ON THE COHOMOLOGY OF THE MOD-2 STEENROD ALGEBRA AND THE NON-EXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

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A very handy E^1 term of the Adams spectral sequence for the sphere spectrum is obtained in [5]. Here we shall use it to calculate the cohomology of the mod-2 Steenrod algebra $H^{s,t}(A)$ in the range $s \leq 3$ and find some relations among the h_i's and c_j's in the range $s \leq 4$. The structure of $H^{3,t}(A)$ and the relations $h_0 h_i^3 \neq 0$ for $i \geq 4$ yield the information $d^2 h_i = h_0 h_{i-1}^2$ for $i \geq 4$ by an easy induction starting from $d^2 h_4 = h_0 h_3^2$. Hence a new proof for the non-existence of the elements of Hopf invariant one is obtained.

1. The structure of (E^1, d^1)

We recall from [5] that the structure of (E^1, d^1) is given by the following

THEOREM 1.1 [5]. (i) *(E^1, d^1) is a graded associated differential algebra (with unit) over Z_2 with*

(ii) *a generator λ_i (of degree i) for every integer $i \geq 0$*

(iii) *for every $m \geq 1$ and $n \geq 0$ a relation*

$$(1.2)_{n,m} \qquad \sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+m} \lambda_{j-1+2n} = 0$$

(iv) *d^1 is given by*

$$(1.3) \qquad d^1\lambda_{n-1} = \sum_{i+j=n} \binom{i+j}{j} \lambda_{i-1} \lambda_{j-1}, \qquad n \geq 2.$$

Given a sequence of non-negative integers $I = (n_1, \cdots, n_r)$, we call r the length, $\sum_{i=1}^r n_i$ the degree, n_1 the leading integer and n_r the ending integer of I. I is said to be admissible if $2n_i \geq n_{i+1}$ for $1 \leq i \leq r-1$. Let λ_I stand for $\lambda_{n_1} \cdots \lambda_{n_r}$; then the additive structure of E^1 is given by the following

THEOREM 1.4 [5]. *The set consisting of the unit and λ_I with I admissible forms a vector basis for E^1.*

In the sequel we shall always express elements in E^1 in admissible forms, i.e., in terms of the above basis. Formulas (1.2) and (1.3) are written in symmetric forms. For the convenience of computation, it would be better to derive admissible expressions for them.

1.5. *The* mod-2 *binomial relations generated by* $\lambda_i \lambda_{2i+1} = 0$ *for* $i \geq 0$.

PROPOSITION 1.5.1. *There is a derivation $D : E^1 \to E^1$ sending λ_i to λ_{i+1} for $i \geq 0$.*

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Proof. Let $\bar{E}^1$ be the free graded associated algebra generated by $\bar{\lambda}_i$ (of degree i), for $i \geq 0$, and $p : \bar{E}^1 \to E^1$ be the algebra homomorphism given by $p(\bar{\lambda}_i) = \lambda_i$, for $i \geq 0$. Clearly there is a derivation $\bar{D} : \bar{E}^1 \to \bar{E}^1$ sending $\bar{\lambda}_i$ to $\bar{\lambda}_{i+1}$ for $i \geq 0$. It is easy to verify that the kernel of p is closed under the action of $\bar{D}$ and $\bar{D}$ induces a derivation D with the desired property.

PROPOSITION 1.5.2. (iii) *is equivalent to* $D^t(\lambda_i \lambda_{2i+1}) = 0$ *for* $t, i \geq 0$.

Proof. $\lambda_i \lambda_{2i+1} = 0$ is $(1.2)_{0,i+1}$ and an easy induction shows that $D^t(\lambda_i \lambda_{2i+1}) = 0$ is $(1.2)_{t,i+1}$.

Let $\lambda_i \lambda_{2i+1+n} = \sum_{n \geqq j \geqq 0} a_{n-j,j} \lambda_{n+i-j} \lambda_{2i+1+j}$ be expressed in admissible form, i.e., $a_{n-j,j} = 1$ or 0 and $a_{n-j,j} = 0$ for $2(i + n - j) < 2i + 1 + j$. Applying D on both sides, we get the following recursive formula.

