NONEXPANSIVE MAPPINGS AND FIXED-POINTS IN BANACH SPACES[1]

BY

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A subset K of a Banach space B has *normal structure* [2] if for each bounded convex subset H of K which contains more than one point there is a point $x \in H$ which is not a diametral point of H (that is, $\sup \{\| x - y \| : y \in H\} < \delta(H)$). We proved in an earlier paper [1] that if K is a bounded, nonempty, weakly compact, convex subset of a Banach space B, and if K has normal structure then every *finite* family $\mathfrak{F}$ of commuting nonexpansive mappings of K into itself has a common fixed-point. (A mapping f on K is nonexpansive if $\| f(x) - f(y) \| \leqq \| x - y \|$ for each $x, y \in K$.) If the norm of B is strictly convex then this theorem holds for infinite families. (For if the norm is strictly convex then the fixed-point set for each $f \in \mathfrak{F}$ is nonempty, bounded, closed and convex. Hence these fixed-point sets are weakly compact and have the finite intersection property; thus there is a point common to all of them.)

Although we do not know whether this theorem is true in general for infinite families, we show in this paper that by appropriately strengthening the condition of normal structure we are able to establish the existence of a common fixed-point for arbitrary families *without assuming strict convexity of the norm.* After proving a consequence of this, an observation is made about characterizations of Hilbert space due to Klee [6] and Phelps [7]. Finally, we show that the stronger version of normal structure introduced in this paper holds in compact convex sets and in closed convex subsets of uniformly convex Banach spaces.

1. Notation and definitions

Throughout the paper, the symbol $\delta(A)$ will denote the diameter of A, that is, $\delta(A) = \sup \{\| x - y \| : x, y \in A\}$, and $\overline{\mathrm{co}}\, A$ will denote the closed convex hull of A. For $x \in B$, $\mathfrak{U}(x; r)$ and $\overline{\mathfrak{U}}(x; r)$ will denote, respectively, the open and closed spherical ball centered at x with radius r.

For subsets H and K of B, H bounded, let

$$r_x(H) = \sup \{\| x - y \| : y \in H\},$$

$$r(H, K) = \inf \{r_x(H) : x \in K\},$$

$$\mathcal{C}(H, K) = \{x \in K : r_x(H) = r(H, K)\}.$$

The set $\mathcal{C}(H, B)$ is frequently referred to as the Chebyshev center of H in B.

Received July 11, 1966.

[1] Research supported by a National Science Foundation grant.

We shall accordingly refer to $\mathcal{C}(H, K)$ as the *Chebyshev center of H in K*. In general, $\mathcal{C}(H, K)$ may be empty, but otherwise it consists of precisely those points of K which serve as centers of balls of minimal radius, $r(H, K)$, which contain H.

LEMMA. *If K is weakly compact and convex, and if H is bounded, then $\mathcal{C}(H, K)$ is a nonempty closed convex subset of K.*

Proof. The argument given for Lemma 1 of [5] establishes this. For $x \in H$, let

$$F(x, n) = \{y \in K : \| x - y \| \leqq r(H, K) + 1/n\}.$$

The sets $C_n = \bigcap_{x \in H} F(x, n)$ form a descending chain of nonempty closed convex sets in K; hence these sets are weakly compact and their intersection, $\mathcal{C}(H, K)$, is nonempty.

DEFINITION. Let K be a bounded closed convex subset of B. We say that K has *complete normal structure* (c.n.s.) if every closed convex subset W of K which contains more than one point satisfies the following condition:

$(*)$ *For every decreasing net $\{W_\alpha : \alpha \in A\}$ of subsets of W which have the property that $r(W_\alpha, W) = r(W, W)$, $\alpha \in A$, it is the case that the closure of $\bigcup_{\alpha \in A} \mathcal{C}(W_\alpha, W)$ is a nonempty proper subset of W.*

We shall subsequently show that any compact convex set and any bounded convex subset of a uniformly convex space have this property. That the above implies normal structure is seen by taking $W_\alpha \equiv W$ in $(*)$.

2. Fixed-point theorem

The first theorem of this section generalizes results of De Marr [4], who assumed K compact, and F. Browder [3] who proved its analogue in uniformly convex spaces.

