SOME PROPERTIES OF A PARTIAL DIFFERENTIAL OPERATOR

BY

J. R. DORROH

1. Introduction

Suppose E is a real or complex Banach space, let M denote a closed positive cone in E (possibly a subspace of E, and possibly all of E). Let S denote an open subset of E such that $S + M \subset S$, and let X denote the Banach space of all bounded and uniformly continuous complex valued functions on S. Suppose p is a continuously Fréchet differentiable function from S into M, and that p', the Fréchet derivative of p, is a bounded function. Let $D(A)$ denote the set of all Fréchet differentiable x in X such that $x'p$ is in X, and let A denote the operator in X with domain $D(A)$ defined by $Ax = x'p$. Various properties are developed for the operator A, A^2, $A + Q$, $A^2 + Q$, and $A^2 + PA + Q$, where P and Q are bounded operators in X, and the results have applications to partial differential equations. If E is real or complex Euclidean n-space, then

$$Ax = \sum p^i D_i x,$$

where p^i denotes the i^{th} component of p, and $D_i x$ denotes the i^{th} place partial derivative of x in the ordinary sense.

Most of the results require that p be a bounded function and are obtained by giving a simple formula for a strongly continuous semi-group (group in case M is a subspace of E) of operators in X which is generated by a closed extension of A. In case E is real Euclidean n-space, the generator is the minimal closed extension of A. In case M is a subspace of E, there is a simple formula for a strongly continuous semi-group generated by a closed extension of A^2. The subspace case is of no interest if E is complex, because then $D(A)$ contains only the constant functions. If E is a real Banach space, then the results can, by [3], be extended to the operators qA, qA^2, $(qA)^2$, etc., where q is a positive function in X which is bounded away from zero.

2. An ordinary differential equation

If g is a function from $S \times [0, \infty)$ or $S \times (-\infty, \infty)$ into a vector space, then g_2 denotes the second place partial derivative of g in the ordinary sense, and g_1 denotes the first place partial derivative of g in the Fréchet sense (see [1, Chapter VIII]).

2.1. THEOREM. *If s is in S, then there is only one function f from $[0, \infty)$ into S ($(-\infty, \infty)$ into S in case M is a subspace of E) such that $f(0) = s$, and $f'(t) = p(f(t))$ for all t in $[0, \infty)$ (all t in $(-\infty, \infty)$).*

Received November 12, 1965.

Proof. The method of successive approximations will establish this result. The range of all the approximate solutions will lie in $s + M$, which is a complete metric space.

2.2. DEFINITION. Let y denote the function from $S \times [0, \infty)$ into S (from $S \times (-\infty, \infty)$ into S if M is a subspace of E) defined by

$$y(s, 0) = s, \qquad y_2(s, t) = p(y(s, t)).$$

2.3. THEOREM. *y is continuous on $S \times [0, \infty)(S \times (-\infty, \infty))$,*

(i) $$\| y(s, t) - y(s', t) \| \leq \| s - s' \| \exp (\| p' \| \cdot | t |)$$

for s, s' in S and t in $[0, \infty)$ (t in $(-\infty, \infty)$), and

(ii) *if p is bounded, then*

$$\| y(s, t) - y(s, t') \| \leq \| p \| \cdot | t - t' |$$

for s in S and t, t' in $[0, \infty)$ (t, t' in $(-\infty, \infty)$).

Proof. The method of successive approximations will establish (i), and (ii) is trivial. The continuity of y follows from (i) and the fact that $y(s, \cdot)$ is continuous on $[0, \infty)$ (on $(-\infty, \infty)$) for each s in S.

