages over that of [2]:

1. if $X$ is abelian $B_X$ is also an abelian associative $H$-space with unit (Section 1),
2. if $X$ is a CW complex and the multiplication is a cellular map then $B_X$ is also a CW complex and the cellular chain complex of $B_X$ is isomorphic to the bar construction on the cellular chain complex of $X$ (Section 2),
3. there is an explicit diagonal approximation $D : B_X \to B_X \times B_X$ which is cellular and in the cell chain complex of $B_X$ induces exactly Cartan’s diagonal approximation for the bar construction (Section 3).

These properties are all easily established and once obtained are applied in Section 4 to give elementary constructions for the Eilenberg-Maclane spaces and to deduce the algebraic and geometric preliminaries to Cartan’s calculations of $H^*(K(\pi, n))$.

Remark. The recent results of J. C. Moore and S. Eilenberg [1] and N. Steenrod and E. Rothenberg [8] which are applied to study $H^*(B_X)$ from information on the homology algebra $H_*(X)$ may also be easily developed using our techniques.

I would like to take this opportunity to thank Professors A. Aeppli, W. Browder, J. Stasheff, and N. Steenrod for their help and encouragement.

1. The construction

In this section we define the $i$-classifying spaces $B^i(X)$ for a given associative $H$-space $X$ with identity $*$ and prove some of their more important properties. Most of these results are well known in one form or another [2], [5], [6], the only novel results being 1.6, 1.7, which exhibit the abelian multiplication in $B_X$ and play a vital role in the applications.

Let $\sigma^n$ be the Euclidian $n$-simplex represented as the set of points $(t_1, \cdots, t_n)$ in $R^n$ with

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1.$$ 

It has faces $\sigma^n_i$ (for which $t_i = t_{i+1}$) $1 \leq i < n$, $\sigma^n_n(t_1 = 0)$, and $\sigma^n_n(t_n = 1)$.

Let $A^i(X)$, $0 \leq i \leq \infty$, be the disjoint union $\sum_{j=0}^{\infty} X \times \sigma^j \times X^j$, ($X^j$ is the $j$-fold Cartesian product). In $A^i$ generate an equivalence relation by

Received March 4, 1966.

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means of generators of two kinds:

\[(x, t_1, \ldots, t_n, x_1, \ldots, x_n)\]

\[
(1) \quad (x, t_1, \ldots, \hat{t}_i, \ldots, t_n, x_1, \ldots, \hat{x}_i, (x_i, x_{i+1}), \ldots, x_n)
\]

if \(t_i = t_{i-1}\) or \(x_i = \ast\) (for \(i = n\) delete the last coordinates if \(t_n = 1\) or \(x_n = \ast\)).

\[(2) \quad (y, 0, t_2, \ldots, t_n, x_1, \ldots, x_n) \sim (yx_1, t_2, \ldots, t_n, x_2, \ldots, x_n).\]

\(E^i(X)\) is defined to be \(A^i(X)/R\). It is topologized by the quotient topology and \(\pi_i\) is the projection

\[\pi_i : A^i \rightarrow E^i.\]

There are two things to notice about \(E^i(X)\).

**Proposition 1.1.** \(X\) acts continuously and associatively as a set of "left transformations" \(X \times E^i(X) \rightarrow E^i(X).\)

**Proof.** Let \(y \in X\), then \(y : A^i(X) \rightarrow A^i(X)\) is given by

\[y(x, t_1, \ldots, t_n, x_1, \ldots, x_n) = (yx, t_1, \ldots, t_n, x_1, \ldots, x_n).\]

This action is continuous in \(X\) and \(A^i(X)\), takes equivalence classes into equivalence classes and hence induces the desired action in \(E^i(X)\), Q.E.D.

Let \(E^\infty(X)\) be the union of the \(E^i(X)\) with the weak topology.

**Proposition 1.2.** \(E^i(X)\) is contained and contractible in \(E^{i+1}(X)\). In particular \(E^\infty(X)\) is contractible.