PROPOSITION 1.5.3.

$$a_{n,i} + a_{n-1,i+1} + a_{n-1,i-1} = a_{n,i+1} \quad \textit{for } i \geq 1,$$

$$a_{n-1,0} + a_{n,0} = 0 \quad \textit{for } n \geq 2$$

$$a_{n,0} + a_{n-1,1} = a_{n,1} \quad \textit{for } n \geq 2$$

$$a_{0,0} = 0, \qquad a_{1,0} = 1, \qquad a_{1,1} = 0 \qquad \pmod 2\text{.}$$

Set $F(X, Y) = \sum_{n,j \geqq 0} a_{n,j} X^n Y^j$. Then Proposition 1.5.3 yields the identity $F(X, Y) = F(X, Y)(X + Y + XY^2) + X + XY \pmod 2$. Hence $F(X, Y) = X/(1 - (1 + Y)X) = X + \sum_{n=1}^{\infty} X^{n+1}(1 - Y)^n \pmod 2$ and

$$a_{n,i} = \binom{n-1}{i} \qquad \pmod 2.$$

THEOREM 1.5.4.

$$\lambda_i \lambda_{2i+1+n} = \sum_{j \geqq 0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j} ;$$

the binomial coefficients are, of course, taken mod 2 *and with the usual convention* $\binom{u}{v} = 0$ *for* $u < v$.

1.6. We recall from (5.7) that the spectral sequence is bigraded such that $E^1_{r,n}$ is the subgroup of E^1 generated by λ_I with length $I = r$ and deg $I = n$, deg $d^m = (m, -1)$. Furthermore E^2 and $H^*(A)$ are linked through

THEOREM 1.6.1 [5]. $E^2_{r,n} = H^{r,r+n}(A)$.

Given any derivation d of E^1 with deg $(1, -1)$, we write

$$d\lambda_n = \sum_{j \geqq 0} b_{n-j,j} \lambda_{n-1-j} \lambda_j$$

in admissible form. Comparing the coefficients of the admissible terms $\lambda_{(n-1,\ ,\ )}$ in $d(\lambda_n \lambda_{2n+1}) = 0$, we have

PROPOSITION 1.6.2.

$$(j - 1)b_{n-j,j} = \binom{n-1-j}{j} b_{n,0} \qquad \textit{for } j > 0.$$

Similarly compare the coefficients of the admissible terms $\lambda_{(n,\ ,\ )}$ in $d(\lambda_n \lambda_{2n+2}) = d(\lambda_{n+1} \lambda_{2n+1})$, we get

PROPOSITION 1.6.3. $b_{n-j,j} = b_{2n-2j,2j+2}$.

Equate the coefficients of the admissible term $\lambda_{n+1} \lambda_n \lambda_1$ in

$$d(\lambda_n \lambda_{2n+3}) = d(\lambda_{n+2} \lambda_{2n+1}),$$

we obtain the identity $b_{n,0} = b_{n+2,0}$.

PROPOSITION 1.6.4.

$$b_{n-j,j} = \binom{n-j-1}{j+1} b_{2,0}.$$

Proof.

$$b_{n-j,j} = b_{2n-2j,2j+2} = \binom{2n-2j-1}{2j+2} b_{2n+2,0} = \binom{2n-2j-2}{2j+2} b_{2,0}$$
$$= \binom{n-j-1}{j+1} b_{2,0} \qquad \text{(mod 2)}.$$

THEOREM 1.6.5. *d^1 is the only nontrivial derivation of E^1 with* deg $(1, -1)$ *and*

$$d^1\lambda_n = \sum_{j \geq 0} \binom{n-j-1}{j+1} \lambda_{n-j-1} \lambda_j .$$

Proof. From (1.3), we have $a_{2,0} = 1$ for d^1.

Remark. (1.5.4) and (1.6.5) are the admissible versions of (1.2) and (1.3) respectively.