2.4. THEOREM. *For s in S and t, u in $[0, \infty)$ (t, u in $(-\infty, \infty)$),*

$$y(y(s, u), t) = y(s, u + t).$$

Proof. Suppose s is in S and u is in $[0, \infty)$ (in $(-\infty, \infty)$), and let

$$f(t) = y(y(s, u), t), \qquad g(t) = y(s, u + t)$$

for t in $[0, \infty)$ (t in $(-\infty, \infty)$). Then

$$f'(t) = y_2(y(s, u), t) = p(f(t)),$$

$$g'(t) = y_2(s, u + t) = p(g(t)),$$

and $f(0) = y(s, u) = g(0)$.

2.5. THEOREM. *y_1 exists and is continuous on $S \times [0, \infty)$ (on $S \times (-\infty, \infty)$) and*

$$y_2(s, t) = y_1(s, t)p(s)$$

for all (s, t) in $S \times [0, \infty)(S \times (-\infty, \infty))$.

Proof. It follows from [1, Theorem 10.8.2, p. 300] that y_1 exists and is continuous on a neighborhood of $(s, 0)$ for all s. The global existence and continuity of y_1 follows from Theorem 2.4 and the chain rule for derivatives [1, Theorem 8.2.1, p. 145]. Since

$$y(s, t) = s + \int_0^t p(y(s, u))\, du,$$

it follows from the rule for differentiation under the integral sign [1, Theorem 8.11.2, p. 172] that

$$y_1(s, t) = I + \int_0^t p'(y(s, u))y_1(s, u)\, du$$

or that

$$y_{12}(s, t) = p'(y(s, t))y_1(s, t), \qquad y_1(s, 0) = I,$$

where I denotes the identity transformation on E. Suppose s is in S, and let

$$f(t) = y_1(s, t)p(s),$$
$$g(t) = y_2(s, t) = p(y(s, t)).$$

Then $f(0) = p(s) = g(0)$,

$$f'(t) = y_{12}(s, t)p(s) = p'(y(s, t))y_1(s, t)p(s) = p'(y(s, t))f(t),$$

and

$$g'(t) = p'(y(s, t))y_2(s, t) = p'(y(s, t))g(t).$$

3. A partial differential equation

Here the results of Section 2 are applied to a partial differential equation. This is the only one of the main results which is not based on the semi-group theory and the only one which does not require that p be bounded.

3.1. THEOREM. *If x is a Fréchet differentiable function from S into the complex numbers, then there is only one function g from $S \times [0, \infty)$ (from $S \times (-\infty, \infty)$) into the complex numbers such that*

$$g_2(s, t) = g_1(s, t)p(s)$$

for all (s, t), and $g(s, 0) = x(s)$ for all s.

Proof. The function $g(s, t) = x(y(s, t))$ is as required.

Suppose ξ and η are two such functions, and let $\phi = \xi - \eta$. Then

$$\phi_2(s, t) = \phi_1(s, t)p(s) \quad \text{and} \quad \phi(s, 0) = 0.$$

Suppose s is in S, $u > 0$, and $\phi(s, u) \neq 0$. Let

$$f(t) = \phi(y(s, u - t), t) \quad \text{for} \quad 0 \leq t \leq u.$$

Then

$$\begin{aligned} f'(t) &= -\phi_1(y(s, u - t), t)y_2(s, u - t) + \phi_2(y(s, u - t), t)) \\ &= \phi_2(y(s, u - t, t)) - \phi_1(y(s, u - t), t)p(y(s, u - t)) \\ &= 0 \end{aligned}$$

and

$$f(0) = \phi(y(s, u), 0) = 0, \qquad f(u) = \phi(s, u).$$

The argument easily extends to give $\phi(s, t) = 0$ for negative t if M is a subspace of E.

