THE HOMOTOPY CATEGORY OF SPECTRA. I

BY

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The objective of this paper is to show that the homotopy category of semisimplicial spectra $\bar{\mathcal{S}}p_E$ in the sense of Kan [4] can be fully embedded in a very convenient manner into an abelian category $\mathcal{A}\bar{\mathcal{S}}p_E$ (Theorem 6.1). We mean by this the following: $\bar{\mathcal{S}}p_E$ coincides with the full subcategory of projectives of $\mathcal{A}\bar{\mathcal{S}}p_E$, $\mathcal{A}\bar{\mathcal{S}}p_E$ has enough injectives and projectives and the injectives and projectives coincide, and there exists a one-to-one correspondence between exact functors on $\mathcal{A}\bar{\mathcal{S}}p_E$ to an abelian category and functors on $\bar{\mathcal{S}}p_E$ to the same category which transform mapping cone sequences into exact sequences. Peter Freyd has proved [7] a general theorem according to which there exists for an additive category satisfying certain conditions a full embedding into an abelian category having properties of the above type; he has applied this to the stable category. It is the work of Freyd which suggested to the authors the considerations of the present paper.

In a letter to a friend of the authors R. L. Knighten stated that he knew some of the results below.

The point of view developed here facilitates the study of some questions concerning the structure of the homotopy category of spectra, such as the Postnikov resolutions and others, which will be dealt with in a subsequent paper. We believe that the homotopy category of spectra is important since it permits the classification of additive generalized cohomology theories.

In §1 we have collected for the convenience of the reader a few notions and results due to D. Kan and G. W. Whitehead, and we have adapted their covering homotopy theorem to our needs.

The main result is contained in Theorem 6.1, and the rest of the paper is devoted to setting up the machinery we need to prove this theorem.

The results contained in this paper have been announced in [1].

1. Preliminaries

The category of semisimplicial spectra $\mathcal{S}p$. The objects are semisimplicial spectra defined as follows: A semisimplicial spectrum X consists of

(i) for *every* integer q a set $X_{(q)}$ with a distinguished element $*$ (called *base point*); the elements of $X_{(q)}$ will be called *simplices* of *degree* q,

(ii) for every integer q and every integer $i \geq 0$ a function

$$d_i : X_{(q)} \to X_{(q-1)}$$

such that $d_i * = *$ (the *i-face* operator), and a function

$$s_i : X_{(q)} \to X_{(q+1)}$$

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such that $s_i * = *$ (the *i-degeneracy* operator). These operators are required to satisfy the following axioms:

I. The following identities hold: $d_i d_j = d_{j-1} d_i$ for $i < j$; $s_i s_j = s_j s_{i-1}$ for $i > j$; $d_i s_j = s_{j-1} d_i$ for $i < j$; $d_i s_j =$ identity for $i = j, j + 1$; $d_i s_j = s_j d_{i-1}$ for $i > j + 1$.

II. For every simplex $\alpha \in X$ all but a finite number of its faces are the base point, i.e. there is an integer n (depending on α) such that $d_i \alpha = *$ for $i > n$.

The morphisms are maps $f : X \to Y$ which are degree-preserving and commute with all face and degeneracy operators.

The *cone functor* $C : \mathcal{S}p \to \mathcal{S}p$ is defined as follows:

$$(CX)_{(q)} = \{(p, \alpha) \mid p \geqq 0, \alpha \in X_{(q-p)}, \text{ where } (p, *) \text{ is identified with } *\}$$

and

$$\begin{aligned} d_i(p, \alpha) &= (p - 1, \alpha), & s_i(p, \alpha) &= (p + 1, \alpha), & i &< p, \\ &= (p, d_{i-p}\, \alpha) & &= (p, s_{i-p}\, \alpha), & i &\geq p. \end{aligned}$$

The *suspension functor* $S : \mathcal{S}p \to \mathcal{S}p$ is defined as follows: SX is obtained from CX by identifying, for every $\alpha \in X$, the simplex $(0, \alpha)$ with the appropriate base point.

For a family $(X_i)_{i \in I}$ of objects of $\mathcal{S}p$, the direct sum $\bigvee_{i \in I} X_i$ and the direct product $\times_{i \in I} X_i$ are defined by $(\bigvee_{i \in I} X_i)_{(q)} = \bigvee_{i \in I} (X_i)_{(q)}$ (i.e. the union of the $(X_i)_{(q)}$'s with the base points identified) and $(\times_{i \in I} X_i)_{(q)} = \times_{i \in I} (X_i)_{(q)}$ with the system $(*)_{i \in I}$ as base point.

For any subspectrum A of a spectrum X, we denote by $X \backslash A$ the spectrum for which $(X \backslash A)_{(q)}$ is the set obtained from $X_{(q)}$ by identifying $A_{(q)}$ to $*$. If $f : A \to Y$ is a map of spectra, where A is a subspectrum of X, we denote by $X \cup_f Y$ the spectrum for which $(X \cup_f Y)_{(q)}$ is obtained from $X_{(q)} \cup Y_{(q)}$ by identifying each $x \in X_{(q)}$ with $f(x) \in Y_{(q)}$. In particular, if we identify X with the subspectrum of CX consisting of the simplices of the form $(0, \alpha)$ and if $f : X \to Y$ is a map of spectra, then we denote by C_f the spectrum $CX \cup_f Y$.

We denote by $\tilde{}\ : \mathcal{S}p \to \mathcal{S}p$ the functor (which is an automorphism of $\mathcal{S}p$) which assigns to each spectrum X the spectrum $\tilde{X}$ defined by $\tilde{X}_{(q)} = X_{(q+1)}$.

For any integer $n > 0$, we denote by $\tilde{}^{\,n}$ the n-th iterate of the automorphism $\tilde{}$, and by $\tilde{}^{\,-n}$ the n-th interate of the inverse of this automorphism.

For any spectrum X we can consider the spectrum $I \cdot X$ [3, ch. 2] and the inclusions $j_1, j_2 : X \to I \cdot X$ which identify X with subspectra of $I \cdot X$. This is the analogue of the cylinder and it permits the definition of the homotopy relation.

Let $\mathcal{S}p_E$ be the full subcategory of $\mathcal{S}p$ consisting of Kan spectra, i.e. spectra satisfying the extension condition (cf. [4, 7.3]).

Let $\mathcal{S}p_G$ be the subcategory of $\mathcal{S}p_E$ whose objects are group spectra, i.e. spectra for which each $X_{(q)}$ is a group and the d_i's and s_i's are homomorphisms, and whose morphisms are maps of spectra which are homomorphisms for each q.

Finally, let $\mathcal{S}p_L$ be the full subcategory of $\mathcal{S}p_G$ consisting of free group spectra.

We denote by $F : \mathcal{S}p \to \mathcal{S}p_L$ the functor which associates to each X the spectrum $F(X)$ defined as follows: $(FX)_{(q)}$ is the group with a generator $F\alpha$ for every $\alpha \in X_{(q)}$ and one relation $F* = *$; the face and degeneracy homomorphisms are given by $d_i F\alpha = F\, d_i\, \alpha$, $s_i F\alpha = Fs_i\, \alpha$ $(i \geq 0)$.

If $K : \mathcal{S}p_G \to \mathcal{S}p_E$ is the inclusion functor, there exists a functorial morphism $i : id_{\mathcal{S}p_E} \to KF$ [4] given by $i(X)(\alpha) = F\alpha$. Whenever there will be no danger of confusion, we shall write simply $F(X)$ instead of $KF(X)$.

