# MEASURES ON NON-SEPARABLE METRIC SPACES

BY

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## 1. Introduction

The main purpose of this note is to give a simpler and more general definition of "weak" or "weak-star" convergence of certain measures on non-separable metric spaces, and to prove its equivalence with the convergence introduced in [1] for the cases considered there.

Let $(S, d)$ be a metric space. Let $\mathcal{B}$ or $\mathcal{B}(S)$ be the class of all Borel sets in $S$, i.e. the smallest $\sigma$-algebra containing all the open sets. One can safely assume that a finite, countably additive measure on $\mathcal{B}$ is concentrated in a separable subset [2]. It has seemed useful to consider finite, countably additive measures on metric spaces, not concentrated in separable subsets, defined on some, but not all, Borel sets [1]. Specifically, one can use the $\sigma$-algebra $\mathcal{U}$ or $\mathcal{U}(S)$ generated by the open balls

$$B(x, \varepsilon) = \{y \in S : d(x, y) < \varepsilon\}$$

for arbitrary $x$ in $S$ and $\varepsilon > 0$. Examples of finite measures on $\mathcal{U}$ not concentrated in separable subsets are the probability distributions of distribution functions of "empirical measures" [1]. For a simpler example, let $S$ be uncountable and $d(x, y) = 1$ for $x \neq y$. Then $\mathcal{U}$ consists of countable sets, which we give measure 0, and sets with countable complement, which we give measure 1.

If $S$ is separable, then all open sets are in $\mathcal{U}$ by the Lindelöf theorem, hence $\mathcal{U} = \mathcal{B}$. I don't know whether $\mathcal{U}$ is always strictly included in $\mathcal{B}$ for $S$ non-separable, but it is in the cases mentioned above, and under the following conditions:

PROPOSITION. *Suppose that the smallest cardinal of a dense set in $S$ is $c$ (cardinal of the continuum). Then $\mathcal{U}$ has cardinal $c$ and $\mathcal{B}$ has cardinal $2^c$. Hence $\mathcal{U}$ is strictly included in $\mathcal{B}$.*

*Proof.* Let $A$ be a dense set in $S$ of cardinal $c$. Let $G$ be the class of balls $B(x, r)$ with $x$ in $A$ and $r$ (positive) rational. We show that $G$ generates $\mathcal{U}$. Let $x \in S$, $r > 0$. Let $x_n \in A$, $x_n \to x$. We can assume $d(x_n, x) < r$ for all $n$. Let $r_n$ be positive rational numbers such that $r_n \to r$ and $r_n < r - d(x_n, x)$ for all $n$. Then

$$B(x, r) = \bigcup_{n=1}^{\infty} B(x_n, r_n),$$

showing that $G$ generates $\mathcal{U}$.

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Let $\omega$ be the cardinal of the set of all integers. Then $G$ has cardinal at most $\omega c$, and $\omega c = c$. Hence the class of complements of sets in $G$ has cardinal at most $c$. The class of countable unions of elements of $G$ has cardinal at most equal to $c^\omega$, and

$$c^\omega = (2^\omega)^\omega = 2^{\omega^2} = 2^\omega = c.$$

Using transfinite induction, we obtain that the cardinal of $\mathfrak{U}$ is at most $\aleph_1 c$ (where $\aleph_1$ is the least uncountable cardinal; we are assuming the axiom of choice, but not the continuum hypothesis). Now $\aleph_1 c = c$. Since each one-point set in $S$ clearly belongs to $\mathfrak{U}$, the cardinal of $\mathfrak{U}$ is exactly $c$. The cardinal of $\mathfrak{B}$ is exactly $2^c$ [3, Remark 3.7 p. 106] and $c < 2^c$. Thus $\mathfrak{U}$ is properly included in $\mathfrak{B}$, q.e.d.

If in the statement of the above proposition we replace $c$ by another uncountable cardinal $\alpha$, then the proof goes through except that possibly $\alpha < \alpha^\omega$, which will happen e.g. if $\alpha = \aleph_\omega$ [3, p. 100], but not if $\alpha = 2^\beta$ for some (infinite) $\beta$. When $\mathfrak{U}$ and $\mathfrak{B}$ have the same cardinal, it remains unclear whether they are equal.