3.2. *Remark.* The function g is bounded if x is bounded and continuous if x is continuous, and g_2 is continuous if $x'p$ is continuous. If E is real or complex Euclidean n-space, then the equation becomes

$$\partial g/\partial t = \sum p^i(s)(\partial g/\partial s^i).$$

4. Semi-groups

For the rest of the paper, p is assumed to be bounded. For each t in $[0, \infty)$ (each t in $(-\infty, \infty)$ if M is a subspace of E), and x in X, let $T(t)x$ denote the function on S defined by

$$[T(t)x](s) = x(y(s, t)).$$

By (i) of Theorem 2.3, $T(t)x$ is a function in X. Clearly, $T(t)$ is a bounded linear operator in X, and $\| T(t) \| \leq 1$. As in [4], a strongly continuous semi-group of operators in X means a semi-group $[S(t);\ 0 \leq t < \infty]$ of bounded operators in X such that $S(0) = I$, the identity operator on X, and $S(\cdot)x$ is continuous on $[0, \infty)$ for each x in X. Such a semi-group is said to be of class (C_0) in [5].

4.1. THEOREM. *$[T(t); 0 \leq t \leq \infty]$ $([T(t); -\infty < t < \infty])$ is a strongly continuous semi-group (group) of operators in X, and its infinitesimal generator (which we shall denote by B) is a closed extension of A.*

Proof. By Theorem 2.4, $[T(t)]$ is a semi-group (group). By (ii) of Theorem 2.3, $[T(t)]$ is strongly continuous. For $h > 0$, and x in X, let

$$B_h x = [T(h)x - x]/h.$$

Suppose x is in $D(A)$, and s is in S. For $t \geq 0$, let

$$f(t) = [T(t)x](s).$$

Then

$$\begin{aligned} |[B_h x - Ax](s)| &= |([f(h) - f(0)]/h) - f'(0)| \\ &\leq \sup |f'(t) - f'(0)| \\ &= \sup |x'(y(s, t))p(y(s, t)) - x'(s)p(s)|, \end{aligned}$$

where the suprema are taken for $0 \leq t \leq h$. Since $x'p$ is uniformly continuous, and

$$\| y(s, t) - s \| \leq \| p \| t,$$

it follows that x is in $D(B)$, the domain of B, and that $Bx = Ax$. Thus B is an extension of A. By [4, Lemma VIII.1.8, p. 620], B is closed.

4.2. *Remark.* It follows from [4, Theorem VIII.1.11, p. 622] that if re $(\lambda) > 0$, then λ is in $\rho(B)$, the resolvent set of B, and

$$R(\lambda, B)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt$$

for all x in X.

4.3. THEOREM. *If the set of all functions in $D(A)$ which have bounded continuous Fréchet derivatives is dense in X, then B is the minimal closed extension of A. This is the case if E is real Euclidean n-space.*

Proof. First we show that if x is in $D(A)$, x' is bounded and continuous, and re $(\lambda) > \| p' \|$, then $R(\lambda, B)x$ is in $D(A)$. Let $w = R(\lambda, B)x$. Then

$$w(s) = \int_0^\infty e^{-\lambda t}x(y(s, t))\,dt$$

for all s in S. Let

$$w_n(s) = \int_0^n e^{-\lambda t}x(y(s, t))\,dt$$

for s in S, and $n = 1, 2, 3, \cdots$. Then $w_n \rightarrow w$ in X. Also,

$$w_n'(s) = \int_0^n e^{-\lambda t}x'(y(s, t))y_1(s, t)\,dt$$

by [1, Theorem 8.11.2, p. 172]. Since

$$\| y_1(s, t) \| \leq e^{\| p' \| t}$$

by (i) of Theorem 2.3, it follows that

$$w_n'(s) \rightarrow \int_0^\infty e^{-\lambda t}x'(y(s, t))y_1(s, t)\,dt$$

uniformly for s in S. Thus, w is Fréchet differentiable on S and

$$w'(s) = \int_0^\infty e^{-\lambda t}x'(y(s, t))y_1(s, t)\,dt$$

by [1, Theorem 8.6.3, p. 157].

$$w'(s)p(s) = \int_0^\infty e^{-\lambda t}x'(y(s, t))p(y(s, t))\,dt,$$

and the integral converges uniformly for s in S, so that $w'p$ is in X, and w is in $D(A)$.