**Proof.** The inclusion \(A^i \subset A^{i+1}\) respects equivalence. The contraction is induced by the map

\[F_t : A^i \rightarrow A^{i+1}\]

given on points by

\[F_t(x, t_1, \ldots, t_j, x_1, \ldots, x_j) = (\ast, \hat{t}, \hat{t} + t_j, \ldots, \hat{t} + t_i, x, x_1, \ldots, x_j)\]

(where \(\alpha = \max(1, \alpha)\)) which also respects equivalence.

**Definition 1.3.** \(B^i(X)\) is the set of equivalence classes of points of \(E^i(X)\) under the action of \(X\). It has the quotient topology, and \(\rho_i\) is the projection

\[\rho_i : E^i(X) \rightarrow B^i(X).\]

From 1.2 it would seem that \(B^\infty(X)\) is in some sense a classifying space for \(X\). This is justified by the following two results.

**Theorem 1.4.** Let \(G\) be a topological group with identity \(\ast\), \(N\) an open neighborhood of \(\ast\), \(h\) a homotopy \(h_t(G, \ast) \rightarrow (G \ast)\) with \(h_0 = \text{id}\), \(h_t(N) \subset N\) and \(h_1(N) = \ast\); then \(\rho_i E^i(G) \rightarrow B^i(G)\) is a Steenrod fiber bundle with fiber and group \(G\).
Proof. For $i = 0$, $E^0 = G$, $B^0 = *$ and the result is true. The proof now goes by induction. Assume the theorem for $j \leq i$. Since $E^i$ is closed in $E^{i+1}$ and $E^{i+1} - E^i$ is equivalent to a product $G \times (B^{i+1} - B^i)$ it suffices to show the theorem in some open set $M$ with

$$B^i \subset M \subset B^{i+1}.$$  

Let $S \subset G^{i+1}$ be the set of points $(g_1 \cdots g_{i+1})$ with some $g_j = *$. Our assumption on $G$ implies there is a homotopy $id \times k_t$ taking

$$(G \times \sigma^{i+1} \times G^{i+1}, G \times \sigma^{i+1} \times S \cup G \times \partial \sigma^{i+1} \times G^{i+1})$$

into itself with $k_0 = id$. Deforming this homotopy we have that $id \times k'_t$ is a fiber preserving retraction of some neighborhood $U$ of $G \times (z \in M \times S \cup 0 \times G (+))$. Set $M = \rho_{i+1} \pi_{i+1}(U)$, $V = \pi_{i+1}(U)$. Then the constructed retraction induces retractions

$$l : V \to E^i, \quad m : M \to B^i \quad (\rho_{i+1} l = m).$$

Moreover, since $G$ is a group $l$ maps fibers homeomorphically onto fibers. Thus we may extend the local product structure in $E^i$ into $\rho^{-1}(M)$, finally comparing the structure in $\rho^{-1}(M)$ with that in $B^{i+1} - E^i$ we see that they differ only by left translation by elements of $G$, Q.E.D.

**Theorem 1.5.** If $X$ is a connected, associative $H$-space with identity and a homotopy $h_t$ as in 1.4, then $\rho_t : E^i(X) \to B^i(X)$ is a quasifibration with fiber $X$.

(For the definition and major properties of quasifibrations see [7].)

**Proof.** This is identical to that of 1.4 up to the last paragraph. $k_t$ need not now map fibers homeomorphically. However, since $X$ is connected it is a homotopy equivalence on fibers. The proof is now completed by means of 2.2, 2.10 and for $B^\infty(X)$ 2.15 of [7].

In case $X$ is abelian there is additional structure in $B^\infty(X)$.