It should be noted that the $\sigma$-algebras $\mathfrak{U}$ in non-separable metric spaces have certain unpleasant properties. For example, they are not always preserved by homeomorphisms or even by uniform isomorphisms. Also, they are not always preserved by "relativization" to a subset of $S$ with the same metric. Finally, if one takes a cartesian product of two metric spaces $S$ and $T$, with any of the usual metrics for the product topology, $\mathfrak{U}(S \times T)$ may not even contain all "rectangles" $A \times B$ where $A \in \mathfrak{U}(S)$, $B \in \mathfrak{U}(T)$.

The Borel $\sigma$-algebras are superior in all these respects, although $\mathfrak{B}(S \times T)$ may not be generated by the rectangles whose sides are Borel sets. Of course, the Borel $\sigma$-algebras are generally too large to carry a finite measure with non-separable support. One might hope for a $\sigma$-algebra which, like $\mathfrak{U}$, would allow such measures, but which had better "functorial" properties.

## 2. Measures on $\mathfrak{U}$

Let $M(S, \mathfrak{U})$ be the set of all finite, countably additive, real-valued set functions (signed measures) on $\mathfrak{U}$, $M^+(S, \mathfrak{U})$ the set of elements of $M(S, \mathfrak{U})$ with nonnegative values, and $P(S, \mathfrak{U})$ the set of elements of $M^+(S, \mathfrak{U})$ with total mass 1 (probability measures).

In [1], "weak-star" convergence of a sequence in $M^+(S, \mathfrak{U})$ to a Borel measure $\mu$ was defined as convergence of the upper and lower integrals of every bounded continuous function $f$ to $\int f\,d\mu$. Here we define a natural convergence in $M(S, \mathfrak{U})$ and prove that if $S$ is complete, the new convergence agrees with the old one whenever the latter is defined (if $\mu$ has separable support, which, as noted above, practically follows from $\mu$ being a Borel measure).

Let $\mathcal{C}(S)$ be the Banach space of all bounded, continuous, real-valued functions on $S$ with supremum norm $\|\ \|_\infty$. Let $C(S, \mathfrak{U})$ be the closed linear subspace of $\mathfrak{U}$-measurable elements of $\mathcal{C}(S)$. Then any $\mu$ in $M(S, \mathfrak{U})$ defines a bounded linear functional

$$f \to \int f \, d\mu$$

on $\mathcal{C}(S, \mathfrak{U})$. Then on $M(S, \mathfrak{U})$, we have the "weak-star" topology of pointwise convergence on $\mathcal{C}(S, \mathfrak{U})$. (Note that $M(S, \mathfrak{U})$ is a proper subset of the dual space $\mathcal{C}(S, \mathfrak{U})^*$ unless $S$ is compact.)

Given a real-valued function $f$ and a measure $\mu$ we define the usual upper and lower integrals:

$$\int^* f \, d\mu = \inf\left\{\int h \, d\mu : h \geq f, \int h \, d\mu \text{ defined}\right\},$$

$$\int_* f \, d\mu = \sup\left\{\int g \, d\mu : g \leq f, \int g \, d\mu \text{ defined}\right\}.$$

THEOREM. *Suppose* $(S, d)$ *is a complete metric space,* $\{\mu_n\}$ *is a sequence of elements of* $M^+(S, \mathfrak{U})$ *and* $\mu$ *in* $M^+(S, \mathfrak{U})$ *is concentrated in a separable subspace. Then* $\mu_n \to \mu$ *for the weak-star topology on* $M(S, \mathfrak{U})$ *if and only if*

$$\lim_{n\to\infty} \int^* f \, d\mu_n = \lim_{n\to\infty} \int_* f \, d\mu_n = \int f \, d\mu$$

*for every* $f$ *in* $\mathcal{C}(S)$.

*Proof.* "If" holds since the upper and lower integrals of functions in $\mathcal{C}(S, \mathfrak{U})$ are integrals.