Now suppose that x is in $D(B)$, and let $\lambda = \| p' \| + 1$. Let $z = (\lambda - B)x$, so that $x = R(\lambda, B)z$. Choose a sequence $\{z_n\}_{n=1}^{\infty}$ from $D(A)$ so that each z_n has a bounded continuous Fréchet derivative, and $z_n \rightarrow z$. Let $x_n = R(\lambda, B)z_n$. Then x_n is in $D(A)$ for each n, and $x_n \rightarrow x$. Also,

$$\lambda x_n - Ax_n = \lambda x_n - Bx_n = z_n,$$

so that

$$\lambda x_n - Ax_n \to z = \lambda x - Bx, \qquad Ax_n \to Bx.$$

Now suppose that E is real Euclidean n-space. The functions

$$x_\varepsilon(s) = \varepsilon^{-n} \int_0^\varepsilon \cdots \int_0^\varepsilon x(s_1 + t_1, \cdots, s_n + t_n)\, dt_1 \cdots dt_n$$

are dense in X, have bounded continuous Fréchet derivatives, and are in $D(A)$.

4.4. *Remark.* In [2], the case where E is the real line, $M = S = E$, was considered, and p was assumed to satisfy a uniform Lipschitz condition rather than to have a bounded continuous derivative. The semi-group generator and its domain were easily describable in that case, and a comparison would seem appropriate. In order to facilitate this, consider the following definition. If x is a complex valued function on S, α is in M, s is in S, and

$$\lim_{(t \to 0+)}[x(s + t\alpha) - x(s)]/t$$

exists, then this limit is called the *derivative of x in the α direction at s*, and is denoted by $[D_\alpha x](s)$. This is the usual definition of directional derivative except that the "direction" is sometimes required to be a unit vector and is usually required to be nonnull. Let $D(A')$ denote the set of all x in X such that $[D_{p(s)} x](s)$ exists for all s in S and such that the function $A'x = [D_{p(\cdot)} x](\cdot)$ is in X. Although it was not described in exactly this way in [2], $B = A'$ in case E is the real line and $M = S = E$. The following example shows that this does not follow if E is real Euclidean 2-space, and that A' need not even be closed.

Let E be real Euclidean 2-space, and let $M = S = E$. Use the notation $s = (\xi, \eta)$, $p = (p^1, p^2)$, $y = (\alpha, \beta)$. Let

$$p^1(\xi, \eta) = 1, \qquad p^2(\xi, \eta) = 2\xi/(1 + \xi^2).$$

Then

$$\alpha(s, t) = \xi + t,$$

and

$$\beta(s, t) = \eta + \log\{1 + (\xi + t)^2]/(1 + \xi^2)\}.$$

Define f on the real line by

$$f(u) = u^{(1/3)} \exp(-u^2),$$

and define x on S by

$$x(\xi, \eta) = f(\eta - \log(1 + \xi^2)).$$

Then x is in X, and

$$x(y(s, t)) = f(\eta - \log(1 + \xi^2)) = x(s)$$

so that x is in $D(B)$ and $Bx = 0$. However,

$$x(s + tp(s)) = f(\eta + 2\xi t/(1 + \xi^2) - \log[1 + (\xi + t)^2]),$$

so that $[D_{p(s)}\, x](s)$ does not exist if $\eta = \log(1 + \xi^2)$, because

$$\frac{\eta + 2\xi t/(1 + \xi^2) - \log[1 + (\xi + t)^2]}{t^3}$$

has no limit as $t \to 0$ in this case. Thus A' is not an extension of B. Since A' is an extension of A, it follows that A' is not closed. The author does not know in general whether or not A' is a restriction of B or if A' and B agree on the intersection of their domains.