There are two equivalent definitions for the concept of homotopy of two maps of a spectrum into a Kan spectrum.

DEFINITION 1 [4]. $f_1, f_2 : X \to Y$, where $Y \in \mathcal{S}p_E$ are said to be *homotopic* if there is a map $f : I \cdot X \to Y$ such that $fj_1 = f_1$, $fj_2 = f_2$.

DEFINITION 2 [6, App. A]. $f_1, f_2 : X \to Y$, where $Y \in \mathcal{S}p_G$ are said to be *homotopic* if $f_1 f_2^{-1} : X \to Y$ can be extended to a map $f : CX \to Y$.

If $Y \in \mathcal{S}p_E$, f_1, f_2 are said to be homotopic if

$$i(Y)f_1,\ i(Y)f_2 : X \to FY$$

are homotopic.

In [6], App. A, it is shown that the two definitions coincide.

Homotopy groups can be defined as in [4] and [5], and homology groups as in [5]. Homotopic maps induce the same homomorphism for the homotopy and homology groups.

DEFINITION 3. A map $f : X \to Y$ where $X, Y \in \mathcal{S}p$, is said to be a *weak homotopy equivalence* if it induces isomorphisms for the homotopy groups.

According to the definition of homotopy groups, $i(X)$ is a weak homotopy equivalence for every X.

The notion of a strong homotopy equivalence for objects of $\mathcal{S}p_E$ is defined as in [4, 8.1]. It is shown in [4] that for spectra in $\mathcal{S}p_E$ weak and strong homotopy equivalences coincide and therefore we shall simply say homotopy equivalences.

We use fibrations in the sense of [5, (5.1)]. By [5, (5.5)], if X is a subspectrum of Y, then the sequence

$$X \xrightarrow{j} Y \xrightarrow{p} Y\backslash X$$

is a fibration, where j denotes the inclusion and p the identification map.

THE HOMOTOPY EXTENSION THEOREM [6, A.8]. *Let $X \in \mathcal{S}p$, $Y \in \mathcal{S}p_G$ and let A be a subspectrum of X. Let*

$$w_0 : X \to Y, \qquad v_0, v_1 : A \to Y$$

be maps such that $w_0 \mid A = v_0$ *and* $v_0 \simeq v_1$. *Then there is a map* $w_1 : X \to Y$ *such that* $w_1 \mid A = v_1$ *and* $w_0 \simeq w_1$.

DEFINITION 4. A map of spectra $p : X \to Y$ is said to be a *Kan fibration*, if for any family $x_0, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_i \in X$ and $y \in Y$ such that $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$ and $px_i = d_i y$ for $i \neq k$ there exists $z \in X$ such that $d_i z = x_i$ for $i \neq k$ and $pz = y$.

PROPOSITION 1.1. *Any epimorphism of group spectra*

$$f : X \to Y$$

(*i.e.* $f_{(q)} : X_{(q)} \to Y_{(q)}$ *is onto*) *is a Kan fibration.*

The proof is similar to that of group spectra (see, for instance, [9]).

PROPOSITION 1.2. *Let* $p : X \to Y$ *be a Kan fibration and* $f : CZ \to Y$ *a map. Then there exists a map* $g : CZ \to X$ *such that* $pg = f$.

Proof. We use an analogon of the Eilenberg-Zilber lemma which is proved in Appendix A.

Let Z^n be the subset of Z consisting of all simplices z such that $d_i z = *$ for $i \geq n$. We remark that Z^1 contains only non-degenerate simplices (except for the base points) and $Z^{n-1} \subset Z^n$.

Let $z \in Z^0$. We consider $x_1 = x_2 = \cdots = *$ in X and $y = f(1, z) \in Y$. The extension condition furnishes a simplex $x \in X$. We set $g(1, z) = x$, and $g(0, z) = d_0 g(1, z)$.

Let $z \in Z^1 - Z^0$. Then $d_0 z \in Z^0$ since $d_i d_0 z = d_0 d_{i+1} z = *$ for all i. We take $x_1 = g(1, d_0 z)$, $x_2 = x_3 = \cdots = *$ in X and $y = f(1, z)$. We have

$$px_1 = pg(1, d_0 z) = f(1, d_0 z) = f\, d_1(1, z) = d_1 f(1, z),$$

$$px_i = * = f(1, d_{i-1} z) = f\, d_i(1, z) = d_i f(1, z)$$

for $i \geq 2$.

The extension condition furnishes a simplex $x \in X$ and we set $g(1, z) = x$. Furthermore, for any $z \in Z^1 - Z^0$ we set $g(0, z) = d_0 g(1, z)$.

It is easily shown that $d_i g(1, z) = g\, d_i(1, z)$ and $d_i g(0, z) = g\, d_i(0, z)$.

For any $z \in Z^1$ and any system of indices $i_1 > i_2 > \cdots > i_r \geq 0$ we set

$$\begin{aligned} g(1, s_{i_1} \cdots s_{i_r} z) &= s_{i_1+1} \cdots s_{i_r+1}\, g(1, z) \\ g(0, s_{i_1} \cdots s_{i_r} z) &= d_0\, g(1, s_{i_1} \cdots s_{i_r} z). \end{aligned} \tag{1}$$

It is checked immediately that $d_i g = g\, d_i$ and, for $i \neq 0$, $s_i g = gs_i$.

Assume inductively that we have defined g on all simplices of the form

$$(0, z), \quad (1, z), \quad (0, s_{i_1} \cdots s_{i_r} z), \quad (1, s_{i_1} \cdots s_{i_r} z)$$

with $z \in Z^n$ nondegenerate, $i_1 > \cdots > i_r \geq 0$ and that g commutes with d_i for any i and with s_i for $i > 0$ and $pg = f$.

Let $z \in Z^{n+1} - Z^n$ be nondegenerate. Then, for any $j \leq n$ and any $i \geq n$ we have $d_i d_j z = d_j d_{i+1} z = *$, hence $d_j z \in Z^n$. $d_j z$ is either nondegenerate or of the form $d_j z = s_{i_1} \cdots s_{i_r} y$ with $i_1 > \cdots > i_r \geq 0$, with y nondegenerate (see Appendix A). In the latter case, it is easily shown that $i_1 < n$. Then, for any $k \geq n$, we have $* = d_k d_j z = d_k s_{i_1} \cdots s_{i_r} y$ which implies either $y = *$ or $d_{k-r} y = *$, i.e. $y \in Z^n$. It follows that $g(1, d_j z)$ is already defined.

We take $x_1 = g(1, d_0 z), \cdots, x_{n+1} = g(1, d_n z)$, $x_{n+2} = \cdots = *$ and $y = f(1, z)$. We have

$$px_i = pg(1, d_{i-1} z) = f(1, d_{i-1} z) = f\, d_i(1, z) = d_i y \qquad (1 \leq i \leq n+1)$$

$$= * = f(1, d_{i-1} z) = f\, d_i(1, z) = d_i f(1, z) = d_i y \qquad (i > n+1).$$

Let $0 < i \leq n+1$.

$$d_i g(1, d_{j-1} z) = g\, d_i(1, d_{j-1} z) = g(1, d_{i-1} d_{j-1} z) = g(1, d_{j-2} d_{i-1} z)$$
$$= g\, d_{j-1}(1, d_{i-1} z) = d_{j-1} g(1, d_{i-1} z).$$

For $0 < i \leq n+1, j \geq n+2$ we have

$$d_i g(1, d_{j-1} z) = d_i g(1, *) = d_i * = * = d_{j-1} *.$$

Thus we can apply the extension condition which yields a simplex x such that $d_i x = x_i$ $(i > 0)$, $p(x) = y$. We set $g(1, z) = x$. Moreover, for any nondegenerate $z \in Z^{n+1} - Z^n$ we set $g(0, z) = d_0 g(1, z)$.