1.7. *An important endomorphism θ of E^1.*

THEOREM 1.7.1. *There is a unique differential algebra endomorphism θ of E^1 which sends λ_j to λ_{2j+1} for $j \geq 0$.*

Proof. Let $\theta : E^1 \to E^1$ be defined by

$$\theta\lambda_I = \lambda_{2I+1}$$

where $2I + 1 = (2n_1 + 1, \cdots, 2n_r + 1)$ if $I = (n_1, \cdots, n_r)$. We have to verify that θ is well defined and satisfies the required condition. It suffices to show that θ is compatible with the defining relations and commutes with d^1.

$$\begin{aligned}\theta(\lambda_i \lambda_{2i+n+1}) = \lambda_{2i+1} \lambda_{4i+2n+3} &= \sum_{j \geq 0} \binom{2n-1-j}{j} \lambda_{2i+2n+1-j} \lambda_{4i+3+j} \\ &= \sum_{u \geq 0} \binom{2n-1-2u}{2u} \lambda_{2i+2n+1-2u} \lambda_{4i+3+2u} \\ &= \sum_{u \geq 0} \binom{n-1-u}{u} \theta\lambda_{i+n-u} \, \theta\lambda_{2i+1+u} .\end{aligned}$$

Hence θ is well defined. To see $d^1 \theta = \theta d^1$, we need only to check this on generators of E^1.

$$\begin{aligned}\theta\, d^1\lambda_n &= \theta\Big(\sum_{j \geq 0} \binom{n-1-j}{1+j} \lambda_{n-1-j} \lambda_j\Big) = \sum_{j \geq 0} \binom{n-1-j}{1+j} \lambda_{2n-2j-1} \lambda_{2j+1} \\ &= \sum_{j \geq 0} \binom{2n-2j-1}{2j+2} \lambda_{2n-2j-1} \lambda_{2j+1} = d^1\lambda_{2n+1} = d^1\theta\lambda_n .\end{aligned}$$

COROLLARY 1.7.2. *θ induces an endomorphism $\theta : H^*(A) \to H^*(A)$ which sends $h_i, c_i, d_i, \cdots$ to $h_{i+1}, c_{i+1}, d_{i+1}, \cdots$.*

Corollary 1.72 tells us that relations, which exist at low degrees, are stabilized by θ. For example, if we know $h_0 h_1 = 0$, then $h_i h_{i+1} = 0$ follows immediately for $i \geq 0$.

PROPOSITION 1.7.3. *$\theta : E^1 \to E^1$ is monic.*

Proof. If I is admissible, so is $2I + 1$. Thus that θ is monic is immediate.

1.8. In this subsection, we look more deeply into the multiplicative structure of E^1 and try to figure out some special properties concerning cycles of E^1. For brevity, we shall even use I for λ_I when no confusion is introduced.

LEMMA 1.8.1. *Let $(n_1, \cdots, n_r, m)$ be a sequence such that $(n_1, \cdots, n_r)$ is admissible and $n_1 + \cdots + n_r + r + n \geq m$, $n \geq n_1$; then in admissible form the leading integers are $\geq n_1$ and $\leq n$, unless $(n_1, \cdots, n_r, m) = 0$, i.e., $\lambda_{n_1} \cdots \lambda_{n_r} \lambda_m = 0$.*

Proof. This is obviously true for $r = 1$ and $m = 0$. Thus we make induction on r, m. Assume $2n_r + 1 < m$, and write

$$(n_r, m) = \sum_{j \geq 0} \binom{m-2n_r-2-j}{j} (m-n_r-1-j, 2n_r+1+j).$$

Since $n_1 + \cdots + n_r + (r-1) + n \geq m - (n_r + 1 + j)$, $n \geq n_1$ and by induction on r, the leading integers of the admissible form of

$$(n_1, \cdots, n_{r-1}, m - n_r - 1 - j)$$

are $\geq n_1$ and $\leq n$, or $(n_1, \cdots, n_{r-1}, m - n_r - 1 - j) = 0$. Note that $2n_r + 1 + j$ is always smaller than m. Again by induction on m, the leading integers of the admissible expression of

$$(n_1, \cdots, n_{r-1}, m - n_r - 1 - j, 2n_r + 1 + j)$$

are $\geq n_1$ and $\leq n$ or $(n_1, \cdots n_{r-1}, m - n_r - 1 - j, 2n_r + 1 + j) = 0$. However we have the identity

$$(n_1, \cdots, n_r, m)$$
$$= \sum_{j \geq 0} \binom{2m - 2n_r - 2 - j}{j} (n_1, \cdots, n_{r-1}, m - n_r - j - 1, 2n_r + 1 + j),$$

thus the Lemma follows immediately.