4.5. *Remark.* If p is a constant function, it is easily seen that $B = A'$, but it does not follow that $B = A$. Again let E be Euclidean 2-space, and let $M = S = E$. Use the notation $s = (\xi, \eta)$, $p = (p^1, p^2)$, $y = (\alpha, \beta)$. Let $p^1(\xi, \eta) = p^2(\xi, \eta) = 1$. Then $\alpha(s, t) = \xi + t$, and $\beta(s, t) = \eta + t$. Let f be a bounded uniformly continuous real or complex-valued function on the real line which is not everywhere differentiable (perhaps nowhere differentiable), and define x on S by $x(\xi, \eta) = f(\xi - \eta)$. Then x is not in $D(A)$ but x is in $D(A') = D(B)$, and $Bx = A'x = 0$.

4.6. DEFINITION. Let $D(B^2)$ denote the set of all x in $D(B)$ such that Bx is in $D(B)$, and let B^2 denote the operator with domain B^2 defined by $B^2x = B(Bx)$.

4.7. THEOREM. *If M is a subspace of E, then B^2 is the infinitesimal generator of a strongly continuous semi-group (which we shall denote by $[V(t), 0 \leq t < \infty]$) of operators in X, with*

$$V(t)x = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-r^2/4t)\, T(r)x \, dr$$

for $t > 0$, so that $\| V(t) \| \leq 1$ for $t \geq 0$.

Proof. If $\lambda > 0$, then λ is in $\rho(B)$, and

$$R(\lambda, B)x = \int_0^{\infty} e^{-\lambda r}\, T(r)x \, dr$$

by [4, Theorem VIII.1.11, p. 622], and also, by [4, Corollary VIII.1.17, p. 628], $-\lambda$ is in $\rho(B)$, and

$$R(-\lambda, B) = -R(\lambda, -B) = -\int_{-\infty}^{0} e^{\lambda r}\, T(r)x \, dr,$$

since $-B$ generates the semi-group $[T(-t);\ 0 \leq t < \infty]$. Also λ^2 is in $\rho(B^2)$, since

$$\lambda^2 - B^2 = (\lambda + B)(\lambda - B).$$

Thus

$$R(\lambda^2, B^2) = R(\lambda, B)R(\lambda, -B) = -R(\lambda, B)R(-\lambda, B)$$
$$= (1/2\lambda)[R(\lambda, B) - R(-\lambda, B)]$$
$$= (1/2\lambda) \int_{-\infty}^{\infty} e^{-\lambda|r|} T(r)x \, dr.$$

Thus, for $\lambda > 0$,

$$R(\lambda, B^2) = (4\lambda)^{-1/2} \int_{-\infty}^{\infty} e^{-|r|\sqrt{\lambda}} T(r)x \, dr.$$

Since $\lambda - B^2$ is invertible, B^2 is closed. $D(B^2)$ is dense in X by [4, Exercise VIII.3.3, p. 653]. If we let

$$V(t)x = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-r^2/4t) T(r)x \, dr,$$

then

$$\int_0^{\infty} e^{-\lambda t} V(t)x \, dt = \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} (4\pi t)^{-1/2} \exp(-\lambda t) \exp(-r^2/4t) \, dt \right\} T(r)x \, dr$$
$$= \int_{-\infty}^{\infty} (4\lambda)^{1/2} e^{-|r|\sqrt{\lambda}} T(r)x \, dr = R(\lambda, B^2)x.$$

Thus $[V(t);\ 0 \leq t < \infty]$ is a strongly continuous semi-group and B^2 is its infinitesimal generator, by [4, Corollary VIII.1.16, p. 627].

4.8. THEOREM. *If M is a subspace of E, and P is a bounded operator in X, then the operator $B^2 + PB$ with domain $D(B^2)$ is the infinitesimal generator of a strongly continuous semi-group*

$$[U(t);\quad 0 \leq t < \infty]$$

of operators in X such that $\| U(t) \| \leq (1 - \gamma)^{-1} e^{\omega t}$ for all $t \geq 0$ if $\omega > \| P \|^2$, and $\gamma = \| P \| \omega^{-1/2}$.