**Theorem 1.6.** If $X$ is abelian then $B^\infty(X)$ is an associative, abelian $H$-space with unit $*$. 

**Proof.** Define a mapping $u : B^\infty(X) \times B^\infty(X) \to B^\infty(X)$ by

$$u((t_1, \cdots, t_n, x_1, \cdots, x_n), (t_{n+1}, \cdots, t_{n+m}, x_{n+1}, \cdots, x_{n+m})) = (t_{\alpha(1)}, \cdots, t_{\alpha(n+m)}, x_{\alpha(1)}, \cdots, x_{\alpha(n+m)})$$

where $\alpha$ is any element of the symmetric group $S_{n+m}$ for which

$$t_{\alpha(1)} \leq t_{\alpha(2)} \leq \cdots \leq t_{\alpha(n+m)}.$$
As a consequence of the first identification relation \( u \) is well defined and it is easy to verify its continuity.

**Corollary 1.7.** If \( G \) is an abelian topological group the same is true of \( \mathcal{B}^\infty(G) \).

**Proof.** It suffices to exhibit inverses. In fact

\[
(t_1, \cdots, t_n, x_1, \cdots, x_n)^{-1} = (t_1, \cdots, t_n, x_1^{-1}, \cdots, x_n^{-1}).
\]

**Example 1.8.** Let \( SP^\infty(Y) \) be the infinite symmetric product of \( Y \). It is an associative abelian \( H \)-space and

\[
E^\infty(SP^\infty(Y)) = SP^\infty(cY), \quad B^\infty(SP^\infty(Y)) = SP^\infty(\Sigma Y),
\]

these equivalences being multiplicative homeomorphisms. (Here \( cY \) is the reduced cone on \( Y \), and \( \Sigma Y \) is the reduced suspension.) This may be seen by identifying the point

\[
\langle(t_1, a_1), \cdots, (t_i a_i), (t_{i+1} a_{i+1}), \cdots, (t_n a_n), \cdots, (t_1 a_1 + k)\rangle
\]

of \( SP^{i+k}(cX) \) with the point

\[
\langle(u(t_1, \langle a_1 \cdots a_i \rangle), (t_{i+1} a_{i+1} \cdots a_{i+1})), \cdots, (t_n, \langle a_1 \cdots a_n \rangle)\rangle
\]

of \( E^\infty(SP^\infty(X)) \). In fact, in this case the fibration is given in [7] and plays an important part in the proof of the main theorem.

**Remark 1.9.** This construction was originally given by Stasheff in [6] in a form equivalent to that in which it appears here. He shows that it is homotopy equivalent to those given by Milnor [5] and Dold-Lashof [7], and from this it follows that for \( X \) connected \( E^i(X) \) is \( i \)-connected.

### 2. The cellular construction

\( X \) is now assumed to be a CW complex with cellular multiplication and \( \ast \) a 0-cell.

**Remark 2.1.** If \( X \) is an associative \( H \)-space, then \( S(X) \), the singular polytope of \( X \), is an associative \( H \)-space with cellular multiplication. Thus, using the equivalence between \( X \) and \( S(X) \), our restriction on \( X \) is always satisfied, at least up to weak \( H \)-equivalence.

The \( k \) topology on \( X \) is defined by letting \( U \) be \( k \)-open only if \( U \cap C \) is relatively open in \( C \) for every compact subset \( C \) of \( X \). In general the \( k \) topology is finer than the original, hence the identity

\[
i : (X, k) \to X
\]

is continuous.

Since \( (X, k) \) has the same compact sets as \( X \) it follows that \( i \) induces isomorphisms of the respective singular complexes. Hence \( i \) is always a weak homotopy equivalence.
Finally, if $X$, $Y$ are CW-complexes then

1. $(X, k) = X$
2. $(X \times Y, k)$ is the “product” CW complex of $X$ and $Y$, that is its cells are products of cells of $X$ and $Y$.