To prove "only if", suppose $\mu_n \to \mu$ on $\mathcal{C}(S, \mathfrak{U})$ and $f$ is in $\mathcal{C}(S)$. Since $\mu$ has separable support it has a natural extension to all Borel sets. We may assume $\| f \|_\infty \leq 1$ and $\mu_n(S) \leq 1$ for all $n$. Let $\varepsilon$ be given, $0 < \varepsilon < 1$. By Ulam's theorem [4], there is a compact set $K$ such that $\mu(S \sim K) < \varepsilon$. Choose $\delta > 0$ so that $d(x, y) < \delta$ and $x$ in $K$ imply $|f(x) - f(y)| < \varepsilon$. Let $C$ be countable and dense in $K$. Let

$$d(y, K) = \inf_{x \in K} d(x, y) = \inf_{x \in C} d(x, y).$$

Then $d(\cdot, K)$ is $\mathfrak{U}$-measurable and continuous (in fact,

$$| d(y, K) - d(z, K) | \leq d(y, z)$$

for all $y$ and $z$). Let

$$g(y) = \min(1, 4d(y, K)/\delta).$$

Then $g \in \mathcal{C}(S, \mathfrak{U})$, so

$$\int g \, d\mu_n \to \int g \, d\mu < \varepsilon.$$

Let $F$ be a finite subset of $K$ such that for any $x$ in $K$, $d(x, z) < \delta/4$ for some $z$ in $F$. Let

$$\phi(t) = \varepsilon t/\delta, \quad 0 \leq t \leq \delta/2$$

$$= 2, \qquad t \geq \delta$$

and let $\phi$ also be linear in the interval $[\delta/2, \delta]$. Let

$$u(x) = \min\,(1, \min\,(f(z) + \varepsilon + \phi(d(x, z)) : z \in F)),$$
$$v(x) = \max\,(-1, \max\,(f(z) - \varepsilon - \phi(d(x, z)) : z \in F)).$$

Then clearly $u, v \in \mathcal{C}(S, \mathfrak{U})$. Let

$$W = \{x : d(x, w) < \delta/4 \text{ for some } w \text{ in } K\}.$$

For any $x$ in $W$, $d(x, z) < \delta/2$ for some $z$ in $F$, so

$$|f(x) - f(z)| < \varepsilon \quad \text{and} \quad \phi(d(x, z)) < \varepsilon.$$

Thus

$$u(x) \leq f(z) + 2\varepsilon \leq f(x) + 3\varepsilon.$$

Given $x$, let $G_x$ be the set of all $z$ in $F$ such that $d(x, z) < \delta$. Then $f(x) \leq f(z) + \varepsilon$ for all $z$ in $G_x$, while for $z$ in $F \sim G_x$, $\phi(d(x, z)) = 2$. Thus $f(x) \leq u(x)$ for all $x$ in $W$. Likewise

$$f(x) \geq v(x) \geq f(x) - 3\varepsilon$$

for all $x$ in $W$. Now since $W \in \mathfrak{U}$,

$$\int^* f\, d\mu_n \leq \int_W u\, d\mu_n + \mu_n(S \sim W),$$

$$\int_* f\, d\mu_n \geq \int_W v\, d\mu_n - \mu_n(S \sim W),$$

$$\limsup \int^* f\, d\mu_n \leq \limsup \int_W u\, d\mu_n + \varepsilon,$$

$$\liminf \int_* f\, d\mu_n \geq \liminf \int_W v\, d\mu_n - \varepsilon,$$

and

$$\limsup \int_W (u - v)\, d\mu_n \leq 6\varepsilon,$$

so

$$\limsup \int^* f\, d\mu_n - \liminf \int_* f\, d\mu_n \leq 8\varepsilon.$$

Since an upper integral is greater than a lower integral of the same function, the limits of $\int^* f\, d\mu_n$ and $\int_* f\, d\mu_n$ exist and are equal. These limits are also approached by $\int_W f\, d\mu$ as $\varepsilon \to 0$ (of course, $W$ depends on $\varepsilon$), thus they equal $\int f\, d\mu$, q.e.d.

## References

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