Proof. Let Q denote the operator with domain $D(B^2)$ defined by $Qx = P(Bx)$ for x in $D(B^2)$. We shall show that Q belongs to the class $\mathcal{P}(B^2)$ of [5, Definition 13.3.5, p. 394]. If $\lambda > 0$, then $QR(\lambda, B^2) = PBR(\lambda, B^2)$, and

$$BR(\lambda, B^2) = (4\lambda)^{-1/2} B[R(\sqrt{\lambda}, B) - R(\sqrt{\lambda}, B)]$$
$$= (4\lambda)^{-1/2} [\sqrt{\lambda} R(\sqrt{\lambda}, B) + \sqrt{\lambda} R(-\sqrt{\lambda}, B)],$$

so that

$$\| QR(\lambda, B^2) \| \leq \| P \| \lambda^{-1/2},$$

and Q is in the class $J(B^2)$ of [5, Definition 13.3.1, p. 391]. If x is in $D(B^2)$, and $t > 0$, then

$$QV(t)x = PBV(t)x,$$

and

$$BV(t)x = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-r^2/4t) T(r) Bx \, dr$$

$$= (4\pi t)^{-1/2} \int_{-\infty}^{\infty} (r/2t) \exp(-r^2/4t) T(r) x \, dr,$$

so that

$$\| QV(t)x \| \leq \| P \| \cdot \| x \| (\pi t)^{-1/2} \int_0^{\infty} (r/2t) \exp(-r^2/4t) \, dr$$

$$= \| P \| \cdot \| x \| (\pi t)^{-1/2}.$$

Thus B is in $\mathcal{P}(B^2)$ and $B^2 + Q$ generates a strongly continuous (class (C_0)) semigroup $[U(t), 0 \leq t < \infty]$ of operators in X by [5, Corollary 1 of Theorem 13.4.1, p. 400]. To get the inequality, set

$$\phi(\xi) = 1, \qquad \psi(\xi) = \| P \| (\pi \xi)^{-1/2}$$

(see [5, Equation 13.4.1, p. 397]),

$$\psi^{(1)} = \psi, \qquad \psi^{(n+1)} = \psi * \psi^{(n)},$$

$$\chi^{(0)} = \phi, \qquad \chi^{(n)} = \phi * \psi^{(n)}.$$

Suppose $\omega \geq \| P \|^2$, and $\gamma = \| P \| \omega^{-1/2}$. If

$$\chi^{(n)}(t) \leq \gamma^n e^{\omega t},$$

then

$$\chi^{(n+1)}(t) = \int_0^t \chi^{(n)}(t - \xi) \psi(\xi) \, d\xi$$

$$\leq \gamma^n e^{\omega t} \int_0^t e^{-\omega \xi} \psi(\xi) \, d\xi$$

$$\leq \gamma^n e^{\omega t} \int_0^{\infty} e^{-\omega \xi} \psi(\xi) \, d\xi = \gamma^{n+1} e^{\omega t}.$$

Also, $\chi^{(0)}(t) = 1$, so that

$$\chi^{(n)}(\xi) \leq \gamma^n e^{\omega \xi}$$

for all n and for all $\xi > 0$, and

$$\phi(\xi) + \sum_{n=1}^{\infty} \phi * \psi^{(n)}(\xi) = \sum_{n=0}^{\infty} \chi^{(n)}(\xi) \leq (1 - \gamma)^{-1} e^{\omega \xi}.$$

This sum dominates $\| U(\xi) \|$ by [5, Theorem 13.4.1, p. 400].