We also define g on simplexes of the form

$$(0, s_{i_1} \cdots s_{i_r} z), \qquad (1, s_{i_1} \cdots s_{i_r} z)$$

where $i_1 > \cdots > i_r \geq 0$ and $z \in Z^{n+1} - Z^n$ is nondegenerate by the formula (1).

We check that $d_i g = g\, d_i$ (all i), and $s_i g = g s_i$ $(i > 0)$.

(a) If z is nondegenerate, it follows from the definition of g that $d_i g(1, z) = g\, d_i(1, z)$ for all i.

$$g s_i(1, z) = g(1, s_{i-1} z) = s_i g(1, z) \qquad (i > 0)$$

by (1)

It is easily shown that $d_i g(0, z) = g\, d_i(0, z)$, $s_i g(0, z) = g s_i(0, z)$.

(b) Let $i_1 > i_2 > \cdots > i_r \geq 0$ and z nondegenerate.

$$g\, d_0(1, s_{i_1} \cdots s_{i_r} z) = g(0, s_{i_1} \cdots s_{i_r} z) = d_0 g(1, s_{i_1} \cdots s_{i_r} z).$$

If $i > 0$, we have

$$g\, d_i(1, s_{i_1} \cdots s_{i_r} z) = g(1, d_{i-1} s_{i_1} \cdots s_{i_r} z).$$

By the commutation rules of d_i and s_i, $d_{i-1} s_{i_1} \cdots s_{i_r}$ is either of the form $s_{j_1} \cdots s_{j_{r-1}}$, with $j_1 > \cdots > j_{r-1} \geq 0$, or of the form $s_{k_1} \cdots s_{k_r} d_l$ with

$k_1 > \cdots > k_r \geq 0$. In the first case,

$$\begin{aligned} g(1, d_{i-1} s_{i_1} \cdots s_{i_r} z) &= g(1, s_{j_1} \cdots s_{j_{r-1}} z) \\ &= s_{j_1+1} \cdots s_{j_{r-1}+1}\, g(1, z) \\ &= d_i\, s_{i_1+1} \cdots s_{i_r+1}\, g(1, z) \\ &= d_i\, g(1, s_{i_1} \cdots s_{i_r} z). \end{aligned}$$

In the second case, according to Appendix A, $d_l z$ is of the form $s_{p_1} \cdots s_{p_m} y$, where $p_1 > \cdots > p_m \geq 0$ and $y \in Z^n$ is nondegenerate. But by the commutation rules of the s_i's the expression $s_{k_1} \cdots s_{k_r} s_{p_1} \cdots s_{p_m}$ can be written as $s_{t_1} \cdots s_{t_{r+m}}$, where $t_1 > \cdots > t_{r+m} \geq 0$ so that we have

$$\begin{aligned} g(1, d_{i-1} s_{i_1} \cdots s_{i_r} z) &= g(1, s_{t_1} \cdots s_{t_{r+m}} y) \\ &= s_{t_1+1} \cdots s_{t_{r+m}+1}\, g(1, y) \\ &= s_{k_1+1} \cdots s_{k_r+1}\, s_{p_1+1} \cdots s_{p_m+1}\, g(1, y) \\ &= s_{k_1+1} \cdots s_{k_r+1}\, g(1, s_{p_1} \cdots s_{p_m} y) \\ &= s_{k_1+1} \cdots s_{k_r+1}\, g(1, d_l z) \\ &= s_{k_1+1} \cdots s_{k_r+1}\, g\, d_{l+1}(1, z) \\ &= s_{k_1+1} \cdots s_{k_r+1}\, d_{l+1}\, g(1, z) \\ &= d_i\, s_{i_1+1} \cdots s_{i_r+1}\, g(1, z) \\ &= d_i\, g(1, s_{i_1} \cdots s_{i_r} z). \end{aligned}$$

Furthermore,

$$\begin{aligned} d_i\, g(0, s_{i_1} \cdots s_{i_r} z) &= d_i\, d_0\, g(1, s_{i_1} \cdots s_{i_r} z) \\ &= d_0\, d_{i+1}\, g(1, s_{i_1} \cdots s_{i_r} z) \\ &= d_0\, g\, d_{i+1}(1, s_{i_1} \cdots s_{i_r} z) \\ &= d_0\, g(1, d_i\, s_{i_1} \cdots s_{i_r} z) \\ &= g(0, d_i\, s_{i_1} \cdots s_{i_r} z) \\ &= g\, d_i(0, s_{i_1} \cdots s_{i_r} z). \end{aligned}$$

It is also proved without difficulty that

$$\begin{aligned} s_j\, g(1, s_{i_1} \cdots s_{i_r} z) &= g s_j(1, s_{i_1} \cdots s_{i_r} z) \\ s_j\, g(0, s_{i_1} \cdots s_{i_r} z) &= g s_j(0, s_{i_1} \cdots s_{i_r} z) \end{aligned}$$

for any $j > 0$.

The verification of the induction is now complete.

By Appendix A and by the fact that $Z = \bigcup_{n \geq 0} Z^n$, it follows that g is defined on all simplices of the form $(0, z)$, $(1, z)$.

For any $n > 1$ and $z \in Z$ we set

$$g(n, z) = s_0^{n-1} g(1, z).$$

We have for every $z \in Z$

$$\begin{aligned} gs_0(1, z) &= g(2, z) = s_0 g(1, z), \\ s_0 g(0, z) &= s_o d_0 g(1, z) = d_0 s_1 g(1, z) = d_0 gs_1(1, z) = d_0 g(1, s_0 z) \\ &= g(0, s_0 z) = gs_0(0, z). \end{aligned}$$

It is also proved easily that

$$d_i g(n, z) = g\, d_i(n, z), \qquad s_i g(n, z) = gs_i(n, z)$$

for all i and any $z \in Z$ and $n > 1$.

PROPOSITION 1.3. *Let A be a subspectrum of X and*

$$j : A \to X, \qquad s : X \to X \backslash A$$

the canonical morphisms. Then the obvious inclusion $F(A) \to \operatorname{Ker} F(s)$ is a weak homotopy equivalence, where Ker $F(s)$ *is the subspectrum consisting of all $x \in F(X)$ such that $F(s)x = *$.*

This follows immediately from the above remark that

$$A \to X \to X \backslash A$$

is a fibration.

COVERING HOMOTOPY THEOREM. *Let $Y, Z \in \mathcal{S}p_G$ and let $p : Y \to Z$ be an epimorphism of groups. Let $X \in \mathcal{S}p$ and let*

$$v_0 : X \to Y, \qquad w_0, w_1 : X \to Z$$

be maps such that $pv_0 = w_0$, $w_0 \simeq w_1$. Then there is a map $v_1 : X \to Y$ such that $pv_1 = w_1$ and $v_0 \simeq v_1$.

Proof. Let $w : CX \to Z$ be an extension of $w_0 \cdot w_1^{-1}$. By Proposition 1.2 there is a map $v : CX \to Y$ such that $pv = w$.
Set $v_1 = (v \mid X)^{-1} v_0$.