Let $\tilde{A}^i$ be $(A^i, k)$ and $\tilde{B}^i = \tilde{A}^i/R$ with the quotient topology. Then 1.1, 1.2 hold without change and $\tilde{B}^i = \tilde{E}^i/X$ is defined. In fact, with the exception of 1.4, all the results of Section 1 continue to hold. Thus by use of excision and the Eilenberg-Zilber theorem it follows that the identity map

$$i : \tilde{B}^i \rightarrow B^i$$

induces isomorphisms in homology.

Now we digress shortly to recall the definition of the bar construction. Given a D.G.A. algebra $A$ over a commutative ground ring $\Gamma$ with unit, and with augmentation $\varepsilon : A \rightarrow \Gamma$, let $\tilde{A} = \ker \varepsilon$, then the bar construction $B(A)$ is

$$\Gamma + \tilde{A} + \tilde{A} \otimes_{\Gamma} \tilde{A} + \cdots + \tilde{A} \otimes_{\Gamma} \cdots \otimes_{\Gamma} \tilde{A} + \cdots.$$  

A typical generator $a_1 \otimes \cdots \otimes a_n$ is written $[a_1 | \cdots | a_n]$, and has dimension $(\sum_{i=1}^n \dim a_i) + n$. $B(A)$ is a graded chain complex with boundary operator defined by

$$\delta[a_1 | \cdots | a_n] = \sum_{j=1}^{n-1} (-1)^j[a_1 | \cdots | a_j \cdot a_{j+1} | \cdots | a_n]$$

$$+ \sum_{j=1}^{n-1} (-1)^{n+j}[a_1 | \cdots | \partial a_j | \cdots | a_n]$$

$$+ \varepsilon(a_1)[a_2 | \cdots | a_n] + (-1)^n \varepsilon(a_n)[a_1 | \cdots | a_{n-1}]$$

where $\delta(j) = \sum_{k<j} \dim a_k$.

A generator of the form $[a_1 | \cdots | a_n]$ is said to have degree $n$, and $F^n(A)$ is the $\Gamma$-submodule of $B(A)$ generated by the elements of degree $\leq n$. Clearly, the $F^n(A)$ give a filtration of $B(A)$. The resulting spectral sequence was studied by Eilenberg-Moore in [1].

Returning to classifying spaces recall that the CW chain complex of $X$ is given by

$$C_i(X) = H_i(X_i, X_{i-1}; Z)$$

and the boundary is that in the exact sequence of the triple $(X_i, X_{i-1}, X_{i-2})$.

**Proposition 2.2.** There is a natural chain isomorphism

$$J : C_\#(X) \otimes C_\#(Y) \rightarrow C_\#(X \times Y, k)$$

(The proof is direct.)

As a result the map $u \cdot : (X \times X, k) \rightarrow X$ ($u$ is the multiplication) induces a chain map

$$u \cdot i_\# J : C_\#(X) \otimes C_\#(X) \rightarrow C_\#(X)$$
which makes \( C_{\#}(X) \) into a D.G.A. algebra. We can now state the main result of this section.

**Theorem 2.3.** \( \check{B}^\infty(X) \) is a CW-complex and there is an isomorphism

\[
T : B((C_{\#}(X))) \to C_{\#}(\check{B}^\infty(X)).
\]

**Proof.** A cell \( \sigma^n \times e^1 \times \cdots \times e^n \) of \( \check{B}^\infty(X) \) is defined to be the set of all points \( (t_1, \ldots, t_n, x_1, \ldots, x_n) \) with \( x_i \in e^i \). It is easy to verify that this decomposition gives \( \check{B}^\infty(X) \) the structure of a CW complex.