4.9. THEOREM. *If M is a subspace of E, and u is a real function in X, then $B^2 + uB$ generates a strongly continuous semi-group of contraction operators in X.*

Proof. For each function z in X, let m_z denote a multiplicative linear functional m on X such that $| m(z) | = \| z \|$. Such a functional exists by [4, Th. 18., p. 274]. For each x, y in X, let

$$[x, y] = m_y(x)[m_y(y)]^*,$$

where the $*$ denotes complex conjugation. Then $[\cdot, \cdot]$ is a *semi inner-product* on X (see [6] or [3]); that is

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z],$$

$$[x, x] = \| x \|^2 \quad \text{and} \quad | [x, y] | \leq \| x \| \cdot \| y \|$$

for all scalars α and β and all functions x, y, and z in X. Since B^2 and B generate contraction semi-groups, they are both *dissipative* with respect to this semi inner-product, by [6, Th. 3.1, p. 686]; that is, re $[B^2x, x] \leq 0$ for x in $D(B^2)$, and re $[Bx, x] \leq 0$ for x in $D(B)$. Moreover, since B generates a group of contraction operators, it is true that re $[Bx, x] = 0$ for x in $D(B)$. Also

$$\text{re } [uBx, x] = \text{re } \{m_x(u)m_x(Bx)[m_x(x)]^*\}$$

$$= m_x(u) \text{ re } [Bx, x] = 0.$$

Therefore, $B^2 + uB$ is dissipative. Since $B^2 + uB$ does generate a strongly continuous semi-group by Theorem 4.8, it follows that the domain of $B^2 + uB$ is dense in X, and that the range of $\lambda I - B^2 - uB$ is all of X for sufficiently large real λ. Thus by an obvious extension of the argument for [6, Th. 3.1, p. 686], $B^2 + uB$ generates a strongly continuous semi-group of contraction operators in X.

5. Applications

Here the semi-group theory is applied to some partial differential equations. The main interest lies in the case where E is real Euclidean n-space, but the solutions will in general be solutions in a weak sense. That is, the solutions will not in general lie in the domain of the partial differential operator, but rather in the domain of its minimal closed extension. All but the first of the applications make use of the connection between the semi-groups and resolvents. Of course, the semi-groups apply directly to some partial differential equations, for instance, the abstract Cauchy problems of [5, pp. 617–633], but there the application is more straightforward.

5.1. Definition. A function g from $S \times [0, \infty)$ (or from $S \times (-\infty, \infty))$ into the complex numbers is said to be of class $C(B)$ if the function $g(\cdot, t)$ is in $D(B)$ for all t in $[0, \infty)$ (for all t in $(-\infty, \infty)$). If g is such a function, then $Bg(s, t)$ means the value of $Bg(\cdot, t)$ at s.

5.2. Theorem. *If q is in X, and x is in $D(B)$ then there is only one function g from $S \times [0, \infty)$ (from $S \times (-\infty, \infty)$ in case M is a subspace of E)*

into the complex numbers which is of class $C(B)$ and satisfies

$$g_2(s, t) = Bg(s, t) + q(s)g(s, t)$$

for all (s, t), $g(s, 0) = x(s)$ for all s.

Proof. Let Q denote the operator in X defined by $Qx = qx$. Then the operator Q with domain $D(B)$ generates a strongly continuous semi-group $[S(t), 0 \leq t < \infty]$ (or a group of operators) by [5, Theorems 13.2.1. and 13.2.2, pp. 389 and 390]. The function $g(s, t) = [S(t)x](s)$ is as required. The uniqueness claim may be established by an argument similar to the one given for Theorem 3.1.

5.3. *Remark.* The equation of Theorem 5.2. is not an abstract Cauchy problem in the sense of [5, pp. 617–633], because the requirements on the solution are not so strong. However, the argument does show that every solution of the equation is also a solution of the associated abstract Cauchy problem.