PROPOSITION 1.4.[1] *For any object $X \in \mathcal{S}p_E$ there exists a homotopy equivalence $\varepsilon_X : F(SX) \to F(X)$ which is natural up to homotopy.*
The proof is to be found in Appendix B.

2. The Puppe sequence

Let $f : X \to Y$ be a morphism in the category $\mathcal{S}p$. Denote by $f' : Y \to C_f$ the natural inclusion. Consider the diagram

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[1] The authors are grateful to Dr. Klaus Dudda for drawing their attention to an error in connection with this proposition in an earlier draft.

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] & F(Y) \arrow[dl, "F(f')"] & & & \\
F(C_f) \arrow[r, "F(f'')"] \arrow[dr, "F(pf'')"] \arrow[ddr, bend right, "\partial = \varepsilon_X F(pf'')"'] & F(C_{f'}) \arrow[r, "F(f''')"] \arrow[d, "F(p)"] \arrow[dr, phantom, "\mathrm{I}"] & F(C_{f''}) \arrow[r, "F(f^{\mathrm{iv}})"] \arrow[d, "F(q)"] \arrow[dr, phantom, "\mathrm{II}"] & F(C_{f'''}) \arrow[r] \arrow[d, "F(r)"] & \cdots \\
& F(SX) \arrow[r, "F(Sf)"] \arrow[d, "\varepsilon_X"] & F(SY) \arrow[r, "F(Sf')"] \arrow[d, "\varepsilon_Y"] & F(SC_f) \arrow[d, "\varepsilon_{C_f}"] & \\
& F(\tilde{X}) \arrow[r, "F(\tilde{f})"] & F(\tilde{Y}) \arrow[r, "F(\tilde{f}')"] & F(\tilde{C}_f) &
\end{tikzcd}$$

where $f'', f''', f^{\mathrm{iv}}, \cdots$ are inclusions and p is defined as follows: the simplices of CY are sent to $*$, i.e. $p(n, y) = *$ (see the definition of C in §1), and the simplices of the form (n, x), with $x \in X$ and $n \neq 0$ are sent to $(n, x) \in SX$. Likewise for q and r. Clearly, we have the following sequence infinite in both directions:

$$\cdots \to F(\tilde{X}^{-1}) \xrightarrow{F(\tilde{f}^{-1})} F(\tilde{Y}^{-1}) \xrightarrow{F(\tilde{f}'^{-1})} F(\tilde{C}_f^{-1}) \xrightarrow{\tilde{\partial}^{-1}} F(X)$$
$$\xrightarrow{F(f)} F(Y) \xrightarrow{F(f')} F(C_f) \xrightarrow{\partial} F(\tilde{X}) \xrightarrow{F(\tilde{f})} F(\tilde{Y}) \to \cdots .$$

3. The homotopy category of spectra

We denote by $\bar{\mathcal{S}}p_E$ the category whose objects are Kan semisimplicial spectra and whose morphisms are homotopy classes of morphisms from $\mathcal{S}p_E$. Similarly, we have the categories $\bar{\mathcal{S}}p_G$, $\bar{\mathcal{S}}p_L$.

We can consider the functors $\bar{I} : \bar{\mathcal{S}}p_L \to \bar{\mathcal{S}}p_E$ and $\bar{F} : \bar{\mathcal{S}}p_E \to \bar{\mathcal{S}}p_L$ induced by I and F.

Proposition 3.1. *Each of the functors $\bar{I}$ and $\bar{F}$ establishes an equivalence between the categories $\bar{\mathcal{S}}p_E$ and $\bar{\mathcal{S}}p_L$.*

Proof. $\bar{F}$ establishes an equivalence. This is an immediate consequence of the fact that each $X \in \bar{\mathcal{S}}p_E$ is isomorphic in $\bar{\mathcal{S}}p_E$ with $F(X)$ and of Corollary (A.11) of [6]. $\bar{I}$ establishes an equivalence for the same reasons.

Proposition 3.2. *The category $\bar{\mathcal{S}}p_E$ possesses arbitrary direct sums and direct products.*

Proof. The direct sum of a family $(X_\alpha)_{\alpha \in A}$ is the object $F(\bigvee_{\alpha \in A} X_\alpha)$ of $\bar{\mathcal{S}}p_E$. To see this, one uses Definition 2 of the homotopy relation. The direct product of a family $(X_\alpha)_{\alpha \in A}$ is the object $\times_{\alpha \in A} X_\alpha$ of $\bar{\mathcal{S}}p_E$. To see this, one uses Definition 1 of the homotopy relation.

Proposition 3.3. *In the category $\bar{\mathcal{S}}p_E$ each object possesses a well-defined multiplication of group-object and a well-defined comultiplication of associative comonoid object.*

Proof. For any object X of $\bar{\mathcal{S}}p_E$, $F(X)$ is a group-object in $\bar{\mathcal{S}}p_E$. Then we consider the following multiplication of X:

$$X \times X \xrightarrow{i(X) \times i(X)} F(X) \times F(X) \xrightarrow{\mu} F(X) \xrightarrow{j(X)} X,$$

where μ is the multiplication given by the group structure of $F(X)$, $j(X)$ is the homotopy inverse of $i(X)$ which exists according to §1. The inverse is given by

$$X \xrightarrow{i(X)} F(X) \xrightarrow{\nu} F(X) \xrightarrow{j(X)} X,$$

where ν is the passage to the inverse in the group $F(X)$. It is readily verified that this defines a structure of group-object on X.

For any object X of $\bar{\mathcal{S}}p_E$, there exists a comultiplication defined as follows:

$$X \xrightarrow{i(X)} F(X) \xrightarrow{\theta} F(X) * F(X) = F(X \vee X)$$

where θ is the comultiplication in the category $\bar{\mathcal{S}}p_L$ defined by $\theta(x) = xx$, xx being a "word" in the free product $F(X) * F(X)$. F commutes with wedges because, for instance, it admits the functor I as a right-adjoint functor. θ is a comultiplication of an associative comonoid in the category $\mathcal{S}p_L$ [3]. It is easy to check that it remains so in the category $\bar{\mathcal{S}}p_L$. Using Proposition 3.1, it follows that the above comultiplication turns X into an associative comonoid object.

Remark. Any morphism $f : X \to Y$ in $\bar{\mathcal{S}}p_E$ is a homomorphism of group objects (of associative comonoid objects), for the multiplication (resp. comultiplication) just defined.

THEOREM 3.4. *The category $\bar{\mathcal{S}}p_E$ is additive.*

Proof. By Theorem 4.17 in [2], the multiplications on $\mathrm{Hom}_{\bar{\mathcal{S}}p_E}(X, Y)$ given by the group structure of Y and the comonoid structure of X coincide and are abelian. This implies also the bilinearity of the composition of morphisms.

4. The Puppe sequence in $\bar{\mathcal{S}}p_E$

LEMMA 4.1. *For any object X of $\bar{\mathcal{S}}p_E$, CX has trivial homotopy groups.*

Proof. Let $r : CX \to *$. We shall prove that r is a weak homotopy equivalence. To this end it is sufficient to prove that

$$i(CX) : CX \to F(CX)$$

is null homotopic (see for instance, [6, (A.15)]). We shall prove that we can define a map $h : C(CX) \to CX$ such that h induces the identity on the "base" of $C(CX)$. The definition of h is as follows:

$$h(\alpha, (\beta, x)) = (\alpha + \beta, x) \qquad \text{for } \alpha, \beta \geq 0, x \in X.$$

It is shown straightforwardly that h commutes with the d_i's and the s_i's and that the "base", i.e. the simplices of the form $(0, (\beta, x))$ are sent onto (β, x).