THEOREM 2.1. *Suppose K is a weakly compact, convex subset of a Banach space B, and suppose that K has complete normal structure. Let $\mathfrak{F}$ be a commutative family of nonexpansive mappings of K into itself. Then there is a point $x \in K$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.*

Proof. Because closed convex subsets of K are weakly compact, we may use Zorn's Lemma to obtain a subset K^* of K which is minimal with respect to being nonempty, closed, convex, and mapped into itself by each member of $\mathfrak{F}$. If K^* consists of a single point, this is the desired fixed-point. We assume, then, that $\delta(K^*) > 0$ and obtain a contradiction.

Let $\mathfrak{A}$ be the family of all nonempty finite subsets of $\mathfrak{F}$. Since each finite subcollection of mappings of $\mathfrak{F}$ has a common fixed-point [1, Theorem 3], for $\sigma \in \mathfrak{A}$, the sets

$$M_\sigma = \{x \in K^* : f(x) = x \text{ for each } f \in \sigma\}$$

are nonempty.

Let σ_0 be an arbitrary but fixed element of $\mathfrak{A}$, and let $r = r(M_{\sigma_0}, K^*)$. Then

$$r \leqq r(K^*, K^*). \tag{1}$$

For $\sigma_0 \subset \sigma$, let

$$H_\sigma = \{x \in K^* : M_\sigma \subset \bar{\mathfrak{U}}(x; r)\}.$$

The collection $\{H_\sigma : \sigma \in \mathfrak{A}, \sigma_0 \subset \sigma\}$ forms an increasing net of convex sets, and in particular,

$$H_{\sigma_0} = \mathfrak{C}(M_{\sigma_0}, K^*).$$

Let $H = \{x \in K^* : x \in H_\sigma, \sigma_0 \subset \sigma\}$. If $x \in H$ and $f \in \mathfrak{F}$ then there exists $\sigma_1 \in \mathfrak{A}$ with $\sigma_0 \subset \sigma_1$, such that $f \in \sigma_1$ and $x \in H_{\sigma_1}$. Hence, for $z \in M_{\sigma_1}$,

$$\| f(x) - z \| = \| f(x) - f(z) \| \leqq \| x - z \| \leqq r$$

and thus $f(x) \in H_{\sigma_1}$. From this it follows that $f(H) \subset H, f \in \mathfrak{F}$. Since the closure $\bar{H}$ of H is thus mapped into itself by each member of $\mathfrak{F}$, and since it is convex, the minimality of K^* implies that $\bar{H} = K^*$.

Let $\varepsilon > 0$ and let $x \in K^*$. Then, since points of H are arbitrarily near x, $\bar{\mathfrak{U}}(x; r + \varepsilon)$ contains M_σ for some $\sigma \in \mathfrak{A}$. This, with the fact that the weakly compact sets $\{\overline{\text{co}}\ M_\sigma : \sigma \in \mathfrak{A}\}$ have the finite intersection property, implies

$$\emptyset \neq \bigcap_{\sigma_0 \subset \sigma} \overline{\text{co}}\ M_\sigma \subset \bigcap_{x \in K^*} \bar{\mathfrak{U}}(x; r + \varepsilon).$$

Therefore $r(K^*, K^*) \leqq r + \varepsilon$. This, along with (1), implies

$$r = r(M_{\sigma_0}, K^*) = r(K^*, K^*) \tag{2}$$

where σ_0 is an arbitrary element of $\mathfrak{A}$. Since the sets $\{M_\sigma, \sigma \in \mathfrak{A}\}$ form a decreasing net in K^* we may apply condition $(*)$ of complete normal structure to conclude that the closure of

$$W = \bigcup_{\sigma \in \mathfrak{A}} \mathfrak{C}(M_\sigma, K^*)$$

is a proper subset of K^*. But $\bar{W}$ is mapped into itself by each member of $\mathfrak{F}$, since (2) implies

$$H_\sigma = \mathfrak{C}(M_\sigma, K^*), \qquad \sigma \in \mathfrak{A}.$$

This contradicts the minimality of K^* and we therefore conclude $\delta(K^*) = 0$, and K^* consists of a single point which is fixed under each of $f \in \mathfrak{F}$.