PROPOSITION 1.8.2.[1] *Let $(n_1, n_2, \cdots, n_{r+1})$ be any sequence admissible or not, and $e_i = n_{i+1} - 2n_i$, $1 \leq i \leq r$. Then in the admissible form the leading integers are $\geq n_1$ and $\leq \max (n_1 + \sum_{i=1}^{s} (e_i - 1); s = 0, 1, \cdots, r)$ unless $(n_1, n_2, \cdots, n_{r+1}) = 0$.*

Proof. We make induction on r. First express $(n_1, \cdots, n_r)$ in admissible form, then apply the preceding lemma.

PROPOSITION 1.8.3. *Let $(n_1, \cdots, n_r)$ be an admissible sequence; then $(d^1\lambda_{n_1})\lambda_{n_2} \cdots \lambda_{n_r}$ has leading integers $\leq n_1 - 1$ in its admissible form.*

Proof. Immediate from Lemma 1.8.1.

THEOREM 1.8.4. *Let $E^1(n)$ be the subgroup of E^1 generated by λ_I with I admissible and the leading integer of $I \leq n$. Then $E^1(n)$ is a subdifferential-algebra of E^1.*

Proof. By Lemma 1.8.1, $E(n)$ is closed under multiplication and by Proposition 1.8.3, $E^1(n)$ is closed under the action of d^1.

THEOREM 1.8.5. *Let $x = \lambda_n x_1 + x'$ be in admissible form, i.e., the leading integers of $x_1(x')$ are $\leq 2n$ $(n - 1)$. If $d^1x = 0$, then $d^1x_1 = 0$.*

Proof. Since $d^1x = 0$, we have $\lambda_n d^1x_1 = (d^1\lambda_n)x_1 + d^1x'$. From Proposition 1.8.3, the leading integers of the right hand side are $\leq n - 1$. However the leading integers of $\lambda_n d^1x_1$ are equal to n unless $d^1x_1 = 0$. Therefore we must have $d^1x_1 = 0$.

Similarly we have

PROPOSITION 1.8.6. *Let $x = \lambda_{2m} x_1 + x' + x''$ such that $\lambda_{2m} x_1 + x'$ is in admissible form and x'' is in* Im (θ). *If $d^1 x = 0$, then $d^1 x_1 = 0$.*

Proof. Since Im (θ) is a sub-differential-algebra of E^1, d^1x'' is in Im (θ).

[1] This was pointed out to the author by E. Curtis.

If $d^1x_1 \neq 0$, then $\lambda_{2m}\, d^1x_1$ is not killed by $d^1(x' + x'')$ and $d^1x \neq 0$ which contradicts our assumption.

PROPOSITION 1.9. *Let x be an element in $E^1_{r,n}$ with $n > 0$. If $d^1x = 0$, then in the homology class of x there is an element y with odd ending integers*[2] *unless x is a boundary.*

Proof. Assume that x is not a boundary. In the homology class of x, we choose y with minimum t such that $y = \sum_{i=0}^{t} y_i \lambda_0^i$ where the ending integers of y_i are $\neq 0$ for $0 \leq i \leq t$. We claim $t = 0$. Suppose not. Let

$$y_t = \sum (n_1, \cdots, n_s);$$

then n_s has to be odd, otherwise $(n_1, \cdots, n_s - 1)\lambda_0^{t+1}$ will appear in d^1x and $d^1x \neq 0$. Let

$$z_t = \sum (n_1, \cdots, n_s + 1)\lambda_0^{t-1};$$

$y + d^1z_t$ would give a contradiction to our choice of y. Hence $s = 0$, $y = y_0$ and the ending integers of y are clearly odd.

COROLLARY 1.10. *Let x be an element in $E^1_{r,n}$ such that $0 < n < r$. If $d^1x = 0$, then x is a boundary.*

Proof. Under the condition $0 < n < r$, in the homology class of x we cannot find y with odd ending integers. Thus x has to be a boundary.