PROPOSITION 4.2. *The square*

$$\begin{tikzcd} F(C_{f'}) \arrow[r, "F(f''')"] \arrow[d, "F(p)"'] & F(C_{f''}) \arrow[d, "F(q)"] \\ F(SX) \arrow[r, "F(Sf)"'] & F(SY) \end{tikzcd}$$

is homotopy anticommutative in $\mathcal{S}p_L$.

Proof. It is sufficient to show that

$$(i(SY) \circ Sf \circ p)(i(SY) \circ q \circ f''') : C_{f'} \to F(SY)$$

is extendable to $C(C_{f'})$ (using Definition 2 of the homotopy relation and Theorem (A.15) of [6], which asserts that if $v : X' \to X$ is a weak equivalence, then $v^* : [X, Z] \to [X', Z]$ is a one-to-one correspondence, where Z is any group spectrum).

To do this, we define a map of spectra

$$\phi : C(C_{f'}) \to F(SY)$$

by setting:

$$\begin{aligned} \phi(\gamma, (\alpha, y)) &= F(\gamma + \alpha, y) && \text{for } \gamma, \alpha \geq 0, y \in Y \\ \phi(\gamma, (\beta, x)) &= F(\gamma + \beta, f(x)) && \text{for } \gamma, \beta \geq 0, x \in X. \end{aligned}$$

Clearly this map is well defined and degree-preserving. We now check that it commutes with the operators d_i. We have

$$\begin{aligned} d_i(\gamma, (\alpha, y)) &= (\gamma - 1, (\alpha, y)) && \text{if } i < \gamma, \\ &= (\gamma, (\alpha - 1, y)) && \text{if } i \geqq \gamma, i - \gamma < \alpha, \\ &= (\gamma, (\alpha, d_{i-\gamma-\alpha}\, y)) && \text{otherwise.} \\ d_i(\gamma, (\beta, x)) &= (\gamma - 1, (\beta, x)) && \text{if } i < \gamma, \\ &= (\gamma, (\beta - 1, x)) && \text{if } i \geq \gamma, i - \gamma < \beta, \\ &= (\gamma, (\beta, d_{i-\gamma-\beta}\, x)) && \text{otherwise.} \end{aligned}$$

Therefore we have by our definition

$$\begin{aligned} \phi\, d_i(\gamma, (\alpha, y)) &= F(\gamma + \alpha - 1, y) && \text{if } i < \gamma, \\ &= F(\gamma + \alpha - 1, y) && \text{if } i \geq \gamma, i - \gamma < \alpha, \\ &= F(\gamma + \alpha, d_{i-\gamma-\alpha}\, y) && \text{otherwise.} \end{aligned}$$

$$\begin{aligned}\phi\, d_i(\gamma, (\beta, x)) &= F(\gamma + \beta - 1, f(x)) && \text{if } i < \gamma,\\ &= F(\gamma + \beta - 1, f(x)) && \text{if } i \geq \gamma, i - \gamma < \beta,\\ &= F(\gamma + \beta, f(d_{i-\gamma-\beta}\, x)) = (\gamma + \beta,\, d_{i-\gamma-\beta} f(x)), && \text{otherwise.}\end{aligned}$$

On the other hand, we may write

$$\begin{aligned} d_i\, \phi(\gamma, (\alpha, y)) = d_i\, F(\gamma + \alpha, y) &= F(\gamma + \alpha - 1, y) && \text{if } i < \gamma + \alpha\\ &= F(\gamma + \alpha, d_{i-\gamma-\alpha}\, y) && \text{otherwise}\\ d_i\, \phi(\gamma, (\beta, x)) = d_i\, F(\gamma + \beta, f(x)) &= F(\gamma + \beta - 1, f(x)) && \text{if } i < \gamma + \beta\\ &= F(\gamma + \beta, d_{i-\gamma-\beta} f(x)) && \text{otherwise.}\end{aligned}$$

A similar calculation yields the commutation of ϕ with the operators s_i. This completes the proof.

PROPOSITION 4.3. *$p : C_{f'} \to SX$ is a weak homotopy equivalence.*

Proof. We have the sequence

$$CY \xrightarrow{k} C_{f'} \xrightarrow{p} C_{f'} \backslash CY = SX$$

where k is the canonical injection. According to Proposition (5.5) in [5], this sequence is a fibration in $\mathcal{S}p$. By the exact homotopy sequence of this fibration [5, p. 245] and by Lemma 4.1, p induces isomorphisms for the homotopy groups.

THEOREM 4.5. *In the diagram in $\mathcal{S}p_E$*

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(f')}$$

$$\begin{tikzcd} F(C_f) \arrow[r, "F(f'')"] \arrow[dr, "F(\partial)"'] & F(C_{f'}) \arrow[r, "F(f''')"] \arrow[d, "F(l)"] \arrow[dl, phantom, "(0)"] \arrow[dr, phantom, "(\mathrm{I})"] & F(C_{f''}) \arrow[r, "F(f^{\mathrm{iv}})"] \arrow[d, "F(m)"] \arrow[dr, phantom, "(\mathrm{II})"] & F(C_{f'''}) \arrow[r] \arrow[d, "F(n)"] & \cdots \\ & F(\tilde{X}) \arrow[r, "F(\tilde{f})"] & F(\tilde{Y}) \arrow[r, "F(\tilde{f}')"] & F(\tilde{C}_f) & \end{tikzcd}$$

where $l = \varepsilon_X F(p)$, $m = \varepsilon_Y F(q)$, $n = \varepsilon_{C_f} F(r)$, the triangle (0) is commutative, the squares (I) and (II) are anticommutative and l, m, n are isomorphisms.

Proof. The theorem follows from §2, and Propositions 1.4, 4.2 and 4.3.

5. Weak kernels and cokernels

DEFINITION. Let $\mathcal{C}$ be an additive category. The morphism $u : X \to Y$ in $\mathcal{C}$ is said to be a weak kernel of the morphism $v : Y \to Z$ if

1°. $vu = 0$

2°. For any morphism $w : U \to Y$ in $\mathcal{C}$ such that $vw = 0$, there exists a (not necessarily unique) morphism $t : U \to X$ such that $ut = w$.

The dual definition gives the weak cokernel.

LEMMA 5.1. *If A is a subspectrum of X and*

$$j : A \to X, \qquad s : X \to X \backslash A$$

are the canonical morphisms, then in the sequence

$$F(A) \xrightarrow{F(j)} F(X) \xrightarrow{F(s)} F(X \backslash A)$$

$F(j)$ is a weak kernel of $F(s)$ and $F(s)$ is a weak cokernel of $F(j)$ in $\bar{\mathcal{S}}p_E$.

Proof. $F(j)$ is a weak kernel. Let B a spectrum and $v : B \to F(X)$ such that $F(s)v \simeq 0$. According to the covering homotopy theorem in §1 there exists a map $v' : B \to F(X)$ such that $F(s)v' = 0$ and $v \simeq v'$. Thus v' factorizes through Ker $F(s)$ and, by using Proposition 1.3 we infer that there exists a morphism $w : B \to F(A)$ such that $F(j)w \simeq v$.