If condition $(*)$ of complete normal structure is replaced by $(*_0)$ where only countable nets (or decreasing sequences) are considered, we say that K has *countable normal structure.*

COROLLARY. *If $\mathfrak{F}$ is countable, or if K is separable, then complete normal structure in Theorem* 2.1 *may be replaced by countable normal structure.*

Proof. If $\mathfrak{F}$ is countable, then the above argument carries through without change under the weakened assumption of countable normal structure.

Each fixed-point set M_f of $f \in \mathfrak{F}$ is closed and nonempty; hence complements of these sets form an open cover $\{\mathfrak{U}\}$ for K (assuming $\bigcap_{f \in \mathfrak{F}} M_f = \emptyset$). If K were separable then some countable subcollection of sets of $\{\mathfrak{U}\}$ would cover K. But this would imply that some countable subcollection of elements of $\mathfrak{F}$ have no common fixed-point—a contradiction.

3. Characterization of Hilbert space

In this section we point out a consequence of a characterization of Hilbert space due to Klee [6].

We say that a Banach space B has *property* (R) if any bounded closed convex subset K of B has the fixed-point property with respect to nonexpansive mappings. All uniformly convex spaces have this property [3] and more generally, so do all reflexive Banach spaces which have normal structure [5].

THEOREM 3.1. *Let B be a reflexive Banach space of dimension at least* 3, *which has property* (R). *Then B is a Hilbert space if and only if every nonempty, bounded, closed and convex subset of B is the convex closure of the fixed-point set of some nonexpansive mapping on B.*

Proof. Suppose K is the convex closure of the fixed-point set M of some nonexpansive mapping T of B into B. If $x \in \mathcal{C}(M, B)$, then for each $z \in M$,

$$\| T(x) - z \| = \| T(x) - T(z) \| \leqq \| x - z \| \leqq r_x(M).$$

Thus, $r_{T(x)}(M) \leqq r_x(M) = r(M, B)$. Hence $T(x) \in \mathcal{C}(M, B)$; this and property (R) imply $\mathcal{C}(M, B) \cap M \neq \emptyset$. Since $\mathcal{C}(M, B) = \mathcal{C}(K, B)$, we have shown that $\mathcal{C}(K, B) \cap K \neq \emptyset$ for any bounded closed convex subset K of B. By a theorem of Klee [6, Corollary 2], B is a Hilbert space.

On the other hand, if B is a Hilbert space, then the *metric projection* (nearest point map) of B onto any bounded closed convex subset K of B is nonexpansive (see Phelps [7]), and has precisely K as its fixed-point set. This completes the proof.

It is not known whether every reflexive Banach space has property (R). Although we have characterized Hilbert space among those reflexive spaces which do, it would be interesting to know if our condition characterizes Hilbert space in general. In these spaces, we have weakened the condition of Phelps [7, Theorem 5.2] that every bounded closed convex set be a fixed-point set of a particular nonexpansive mapping—the metric projection.

4. Complete normal structure

In this section we take a closer look at the concept of complete normal structure as defined in Section 1.

THEOREM 4.1. *If K is a bounded, closed, convex subset of a uniformly convex Banach space then K has c.n.s.*

Proof. Let W be a closed convex subset of K which contains more than one point, and let $\{W_\alpha : \alpha \in A\}$ be a decreasing net of subsets of W. For $\alpha \in A$, suppose there are distinct points $z_1, z_2 \in \mathcal{C}(W_\alpha, W)$. Then if $x \in W$,

$$\| z_1 - x \| \leqq r(W_\alpha, W) \quad \text{and} \quad \| z_2 - x \| \leqq r(W_\alpha, W).$$

Let $m = \frac{1}{2}(z_1 + z_2)$. The uniform convexity of the space implies the existence of a positive number δ such that $\| m - x \| \leqq (1 - \delta) r(W_\alpha, W)$. But this implies that for some $m \in W$,

$$r_m(W_\alpha) < r(W_\alpha, W)$$

which contradicts the way in which these numbers are defined. Thus, *for each* $\alpha \in A$, $\mathcal{C}(W_\alpha, W)$ *consists of a single point.* If $r(W_\alpha, W) = r(W, W)$ for each $\alpha \in A$, it follows that

$$\mathcal{C}(W_\alpha, W) = \mathcal{C}(W, W), \qquad \alpha \in A,$$

and hence $\bigcup_{\alpha \in A} \mathcal{C}(W_\alpha, W)$ will consist of precisely one point; thus it must be a proper subset of W.