COROLLARY 1.10′. *$H^{s,t}(A) = 0$ for $0 < t - s < s$.*

Proof. $H^{s,t}(A) = E^2_{s,t-s} = 0$ for $0 < t - s < s$.

PROPOSITION 1.11. *Let x be an element with odd ending integers. If x is a boundary, there is y with odd ending integers and $d^1y = x$.*

Proof. Same argument as that used in the proof of Proposition 1.9.

2. The structure of $E^2_{r,n}$ for $r \leq 3$

In this section we shall use the machinery developed in the preceding section to compute $E^2_{r,n}$ in the range $r \leq 3$.

LEMMA 2.1.

$$\binom{n-j}{j} = 0 \qquad \pmod 2$$

for $j > 0$, if and only if $n = 2^i - 1$ for a non-negative integer i.

Proof is easy and is omitted.

[2] When we speak of the ending (leading) integers of an element we always mean those in the admissible expression of the element.

PROPOSITION 2.2. *$h_i = \mathrm{cl}\,(\lambda_{2^i-1})$, the homology class of λ_{2^i-1}, for $i \geq 0$ form a basis for $\sum_n E^2_{1,n}$.*

Proof. Since

$$d^1\lambda_n = \sum_{j\geq 0} \binom{n-1-j}{1+j} \lambda_{n-1-j}\,\lambda_j,$$

$d^1\lambda_j = 0$ is equivalent to

$$\binom{n-k}{k} = 0 \qquad (\text{mod } 2)$$

for $k > 0$. Therefore by Lemma 2.1, the proposition is immediate.

PROPOSITION 2.3. *Let $\theta_2 : \sum_n E^2_{2,n} \to \sum_n E^2_{2,2n+2}$ be induced by θ; then θ_2 is monic.*

Proof. Let x be a cycle in $E^1_{2,n}$. If θx is a boundary, i.e., $\theta x = d^1\lambda_{2n+3} = d^1\theta\lambda_{n+1} = \theta\, d^1\lambda_{n+1}$. Since, by Proposition 1.7.3, θ is monic, we have $x = d^1\lambda_{n+1}$. Hence θ_2 is monic.

PROPOSITION 2.4. *Let x be a cycle in $E^1_{2,2m+1}$ with odd ending integers; then x is 0 or $\lambda_0\,\lambda_{2^i-1}$ for a non-negative integer i.*

Proof. We can assume that $x \neq 0$. Let $x = \lambda_{2m-2v}\,\lambda_{2v+1} + x'$ be the admissible form of x. Equate the admissible terms $(2m - 2v - 1, \ , \ )$ in $d^1x = 0$; we must have $\lambda_0\,\lambda_{2v+1} = 0$, that is, $v = 0$. Thus we can write $x = \lambda_0\,\lambda_{2m+1} + x''$ such that the leading integers of x'' are $\leq 2\mathrm{m} - 2$. Claim that $2m + 1 = 2^i - 1$ for a non-negative integer i and $x'' = 0$. If $2m + 2$ is not a power of 2, i.e., $2m + 1 = 2^{j+1}k + 2^j - 1$ for a $j \geq 1$, then we have

$$(2.5) \qquad d^1x'' = d^1(\lambda_0\,\lambda_{2n+1}) \begin{array}{ll} = (2^{j+1}k - 2, 1, 2^j - 1) + a' & \text{for } j \neq 2 \\ = (8k - 4, 3, 3) + a' & \text{for } j = 2, \end{array}$$

where the leading integers of a' are $< 2^{j+1}k - 2(8k - 4)$ for $j \neq 2$ $(= 2)$. Since $(1, 2^j - 1)$ and $(3,3)$ are not boundaries with the exception that $j = 0, 2$, the maximum leading integer $2m - 2u$ of x'' has to be larger than $2^jk - 2(8k - 4)$ for $j \neq 2$ $(= 2)$. Equate the admissible terms $(2m - 2u - 1, \ , \ )$ in (2.5); again we get $u = 0$. However this yields the contradiction that $2m = 2m - 2u \leq 2m - 2$. Hence $2m + 2$ is a power of 2 and $x'' = 0$ follows from the fact that x'' has odd ending integers, $d^1x'' = 0$ and the leading integers of x'' are $\leq 2m - 2$.