$F(s)$ is a weak cokernel. Consider the commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(j)"] & F(X) \arrow[r, "F(s)"] & F(X \backslash A) \\ A \arrow[u, "i(A)"] \arrow[r, "j"'] & X \arrow[u, "i(X)"] \arrow[r, "s"'] & X \backslash A. \arrow[u, "i(X \backslash A)"] \end{tikzcd}$$

Let $u : F(X) \to G$ be such that $u \circ F(j) \simeq 0$, where G is a group-spectrum (the assumption that G is a group spectrum does not restrict the generality). Then $ui(X)j \simeq 0$ and, according to the homotopy extension theorem (see §1), there exists $u' : X \to G$ such that $uj = 0$ and $u' \simeq ui(X)$. This implies that there exists $w' : X \backslash A \to G$ such that $w's = u'$. We set $w = k \circ w'$, where k is the homotopy inverse of $i(X \backslash A)$.

PROPOSITION 5.2. *Every morphism in the category $\bar{\mathcal{S}}p_E$ has a weak kernel and a weak cokernel and every morphism in $\bar{\mathcal{S}}p_E$ is a weak kernel and a weak cokernel.*

Proof. Let $f : X \to Y$ be an arbitrary morphism in $\bar{\mathcal{S}}p_E$.

We consider the sequence

$$F(\tilde{C}_f^{-1}) \xrightarrow{F(\tilde{\partial}^{-1})} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(f')} F(C_f).$$

We shall prove that $F(f')$ is a weak cokernel of $F(f)$, $F(\tilde{\partial}^{-1})$ is a weak kernel of $F(f)$, $F(f)$ is a weak kernel of $F(f')$, and $F(f)$ is a weak cokernel of $F(\tilde{\partial}^{-1})$. This is clearly sufficient, in view of the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i(X)"'] & Y \arrow[d, "i(Y)"] \\ F(X) \arrow[r, "F(f)"'] & F(Y), \end{tikzcd}$$

where $i(X)$ and $i(Y)$ are isomorphisms.

1°. $F(f')$ is a weak cokernel of $F(f)$. This is equivalent with the assertion that $F(f^{iv})$ is a weak cokernel of $F(f''')$, by Theorem 4.4. Consider the diagram

$$\begin{tikzcd}
& & F(C(C_{f'})) \arrow[d, "F(b)"] \\
F(C_{f'}) \arrow[r, "F(f''')"] & F(C_{f''}) \arrow[r, "F(f^{iv})"] \arrow[dr, "F(l)"'] & F(C_{f'''}) \arrow[d, "F(a)"] \\
& & F(C_{f''}\backslash C_{f'}) \approx F(C_{f'''}\backslash CC_{f'})
\end{tikzcd} \tag{2}$$

where f''' and b are inclusions, l and a are indentification maps. According to Lemma 4.1, $F(a)$ is an isomorphism in $\bar{\mathcal{S}}p_E$. According to Lemma 5.1, $F(l)$ is a weak cokernel of $F(f''')$, whence it follows that $F(f^{iv})$ is a weak cokernel of $F(f''')$.

2°. $F(\tilde{\partial}^{-1})$ is a weak kernel of $F(f)$. This is equivalent to the assertion that $F(f'')$ is a weak kernel of $F(f''')$. This results from arguments similar to those at 3°.

3°. $F(f)$ is a weak kernel of $F(f')$. This is equivalent to the assertion that $F(f''')$ is a weak kernel of $F(f^{iv})$, and this follows from diagram (2) and Lemma 5.1.

4°. $F(f)$ is a weak cokernel of $F(\tilde{\partial}^{-1})$. This is equivalent to the assertion that $F(f''')$ is a weak cokernel of $F(f'')$ and this follows from arguments similar to those at 1°.

6. The main theorem

THEOREM 6.1. *There exists a full embedding J of the category $\bar{\mathcal{S}}p_E$ into an abelian category $\mathcal{A}\bar{\mathcal{S}}p_E$ having the following properties:*

1°. *$\mathcal{A}\bar{\mathcal{S}}p_E$ has enough injectives and projectives and the injectives and projectives coincide.*

2°. *Every object of the form $J(A)$ with $A \in \bar{\mathcal{S}}p_E$ is injective (and projective) and every injective (or projective) object of $\mathcal{A}\bar{\mathcal{S}}p_E$ is isomorphic with an object of the form $J(A)$.*

3°. *The category $\mathcal{A}\bar{\mathcal{S}}p_E$ verifies the conditions $AB3$, $AB4$ and their duals (in the sense of Grothendieck).*

4°. *For any abelian category $\mathcal{A}$ and for any additive functor $T : \bar{\mathcal{S}}p_E \to \mathcal{A}$ there exist functors $R, M, L : \mathcal{A}\bar{\mathcal{S}}p_E \to \mathcal{A}$, each of them unique up to an isomorphism, which extend T and such that R is right exact, L is left exact and M preserves images.*

5°. *The following assertions are equivalent:*

(a) *The sequence*

$$J(X) \xrightarrow{J(f)} J(Y) \xrightarrow{J(g)} J(Z)$$

of $\mathcal{A}\bar{\mathcal{S}}p_E$ *is exact.*

(b) g *is a weak cokernel of* f *in* $\bar{\mathcal{S}}p_E$.

(c) f *is a weak kernel of* g *in* $\bar{\mathcal{S}}p_E$.

6°. *If* $\mathcal{A}$ *is an abelian category there exists a one-to-one correspondence between the exact functors* $G : \mathcal{A}\bar{\mathcal{S}}p_E \to \mathcal{A}$ *and the functors* $H : \bar{\mathcal{S}}p_E \to \mathcal{A}$ *which transform the sequence*

$$F(X) \xrightarrow{F(f)} F(Y) \to F(C_f)$$

into an exact sequence for each f.

Proof. The assertions 1°, 4°, 5° as well as the following assertion (contained in 2°):

2′°. Any object of the form $J(A)$ where $A \in \bar{\mathcal{S}}p_E$ is injective (and projective) and any object of $\mathcal{A}\bar{\mathcal{S}}p_E$ admits an injective (or projective) resolution by objects of the form $J(A)$, are immediate consequences of a general theorem of Peter Freyd [7], according to which each additive category having weak kernels and cokernels and in which each morphism is a weak kernel and a weak cokernel admits an embedding into an abelian category with the properties 1°, 2′°, 4°, 5°.

Since the work of Freyd is not yet published,[2] we give brief indications about these facts for the convenience of the reader. The objects of the category $\mathcal{A}\bar{\mathcal{S}}p_E$ are morphisms

$$A \xrightarrow{f} B$$

in $\bar{\mathcal{S}}p_E$; the morphisms from

$$A \xrightarrow{f} B \quad \text{to} \quad A' \xrightarrow{f'} B'$$

are equivalence classes of couples of morphisms (u, v) such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is commutative, the equivalence being defined as follows: (u, v) is equivalent with (u', v') if $f'u = f'u'$ (or, equivalently, if $vf = v'f$).

[2] *Added in Proof.* Freyd's papers [7] and [8] have now appeared in the Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, Springer-Verlag, New York 1966.

The functor J sends the object A of $\bar{\mathcal{S}}p_E$ onto the object

$$A \xrightarrow{1_A} A$$

of $\mathcal{A}\bar{\mathcal{S}}p_E$.

Every diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "1_B"] \\ A' \arrow[r] & B \end{tikzcd} \qquad \left[\text{resp. } \begin{tikzcd} A \arrow[r] \arrow[d, "1_A"'] & B \arrow[d] \\ A \arrow[r] & B' \end{tikzcd}\right]$$

represents a monomorphism (resp. an epimorphism) in $\mathcal{A}\bar{\mathcal{S}}p_E$.