The boundary operator on such a cell is given by

\[
\partial[\sigma^n \times e^1 \times \cdots \times e^n] = [\partial \sigma^n \times e^1 \times \cdots \times e^n] + (-1)^n[\sigma^n \times \partial(e^1 \times \cdots \times e^n)].
\]

From the identifications of Section 1 it follows that

\[
[\sigma^j \times e^1 \times \cdots \times e^n] = [\sigma^{n-1} \times e^1 \times \cdots \times (e^j e^{j+1}) \times \cdots \times e^n] \quad \text{for} \quad 1 \leq j < n.
\]

Similarly,

\[
[\sigma^1 \times e^1 \times \cdots \times e^n] = \varepsilon(e^1)[\sigma^{n-1} \times e^2 \times \cdots \times e^n],
\]

\[
[\sigma^n \times e^1 \times \cdots \times e^n] = \varepsilon(e^n)[\sigma^{n-1} \times e^1 \times \cdots \times e^{n-1}].
\]

Thus if we identify \( \sigma^n \times e^1 \times \cdots \times e^n \) with \( |e^1| \cdots |e^n| \) in \( B(C_{\#}(X)) \) the desired isomorphism is obtained, Q.E.D.

**Corollary 2.4.** The chain complex \( C_{\#}(\check{B}^i(X)) \) is chain isomorphic under \( T \) with \( F^i B(C_{\#}(X)) \).

If \( X \) is abelian the multiplication

\[
\check{B}^\infty(X) \times \check{B}^\infty(X) \to \check{B}^\infty(X)
\]
given in Section 1 is still continuous and is, moreover, cellular. In fact

\[
\sigma^n \times e^1 \times \cdots \times e^n \cdot \sigma^m \times e^{n+1} \times \cdots \times e^{n+m} = \sum_{\alpha \in \text{Shuff}_{(n,m)}} (-1)^{\gamma(\alpha)} \sigma^{m+n} \times e^{\alpha-1(1)} \times \cdots \times e^{\alpha-1(n+m)}.
\]

Here, \( \alpha \) is an \( n, m \) shuffle, \( \gamma(\alpha) = \alpha + m \sum_{k \leq n} \dim(e^k) + s \) where \( s \) is the sign associated with the map

\[
\text{Shuff } \alpha : (C_{\#}(X))^{n+m} \to (C_{\#}(X))^{n+m}.
\]

This is shown by subdividing \( \sigma^n \times \sigma^m \) into simplexes via the standard decomposition (see p. 68 of [3]), and observing that the restriction to the interior of each of these simplexes is a homeomorphism onto the interior of its image cell, in
the composition
\[ \sigma^{n+m} \times e^1 \times \cdots \times e^{n+m} \rightarrow (\sigma^n \times e^1 \times \cdots \times e^n) \times (\sigma^m \times e^{n+1} \times \cdots \times e^{n+m}) \rightarrow \tilde{B}^i(X). \]

3. The diagonal map

A homotopy of the diagonal map \( \Delta : \sigma^n \rightarrow \sigma^n \times \sigma^n \) is defined by
\[ f_i(t_1, \ldots, t_n) = \{(1 + t)t_1, \ldots, (1 + t)t_n; (1 + t)(t_1 - 1) + 1, \ldots, (1 + t)(t_n - 1) + 1\} \]
where \( a = \min (a, 1), b = \max (b, 0). \) On a typical point \((t_1, \ldots, t_n)\) with \( t_i \leq \frac{1}{2} \leq t_{i+1}\)
\[ f_i(t_1, \ldots, t_n) = \{(2t_1, \ldots, 2t_i, 1, \ldots, 1), (0, \ldots, 0, 2t_{i+1} - 1, \ldots, 2t_n - 1)\}, \]
and on the chain level
\[ f_i(\sigma^n) = \sum (F_{ij} \sigma^n) \otimes (L_{n-j} \sigma^n) \]
the usual diagonal approximation.