THEOREM 2.6. *$h_i\,h_j$ for $j \geq i$ and $j \neq i + 1$ form a basis for $\sum_n E^2_{2,n}$.*

Proof. Let w be an element in $E^2_{2,n}$ such that w is not in $\mathrm{Im}\,(\theta_2)$. Then n is either 0 or odd. By the preceding proposition, $w = h_0\,h_i$ for a non-negative integer $i \neq 1$. Since $\theta h_i = h_{i+1}$ and θ_2 is monic, the theorem follows easily.

PROPOSITION 2.7. *Let x be a cycle in $E^1_{3,2n+1}$; then there exists an element y*

in $E^1_{3,n-1}$ *such that* θy *is homologous to* x *if* $2n + 2$ *is not a power of* 2 *and* θy *is homologous to* x *or* $x + (0, 0, 2^r - 1)$ *if* $2n + 1 = 2^r - 1$.

Proof. By Proposition 1.9, we can assume that x has odd ending integers. Write $x = \lambda_{n_1} x_1 + \cdots + \lambda_{n_s} x_s$ in admissible form.If all n_i are odd, then x is already in Im (θ). Assume that not all n_i are odd. Let n_j be maximum among those even n_i. By Proposition 1.8.6, $dx_j = 0$ and by Proposition 2.5, $x_j = (0, 2^k - 1)$ for a $k \geq 2$. $k = 2$, otherwise $(n_j - 1, 2^k - 4, 2, 1)$ will appear in $d^1(\lambda_{n_j} x_j)$ and $d^1 x \neq 0$. Therefore x has to be of the form

$$x = (2n - 1, 1, 1) + (2n - 2, 2, 1) + (\text{terms with leading integers} \leq 2n - 3).$$

Through the defining relations of E^1, we can write

$$x = (0, 0, 2n + 1) + x'(\text{terms with leading integers} \leq 2n - 3).$$

If $2n + 1 = 2^r - 1$, then $x + \lambda_0^2 \lambda_{2^r-1} = x'$ is in Im (θ) otherwise, as what we have just proved above for x, the maximum leading integer of x' is $2n - 1$. If $2n + 1 = 2^{m+1}k + 2^m - 1$ for an $m \geq 1$, the part with maximum even leading integers of $d^1(0, 0, 2n + 1)$ is

$$\begin{array}{ll} (8k - 7, 2, 3, 3) & m = 2 \\ (16k - 8, 6, 1, 7) & m = 3 \\ (2^{m+1}k - 4, 2, 1, 2^m - 1) & m > 3. \end{array}$$

Let $2t$ be the maximum even leading integer of x'. Since $(2, 3, 3)$, $(6, 1, 7)$ and $(2, 1, 2^m - 1)$ for $m > 3$ are not boundaries, $2t$ is larger than $8k-7$ $(16k - 8, 2^{m+1}k - 4)$ for $m = 2$ $(m = 3, m > 3)$. Equate the terms $(2t, \ , \ , \ )$ and $(2t - 1, \ , \ , \ )$ in $d^1(0, 0, 2n + 1) = d^1 x'$. Again we lead to $2t = 2n - 2$ which contradicts the fact that $2t \leq 2n - 3$. If $2n + 1 = 1 + 4k$, we lead to the contradiction that $(1, 1, 1)$ is a boundary. Henceforth we draw the conclusion that if $2n + 2$ is not a power of 2, x is already in Im (θ), and if $2n + 1 = 2^r - 1$ either x or $x + (0, 0, 2^r - 1)$ is in Im (θ).

In the following when we write $x = \lambda_n x_1 + x'$ we always mean that this is in admissible form, i.e., the leading integers of $x_1(x')$ are $\leq 2n(n - 1)$ except otherwise explained.

PROPOSITION 2.8. *Let* x *be an element in* $E^1_{3,2n}$ *with odd ending integers. If* $x = (2m, 3, 3) + x'$ *and its maximum leading integer* $2m$ *is smallest among those homologous to* x *with odd ending integers, then* x *is not a cycle for* $m > 1$.

Proof. Case 1. $1 < m = 2k + 1$.

$$d^1(2m, 3, 3) = d^1(4k + 2, 3, 3) = (4k - 1, 2, 3, 3) + a'.$$

$(4k - 1, 2, 3, 3)$ is not killed by $d^1 x'$, hence x is not a cycle.

Case 2. $m = 4k + 2$. Then $x + d^1(0, 2m + 7)$ would give a contradiction about the minimum of $2m$.

Case 3. $m = 4k$.

$$d^1(2m, 3, 3) = d^1(8k, 3, 3) = (8k - 4, 3, 3, 3) + b'$$

and $(8k - 4, 3, 3, 3)$ is retained in d^1x.

Let $x = \lambda_{2m} x_1 + x'$ be a boundary in $E^1_{4,2t+1}$ with odd ending integers. Among all the z's in $E^1_{3,2t+2}$ with odd ending integers such that $d^1z = x$, we choose $y = \lambda_n y_1 + y'$ with minimum n. Then we have the following

PROPOSITION 2.9. (i) *n is even and $n \geq 2m$,*

(ii) *if $n = 2m$, x_1 is a boundary,*

(iii) *if $n > 2m$, we can assume that $y_1 = (1, 2^j - 1), (3, 3)$.*

Proof. (i) Suppose on the contrary that we have odd n. By Proposition 2.4, $y_1 = (0, 2^i - 1)$ with an $i \geq 2$. Then $y + d^1(n + 1, 2^j - 1)$ has a smaller n which contradicts our choice of y. Hence $n = 2r \geq 2m$. (ii) is trivial.

(iii) If $2r > 2m$, then equating the admissible terms $(n, \ \ , \ \ , \ \ )$ and $(n - 1, \ \ , \ \ , \ \ )$ we get $d^1y_1 = 0$ and $\lambda_0 y_1$ is a boundary. y_1 is not a boundary, otherwise n can be made smaller. Therefore $y_1 = (2^i - 1, 2^j - 1)$ with $i - j \neq \pm 1$ and $(0, 2^i - 1, 2^j - 1)$ homologous to 0 implies $i = j = 2$ or $\min(i, j) = 1$. In case $j = 1$, $y + d^1(n, 2^i + 1)$ gives the desired element.

PROPOSITION 2.10. *For $n > 0$, $E^2_{3,2n}$ is generated by c_0, $c_0 = \mathrm{cl}\,(2, 3, 3)$, $2n = 8$; $h_0 h_i h_j$, $2n = 2^i + 2^j - 2$; 0, otherwise.*

Proof. We shall show that any cycle y in $E^1_{3,2n}$ is homologous to 0, $(2, 3, 3)$, $(0, 2^i - 1, 2^j - 1)$. We prove this assertion by induction on the maximum leading integer of y. Assume that y is not a boundary, y has odd ending integesrs and its maximum leading integer is minimum among elements homologous to y with odd ending integers. Then by Propositions 2.9 and 2.8, $y = (2, 3, 3)$ or $y = (2r, 1, 2^k - 1) + y'$, for a $k > 2$. Hence we need only to consider the latter case. Write

$$y = (0, 2r + 1, 2^k - 1) + y'' \text{ (terms with leading integers } \leq 2r - 1).$$

If $2m + 1 = 2^t - 1$, by induction the assertion is true for y'' and hence true for y. If $2m + 2$ is not a power of 2, by the method of equating admissible terms and Proposition 2.9 we would lead to contradiction.

THEOREM 2.11. *$E^2_{3,n}$ is generated by c_i, $n = 2^{i+1} + 2^{i+2} + 2^{i+3} - 3$; $h_i h_j h_k$, $n = 2^i + 2^j + 2^k - 3 \neq 2^m + 2^{m+1} + 2^{m+2} - 3$; 0, otherwise.*

Proof. Since $\theta h_i = h_{i+1}$, $\theta c_i = c_{i+1}$, the theorem follows from Propositions 2.7 and 2.10 immediately.

3. Some relations in $E^2_{r,n}$ for $r \leq 4$

The following two propositions are immediate from the defining relations of E^1.