Given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

let $K \to A$ be a weak kernel of vf ($B' \to K'$ a weak cokernel of vf). Then

$$\begin{tikzcd} K \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & B \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d] \\ A' \arrow[r] & K' \end{tikzcd}\right)$$

is a kernel (resp. cokernel) of (u, v).

The definition of the functors R, L, M is as follows:

$$R(A \to B) = \text{Coker}\,(T(K) \to T(A)),$$

where $K \to A$ is a weak kernel of $A \to B$.

$$M(A \to B) = \text{Im}\,(T(A) \to T(B))$$

$$L(A \to B) = \text{Ker}\,(T(B) \to T(C))$$

where $B \to C$ is a weak cokernel of $A \to B$.

We now complete the proof of the theorem.

For assertion 3°, it is straightforward to verify that if

$$(A_i \xrightarrow{f_i} B_i)_{i \in I}$$

is a family of objects of $\mathcal{A}\bar{\mathcal{S}}p_E$, then

$$\oplus_{i \in I} A_i \xrightarrow{\oplus_{i \in I} f_i} \oplus_{i \in I} B_i$$

is the direct sum, where the existence of $\oplus_{i \in I} A_i$ and $\oplus_{i \in I} B_i$ is guaranteed by

Proposition 3.2. Likewise, the direct product of this family is

$$\prod_{i\in I} A_i \xrightarrow{\prod_{i\in I} f_i} \prod_{i\in I} B_i\,.$$

The existence of direct sums and products implies in an abelian category the existence of arbitrary inductive and projective limits.

To show that the direct sum of a family monomorphisms is a monomorphism let (u_i, v_i)

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & A_i' \arrow[d, "v_i"] \\ B_i \arrow[r, "g_i"'] & B_i' \end{tikzcd} \qquad (i \in I)$$

be a family of monomorphisms. This diagram may be decomposed as

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "1_{A_i}"'] \arrow[dr, phantom, "\mathrm{I}"] & A_i' \arrow[d, "v_i"] \\ A_i \arrow[r, "v_i f_i"] \arrow[d, "u_i"'] \arrow[dr, phantom, "\mathrm{II}"] & B_i' \arrow[d, "1_{B_i'}"] \\ B_i \arrow[r, "g_i"'] & B_i'\,. \end{tikzcd}$$

The diagram I represents a monomorphism, since (u_i, v_i) is a monomorphism; moreover, diagram I represents an epimorphism, as noted above. Thus I is an isomorphism. Hence, to prove that

$$\oplus_{i\in I} (u_i, v_i) = (\oplus_{i\in I} u_i, \oplus_{i\in I} v_i)$$

is a monomorphism, it is sufficient to show that $(\oplus_{i\in I} u_i, \oplus_{i\in I} 1_{B_i'})$ is a monomorphism, which is indeed the case since $\oplus_{i\in I} 1_{B_i'} = 1_{\oplus_{i\in I} B_i'}$.

Remark. Analogously, the sum of a family of epimorphisms is an epimorphism. Dually, it can be shown that a product of a family of epimorphisms (monomorphisms) is an epimorphism (monomorphism).

It remains to show that any projective object of the category $\mathfrak{A}\mathcal{S}p_E$ is isomorphic with an object of the form $J(A)$ where $A \in \mathcal{S}p_E$. To do this, we use the following two propositions.

PROPOSITION 6.2. *Any retract in the category $\mathcal{S}p_E$ admits a complement.*

Proof. Let $f : X \to Y$ be a retract in $\mathcal{S}p_E$, i.e. there exists $p : Y \to X$ such that $pf = 1_X$. Consider the Puppe sequence

$$JF(X) \xrightarrow{JF(f)} JF(Y) \to JF(C_f) \to JF(\tilde{X}) \xrightarrow{JF(\tilde{f})} JF(\tilde{Y})$$

which according to assertion 5°, which has been proved, is exact. The existence of p implies that $JF(f)$ is a monomorphism. Then $JF(\tilde{f})$ is also a monomorphism and the sequence

$$0 \to JF(X) \xrightarrow{JF(f)} JF(Y) \to JF(C_f) \to 0$$

is exact. Since $JF(p)$ yields a splitting of this sequence, it follows that

$$JF(Y) = JF(X) \oplus JF(C_f), \quad \text{whence} \quad F(Y) = F(X) \oplus F(C_f)$$

(see the definition of direct sums above), i.e. the given retract has a complement.

PROPOSITION 6.3 (P. Freyd [8]). *If in the additive category $\mathcal{C}$ for every object A there exists the sum $\oplus_{n=1,2,\cdots} A_n$, where $A_n \approx A$ for any n, and any retract admits a complement, then for any morphism $v : P \to P$ such that $v^2 = v$, there exists an object Q and morphisms $t : P \to Q$, $s : Q \to P$ such that $ts = 1_Q$ and $st = v$.*

To finish the proof of 2°, let X be a projective object of $\mathcal{A}\bar{\mathcal{S}}p_E$. According to 2′°, there exists an epimorphism $p : J(P) \to X$, where $P \in \bar{\mathcal{S}}p_E$; the projectivity of X yields a monomorphism $u : X \to J(P)$ such that $pu = 1_X$. Thus $(up)^2 = up$. Since J is a full embedding, there exists $v : P \to P$ such that $J(v) = up$. We clearly have $v^2 = v$. By Proposition 6.3 there exist an object $Q \in \bar{\mathcal{S}}p_E$ and morphisms $t : P \to Q$, $s : Q \to P$ such that $ts = 1_Q$ and $st = v$. We assert that $J(Q)$ is isomorphic with X. For, consider the diagram

$$\begin{tikzcd} J(P) \arrow[r, "p"] & X \arrow[r, "u"] & J(P) \\ J(P) \arrow[u, "J(1_P)"] \arrow[r, "J(t)"'] & J(Q) \arrow[u, "q"] \arrow[r, "J(s)"'] & J(P) \arrow[u, "J(1_P)"'] \end{tikzcd}$$

where $q = p \circ J(s)$. We check the commutativity of this diagram.

$$u \circ q = u \circ p \circ J(s) = J(s) \circ J(t) \circ J(s) = J(s),$$

$$q \circ J(t) = p \circ J(s) \circ J(t) = p \circ u \circ p = p.$$

Since p is an epimorphism and $J(s)$ a monomorphism, we infer that q is an isomorphism.

To prove assertion 6°, let $H : \bar{\mathcal{S}}p_E \to \mathcal{A}$ be a functor which carries the sequence $F(X) \to F(Y) \to F(C_f)$ into an exact sequence for each f. It is immediate that H is additive. Let then L be the left-exact extension of H provided by 4°. It can be shown that H preserves epimorphisms. The proof paraphrases that given by Freyd for the stable category (Lemma 4.1 of [7]).

COROLLARY 6.4. *The functors* Hom $(J(A),\ )$ *and* Hom $(\ , J(A))$ *are exact in $\mathcal{A}\bar{\mathcal{S}}p_E$ for any object A of $\bar{\mathcal{S}}p_E$.*

This follows from the fact that $J(A)$ is both injective and projective.

COROLLARY 6.5. *Every representable exact functor on the category $\mathfrak{A}\bar{\mathcal{S}}p_E$ is represented by an object of the form $J(A)$, where $A \in \bar{\mathcal{S}}p_E$.*

This follows from assertion 2° of the theorem.