Shuff \((\Delta \times f_i \times \Delta) : A^i \rightarrow A^i \times A^i\) commutes with the identifications and hence induces a homotopy
\[ F_i : \tilde{B}^i(X) \rightarrow \tilde{B}^i(X) \times \tilde{B}^i(X). \]
For a typical point
\[ F_i(t_1, \ldots, t_n, x_1, \ldots, x_n) = \{(2t_1, \ldots, 2t_i, x_1, \ldots, x_i), (2t_{i+1} - 1, \ldots, 2t_n - 1, x_{i+1}, \ldots, x_n)\}, \]
and an easy calculation now gives

**Theorem 3.1.**
\[ F_i\sigma^i[e^1 | \cdots | e^n] = \sum_{j=0}^n (-1)^{\tau(j)}[e^1 | \cdots | e^j] \otimes [e^{j+1} | \cdots | e^n] \]
is a diagonal approximation where \( \tau(j) = (n - j)(\sum_{k \geq j} \dim E^k). \)

4. Application to Eilenberg-MacLane spaces

Let \( \pi \) be a group. Give it the discrete topology. Then it is a CW complex consisting of 0-cells and the multiplication is cellular. Hence we can apply the theory of the last three sections.

In particular \( E^\infty(\pi) \) is the universal covering space of \( B^\infty(\pi) \) with \( \pi \) as group of cover transformations. Hence \( B^\infty(\pi) \) is a \( K(\pi, 1) \), as is also \( \tilde{B}^\infty(\pi) \).

If \( \pi \) were abelian then \( \tilde{B}^\infty(\pi) \) would be an abelian topological group with cellular multiplication, and we could iterate the construction obtaining \( \tilde{B}^\infty(\tilde{B}^\infty(\pi)) = K(\pi, 2) \), etc. thus we have
THEOREM 4.1. If $\pi$ is an abelian group then there is a $K(\pi, n)$ for each $n = 0, 1, 2, \ldots$ which is a topological abelian group and a CW complex with cellular multiplication. Moreover,

$$C_\#(K(\pi, n)) \cong B(C_\#(K(\pi, n - 1))),$$

the isomorphism being of D.G.A.-algebras.

Taking into account the diagonal maps we have

THEOREM 4.2. (i) $H_\#(K(\pi, n)) \cong H_\#(B(C_\#(K(\pi, n - 1))))$

(ii) $H^\#(K(\pi, n)) \cong H^\#(B(C_\#(K(\pi, n - 1))))$,

the isomorphisms being ring homomorphisms respectively of Pontrjagin and cohomology rings.

Other results on suspension and transgression may be obtained from the fact that $B^1(X) = \Sigma(X)$, but we do not make them explicit here.

5. The $H$-type of an abelian $H$-space.

Suppose again that $X$ is an abelian associative $H$-space with unit. There is then a map

$$F : X \to \Omega(B^\#(X))$$

defined by $[F(x)]t = (1 - t, x) \in \Sigma X \subseteq B^\#(X)$. Suppose we define an abelian multiplication in the space $E$ of paths by $(\gamma \cdot \tau)t = \gamma(t) \cdot \tau(t)$, then $F$ may be extended to a map $G : E^\#(X) \to E$ as follows: We first define a path $x_s$ by

$$x_s(t) = (1 - (1 - s)t, x),$$

then $G : (x, t_1, \ldots, t_n, x^1, \ldots, x^n) = F(x) \cdot x^1_{t_1} \cdots x^n_{t_n}$ and it is easy to check that

(1) $G$ is a continuous homomorphism,

(2) \[
\begin{array}{ccc}
E^\# & \xrightarrow{G} & E \\
\rho \downarrow & & \downarrow \pi \\
B^\#(X) & & \\
\end{array}
\]

(3) $G | X = F$.

Thus, from the homotopy exact sequences of the 2 quasifibrations it follows that $F$ is a weak homotopy equivalence (if $X$ is connected). Moreover, as $B^\#(X)$ is again a connected, associative, abelian $H$-space with unit we may iterate the constructions and we have

THEOREM 5.1. $X$ (as above) is weakly homotopic to an $n$th loop space $\Omega^n(Y)$ for $n = 1, 2, 3, \ldots$. Moreover, there is an $H$-structure on $\Omega^n(Y)$ so that $X$ is actually $H$-equivalent to $\Omega^n(Y)$.
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