PROPOSITION 3.1. $\lambda_{2^i-1}\lambda^r_{2^i+r-1} = 0, \lambda^i_{2^i-1}\lambda_{2^i-1+r2^i} = 0$ *for* $r > 0$.

PROPOSITION 3.2. $\lambda^r_{2^i-1}\lambda_{2^i-1+(r+1)2^i} = \lambda^{r+1}_{2^{i+1}-1}$ *for* $r > 0$.

THEOREM 3.3. (1) *The only relations among* h_i*'s in* $E^2_{r,n}$ *for* $r \leq 3$ *are* $h_i h_{i+1} = 0, h_i h_j = h_j h_i, h_i h_{1+i} h_j = 0, h_i h^2_{i+2} = 0, h^2_i h_{i+2} = h^3_{i+1}$ *for* $i, j \geq 0$.

Proof. All these relations follow easily from Propositions 3.1 and 3.2. We leave the other part to the readers.

PROPOSITION 3.4. $h_0 h^3_i \neq 0$ *for* $i \geq 4$.

Proof. Let

$$x = (0, 2^i - 1, 2^i - 1, 2^i - 1, 2^i - 1) = (2^i - 2, 1, 2^i - 1, 2^i - 1) + a'.$$

If x is a boundary, choose $y = \lambda_{2r} y_1 + y'$ with $d^1 y = x$ as we have done in Proposition 2.9. Since $(1, 2^i - 1, 2^i - 1)$ is not a boundary for $i \geq 4$, by Proposition 2.9, $y_1 = (3, 3)$ or $(1, 2^j - 1)$ with $i \geq j \geq 3$. If $y_1 = (3, 3)$, then $(2^{i+1} + 2^i - 12, 3, 3, 3)$ will appear in $d^1(\lambda_{2r} y_1)$ and retain in $d^1 y$. Thus we lead to the contradiction that $d^1 y \neq x$. Therefore we must have $y_1 = (1, 2^j - 1)$ with $i \geq j \geq 3$ and $2r = 2^{i+1} + 2^i - 2^j - 2$. Write

$$y = (0, 2r + 1, 2^j - 1) + y'' \text{ (terms with leading integers } \leq 2r - 1).$$

If $j = i$, i.e., $2r + 1 = 2^{i+1} - 1$, y'' gives contradiction to our choice of y. Hence $i > j \geq 3$ and

$$\begin{aligned} d^1(0, 2r + 1, 2^j - 1) & \\ &= (2^{i+1} + 2^i - 2^{j+1} - 2, 1, 2^j - 1, 2^j - 1) + b', \quad j \geq 4 \\ &= (2^{i+1} + 2^i - 18, 7, 7, 7) + b', \quad j = 3. \end{aligned}$$

However the term

$$(2^{i+1} + 2^i - 2^{j+1} - 2, 1, 2^j - 1, 2^j - 1)$$

for $j \geq 4((2^{i+1} + 2^i - 18, 7, 7, 7)$ for $j = 3)$ keeps in $d^1 y$. Again we lead to contradiction.

Similarly we have

PROPOSITION 3.5. $h_0 c_i \neq 0$ *for* $i \geq 2$.

THEOREM 3.6. $d^2 h_{i+1} = h_0 h^2_i$ *for* $i \geq 3$.

Proof. It is well known that h_3 survives in E^∞. Let h_3' be the element corresponding to h_3 in the stable homotopy group of sphere. Since $\dim h_3' = 7$,

$2h_3'^2 = 0$. Hence $h_0 h_3^2$ can not survive in E^∞. The only chance that $h_0 h_3^2$ is killed, is in the process from E^2 to E^3. Hence $d^2 h_4 = h_0 h_3^2$. By induction suppose $d^2 h_n = h_0 h_{n-1}^2$ for $i > n \geq 4$. Then from $h_i h_{i+1} = 0$, we have $(d^2 h_i) h_{i-1} = h_0 h_{i-1}^3$. Since $i - 1 \geq 4$, $h_0 h_{i-1}^3 \neq 0$. Hence $d^2 h_i \neq 0$. By Theorem 2.11 $E^2_{3,2^i-2}$ is generated by $h_0 h_{i-1}^2$, thus $d^2 h_i = h_0 h_{i-1}^2$ follows easily.

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