COROLLARY 6.6. *The functor J carries the Puppe sequence onto an exact sequence.*

Denote by $H^{-n}(A, B) = \mathrm{Hom}_{\bar{\mathcal{S}}p_E}(A, \tilde{\bar{B}}^n) = \mathrm{Hom}_{\bar{\mathcal{S}}p_E}(\tilde{A}^{-n}, B)$ and by $H^n(A, f) = H^n(A, F(C_f))$ and $H^n(f, B) = H^n(F(C_f), B)$ where $f : X \to Y$ is a morphism in $\mathcal{S}p_E$.

COROLLARY 6.7. *The following sequences are exact:*

$$\cdots \to H^{n+1}(A, f) \to H^n(A, X) \to H^n(A, Y) \to H^n(A, f) \to \cdots$$

$$\cdots \leftarrow H^n(X, A) \leftarrow H^n(Y, A) \leftarrow H^n(f, A) \leftarrow H^{n+1}(X, A) \leftarrow \cdots$$

Appendix A

THE EILENBERG-ZILBER LEMMA FOR SPECTRA. *Let X be a semi-simplicial spectrum and $x \in X$. Then x can be written uniquely as*

$$x = s_{i_1} s_{i_2} \cdots s_{i_r} y,$$

where y is non-degenerate and $i_1 > i_2 > \cdots > i_r \geq 0$.

Proof. We first remark that, if we have

$$x = s_{i_1} \cdots s_{i_r} y,$$

then, using the commutation relations of the s_i's, x may be written as

$$x = s_{j_1} \cdots s_{j_r} y,$$

where $j_1 > j_2 > \cdots > j_r \geq 0$.

To prove the existence of a representation as in the lemma, let n be the integer such that $d_i x = *$ for $i \geq n$. Then clearly if $x \neq *$, for any representation

$$x = s_{i_1} \cdots s_{i_r} y, \qquad i_1 > \cdots > i_r \geq 0,$$

we must have $r \leq n + 1$. This implies that there exists at least one representation of the required form. The uniqueness of this representation is proved in the same manner as in the case of semi-simplicial complexes.

Appendix B

Our objective is to prove Proposition 1.4. To do this, we introduce a definition for the suspension of a semisimplicial complex in addition to the one given in [5, p. 241].

Let K be a semisimplicial complex. The "left" suspension of K is the complex $S_1 K$ which has as n-simplices the appropriate degeneracy of the base point and all pairs (p, σ) such that $p \geq 0$, $\sigma \in K$, $\sigma \neq *$ and $p + \dim \sigma = n$. The degeneracy and face operators are given by

$$\begin{aligned} d_i(p, \sigma) &= (p - 1, \sigma), & s_i(p, \sigma) &= (p + 1, \sigma), & i &< p, \\ &= (p, d_{i-p}\sigma) & &= (p, s_{i-p}\sigma), & i &\geq p, \end{aligned}$$

whenever this has a meaning and $d_i(p, \sigma) = *$ otherwise.

Now, given a semisimplicial spectrum X, we associate with it two semisimplicial spectra ΣX and $\Sigma_1 X$ as follows: Let $\mathrm{Ps}\, X = \{X_i, \lambda_i\}$ be the prespectrum associated to X[4, p. 468]. Consider the prespectra $\{SX_i, S\lambda_i\}$ and $\{S_1 X_i, S_1 \lambda_i\}$. (To see that $\{S_1 X_i, S_1\lambda_i\}$ is a prespectrum, notice that the map

$$(q, (\alpha, p)) \to ((q, \alpha), p), \qquad p, q \geq 0, \alpha \in K$$

establishes an isomorphism between $S_1 SK$ and $SS_1 K$ for every semisimplicial complex K.)

Set

$$\Sigma X = \mathrm{Sp}\,\{S X_i, S\lambda_i\}, \qquad \Sigma_1 X = \mathrm{Sp}\,\{S_1 X_i, S_1 \lambda_i\},$$

where Sp is the functor defined in [4, p. 468].

LEMMA B.1. *For every spectrum X there exists a natural map*

$$\varphi_X : \Sigma X \to \tilde{X}$$

which is a weak homotopy equivalence.

Proof. Consider the prespectrum $\{Y_i\}$ where $Y_i = X_{i+1}$. The maps λ_i of the prespectrum $\{X_i\}$ determine a map of prespectra

$$f : \{SX_i, S\lambda_i\} \to \{Y_i\}.$$

On the other hand, it is easy to show that the spectra $\mathrm{Sp}\,\{Y_i\}$ and $\tilde{X}$ are isomorphic. Set $\varphi_X = \mathrm{Sp}\, f$. One proves without difficulty that φ_X induces isomorphisms for the homotopy groups of the prespectra associated to ΣX and $\tilde{X}$.

LEMMA B.2. *For every spectrum X there exists a natural map*

$$\psi_X : \Sigma_1 X \to SX$$

which is a weak homotopy equivalence.

Proof. Let $\mathrm{Ps}\, X = \{X_i\}$ and $\mathrm{Ps}\, SX = \{(SX)_i\}$. It is straightforward to verify that any n-simplex (p, α) of $S_1 X_i$ is also an n-simplex of $(SX)_i$. Thus we have a map of prespectra $j : \{S_1 X_i\} \to \{(SX)_i\}$. But Sp Ps SX is isomorphic with SX [4, p. 469]. Set $\psi_X = \mathrm{Sp}\, j$. It can be checked readily that ψ_X induces isomorphisms for the homotopy groups of the prespectra associated to $\Sigma_1 X$ and SX.

LEMMA B.3. *For any spectrum X there exists a natural morphism in $\bar{\mathrm{S}}\mathrm{p}_E$*

$$\theta_X : F(\Sigma X) \to F(\Sigma_1 X)$$

which is an isomorphism.

Proof. Let R and Sin be the geometric realization functor and the singular functor [4], and let $\mathrm{Ps}\, X = \{X_i\}$. Then we have maps of prespectra

$$f : \{SX_i\} \to \{\mathrm{Sin}\, RSX_i\}, \qquad f_1 : \{S_1 X_i\} \to \{\mathrm{Sin}\, RS_1 X_i\}$$

which are weak homotopy equivalences. Moreover, we have maps of prespectra

$$g : \{\mathrm{Sin}\, RSX_i\} \to \{\mathrm{Sin}\, SRX_i\}, \qquad g_1 : \{\mathrm{Sin}\, RS_1 X_i\} \to \{\mathrm{Sin}\, SRX_i\}$$

which are isomorphisms, since by Proposition 2.3 of [4], S commutes with R and one can verify that, for every semisimplicial complex K, the spaces $RS_1 K$ and SRK are homeomorphic in a natural manner. Now, by Proposition 9.2 of [4], a map in $\mathcal{S}p_E$ is a weak homotopy equivalence if and only if it is a homotopy equivalence. Thus we may set in $\bar{\mathcal{S}}p_E$

$$\theta_X = F(\mathrm{Sp}\, f_1)^{-1} \circ F(\mathrm{Sp}\, g_1)^{-1} \circ F(\mathrm{Sp}\, g) \circ F(\mathrm{Sp}\, f).$$

To complete the proof of Proposition 1.4, it is sufficient to set in $\bar{\mathcal{S}}p_E$

$$\varepsilon_X = F(\varphi_X) \circ \theta_X^{-1} \circ F(\psi_X)^{-1}.$$

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