THEOREM 4.2. *If K is a compact convex subset of a Banach space then K has c.n.s.*

Proof. Again let W be a closed convex subset of K which contains more than one point, and let $\{W_\alpha : \alpha \in A\}$ be a decreasing net of subsets of W for which $r(W_\alpha, W) = r(W, W)$, $\alpha \in A$. We order A in the natural way, that is, $\alpha \geqq \beta$ provided $W_\alpha \subset W_\beta$. There is no loss in generality if it is assumed that the sets W_α are closed (or, in fact, closed and convex) since $r(W_\alpha, W)$ remains unchanged.

We are actually able to prove the following:

$$\overline{\bigcup_{\alpha \in A} \mathcal{C}(W_\alpha, W)} \subset \mathcal{C}(\textstyle\bigcap_{\alpha \in A} W_\alpha, W).$$

To see this let $\varepsilon > 0$ and let $r = r(\bigcap_{\alpha \in A} W_\alpha, W)$. Because the sets W_α are compact, for α sufficiently large,

$$(1) \qquad \sup_{y \in W_\alpha} [\inf \{ \| x - y \| : x \in \textstyle\bigcap_{\alpha \in A} W_\alpha \}] < \varepsilon.$$

Hence if $x \in \mathcal{C}(\bigcap_{\alpha \in A} W_\alpha, W)$ then $W_\alpha \subset \bar{\mathfrak{U}}(x; r + \varepsilon)$ for α sufficiently large. Therefore $r(W_\alpha, W) \leqq r + \varepsilon$; since ε is arbitrary $r(W, W) = r(W_\alpha, W) \leqq r$, $\alpha \in A$. But clearly $r(W_\alpha, W) \geqq r$ since $\bigcap_{\alpha \in A} W_\alpha \subset W_\alpha$. Thus

$$r(W_\alpha, W) = r(\textstyle\bigcap_{\alpha \in A} W_\alpha, W), \qquad \alpha \in A.$$

If $x \in \mathcal{C}(W_\alpha, W)$ then since

$$\textstyle\bigcap_{\alpha \in A} W_\alpha \subset W_\alpha \subset \bar{\mathfrak{U}}(x : r),$$

we conclude

$$\overline{\bigcup_{\alpha \in A} \mathcal{C}(W_\alpha, W)} \subset \mathcal{C}(\textstyle\bigcap_{\alpha \in A} W_\alpha, W).$$

This is all that is needed to complete the proof of the theorem. Since $\bigcap_{\alpha \in A} W_\alpha$ is compact, some point x in $H = \overline{\text{co}} \bigcap_{\alpha \in A} W_\alpha$ is not a diametral point of H (see [4, Lemma 1]). Because $r(H, W) = r(\bigcap_{\alpha \in A} W_\alpha, W)$, we see that

$r(H, W) < \delta(H) \leqq \delta(W)$. Therefore $\mathcal{C}(H, W)$ is a proper subset of W, because $\delta(\mathcal{C}(H, W) \cap H) \leqq r(H, W)$. Since $\mathcal{C}(H, W) = \mathcal{C}(\bigcap_{\alpha \in A} W_\alpha, W)$ we see that $\mathcal{C}(\bigcap_{\alpha \in A} W_\alpha, W)$, and hence the closure of

$$\bigcup_{\alpha \in A} \mathcal{C}(W_\alpha, W)$$

is a proper subset of W, completing the proof.

We know of no example of a weakly compact convex set K which does not possess normal structure (or complete normal structure).

The authors thank the referee for a suggestion which improved the original version of this manuscript.

Added in Proof. Mr. Stanley Weiss has recently provided us with an example, which he attributes to R. C. James, of a weakly compact convex set which does not possess normal structure.

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