5.4. THEOREM. *If z is in X, q is in X, and* re (q) *is negative and bounded away from zero (the assumption of negativity is unnecessary if M is a subspace of E), then there is only one function x in $D(B)$ such that*

$$Bx + qx = z.$$

Proof. Let $d = \inf | \operatorname{re}(q) |$. If λ is a real number having the same sign as re (q), and $| \lambda | > \| q \|^2/2d$, then $\| q - \lambda \| < | \lambda |$, so that $-\lambda$ is in the resolvent set of $B + Q$, where Q is that operator in X defined by $Qx = qx - \lambda x$, and $x = -R(-\lambda, B + Q)z$ is the only function in $D(B)$ such that

$$Bx + qx = z.$$

5.5. THEOREM. *Suppose M is a subspace of E, q is a function in X whose real part is negative and bounded away from zero, and z is in X. Then there is only one function x in $D(B^2)$ such that*

$$B^2x + qx = z.$$

Proof. Let $d = \inf | \operatorname{re}(q) |$, $\lambda > \| q \|^2/2d$, and $Qx = qx + \lambda x$ for x in X. Then λ is in the resolvent set of $B^2 + Q$ and $x = -R(\lambda, B^2 + Q)z$ is the only function in $D(B^2)$ such that $B^2x + qx = z$.

5.6. THEOREM. *Suppose M is a subspace of E, u, v, and z are functions in X, and there exist numbers $\omega > \| u \|^2$, $r > 0$, and $\lambda > \omega + r(1 - \gamma)^{-1}$, where $\gamma = \| u \| \omega^{-1/2}$, such that all the values of v lie in the circular disk with center $-\lambda$ and radius r. Then there is only one function x in $D(B^2)$ such that*

$$B^2x + uBx + vx = z.$$

Proof. Let P and Q denote the operators defined on X by $Px = ux$ and

$Qx = vx + \lambda x$, respectively. Then

$$\lambda > \omega + \| Q \| (1 - \gamma)^{-1},$$

so that λ is in the resolvent set of $B^2 + PB + Q$, and

$$x = -R(\lambda, B^2 + PB + Q)z$$

is the only function in $D(B^2)$ such that $B^2x + uBx + vx = z$.

5.7. THEOREM. *If M is a subspace of E, u, v, and z are functions in X, u is real, and the real part of v is negative and bounded away from zero, then there is only one function x in $D(B^2)$ such that*

$$B^2x + uBx + vx = z.$$

Proof. Let $d = \inf |\operatorname{re}(v)|$, $\lambda > \| v \|^2/2d$, and $Qx = vx + \lambda x$ for x in X. Then $\lambda > \| Q \|$, so that λ is in the resolvent set of $B^2 + uB + Q$, and $-R(\lambda, B^2 + uB + Q)z$ is the only function x in $D(B^2)$ such that $B^2x + uBx + vx = z$.

5.8. *Remark.* If E is a real space, and q is a positive function in X which is bounded away from zero, then it follows from [3] that qB is the infinitesimal generator of a strongly continuous semi-group of contraction operators, and that qB^2 is also, if M is a subspace of E. This will allow the appropriate theorems to extend to the operators qB, qB^2, $(qB)^2$, $qB + Q$, $qB^2 + Q$, and $(qB)^2 + P(qB) + Q$.

REFERENCES

1. J. DIEUDONNE, *Foundations of modern analysis*, New York and London, Academic Press, 1960.
2. J. R. DORROH, *Some properties of a singular differential operator*, Illinois J. Math., vol. 10 (1966), pp. 136–146.
3. ———, *Contraction semi-groups in a function space*, Pacific J. Math., vol. 19 (1966), pp. 35–38.
4. N. DUNFORD AND J. SCHWARTZ, *Linear operators, I. General Theory* New York, Interscience, 1958.
5. E. HILLE AND R. S. PHILLIPS, *Functional analysis and semi-groups*, revised edition, Amer. Math. Soc. Colloquium Publications, vol. 31, 1957.
6. G. LUMER AND R. S. PHILLIPS, *Dissipative operators in a Banach space*, Pacific J. Math., vol. 11 (1961), pp. 679–698.

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA