HIGHER ORDER WHITEHEAD PRODUCTS AND POSTNIKOV SYSTEMS

BY

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In an arcwise connected CW-complex, higher order Whitehead products are determined, as are all homotopy operations, by the Postnikov invariants of the space. This fact has been used implicitly in [3] to prove that certain products are non-zero. In this note we calculate this relationship explicitly. This is done by relating Whitehead products and classical obstruction theory. Let $P^n(C)$ denote n-dimensional complex projective space. As an application we show that if $\iota \in \pi_2(P^n(C))$ is a generator, then the set of $(n+1)^{\text{st}}$ order Whitehead products $[\iota, \cdots, \iota]$ equals $(n+1)!\ \sigma$, where σ is a generator of $\pi_{2n+1}(P^n(C))$.

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Let T denote the subset of the cartesian product, $\times_{i=1}^{k} S^{n_i}$, consisting of those points with at least one coordinate at a base point. We assume throughout that $n_i > 1$, all i, and $k \geq 2$. Choose a generator $\mu \in H_N(\times S^{n_i}; Z)$, where $N = \sum n_i$. Given a map $g : T \to X$, the k^{th} order Whitehead product, $W(g) \in \pi_{N-1}(X)$, is defined by $W(g) = g_* \partial H j_*(\mu)$, where j_* is induced by the inclusion,

$$j : (\times S^{n_i}, *) \to (\times S^{n_i}, T),$$

H is the Hurewicz homomorphism, and ∂ is the boundary in the homotopy sequence of the pair $(\times S^{n_i}, T)$. These products were defined and studied in [2]. It was shown there that g can be extended to the cartesian product if and only if $W(g) = 0$.

On the other hand classical obstruction theory yields an element,

$$o(g) \in H^N(\times S^{n_i}, T; \pi_{N-1}(X)),$$

such that g can be extended if and only if $o(g) = 0$. (We use here the fact that the $(N-2)$ skeleton of $\times S^{n_i}$ equals T.)

Let $\langle\ ,\ \rangle$ denote the Kronecker pairing

$$H^N(\times S^{n_i}; \pi_{N-1}(X)) \otimes H_N(\times S^{n_i}; Z) \to \pi_{N-1}(X).$$

The following lemma is then evident.

LEMMA 1. $\langle j^* o(g), \mu \rangle = W(g)$.

Given a fibre space (E, p, B) with fibre F and a map $g : T \to E$ such that pg can be extended to $h : \times S^{n_i} \to B$, the usual obstruction theory for cross-

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sections of a fibre space extends in an obvious manner to yield a cohomology class,

$$u(g) \in H^N(\times S^{n_i}, T; \pi_{N-1}(F))$$

such that g can be extended to cover h if and only if $u(g) = 0$; $u(g)$, of course, depends upoon the choice of h. The inclusion map $l : F \to E$ induces

$$l_* : \pi_{N-1}(F) \to \pi_{N-1}(E)$$

which in turn induces

$$l_\# : H^N(\times S^{n_i}, T; \pi_{N-1}(E)) \to H^N(\times S^{n_i}, T; \pi_{N-1}(F)).$$

The next lemma follows at once from the definition of $u(g)$.

LEMMA 2. $l_\# u(g) = o(g)$.

In particular if $\pi_{N-1}(B) = \pi_N(B) = 0$, then $p_* W(g) = 0$ and there exists an extension, h, unique up to homotopy. If in addition E is induced by $\zeta \in H^N(B; G)$, $j^* u(g) = h^*(\zeta)$. Combining this with the above lemmas we have:

THEOREM 3. *Let E be the fibre space induced by $\zeta \in H^N(B; G)$ and suppose $\pi_N(B) = \pi_{N-1}(B) = 0$. Given $g : T \to E$*

$$W(g) = \langle l_\# h^*(\zeta), \mu \rangle$$

where h is any extension of pg to the cartesian product.

We note that the above hypothesis implies that $l_\#$ is an isomorphism.

Applications

The Postnikov system of a space X is a sequence of spaces and maps (X_n, p_n, q_n), $q_n : X \to X_n$, $p_n : X_n \to X_{n-1}$, such that

(i) p_n is a fibre map with $K(\pi_n(X), n)$ as fibre,
(ii) q_n is an n-equivalence,
(iii) X_0 is a point,
(iv) $p_n q_n = q_{n-1}$,

X_n is induced from X_{n-1} by $k_{n+1} \in H^{n+1}(X_{n-1}; \pi_n(X))$.

Since $\pi_N(X_{N-2}) = \pi_{N-1}(X_{N-2}) = 0$, Theorem 3 implies

COROLLARY 1. *If $g : T \to X$, $(q_{N-1})_* W(g) = \langle l_\# h^*(k_N), \mu \rangle$ where h is any extension of $q_{N-2}\, g$ to the cartesian product.*

This determines $W(g)$ since q_{N-1} is an $(N - 1)$-equivalence.

We recall that

$$[f_1, \cdots, f_k] = \{W(g) : gj_i \cong f_i, \quad \text{for each} \quad i\},$$

where j_i is the canonical injection of $S^{n_i} \to T$.

COROLLARY 2. *The set of* $(n+1)^{st}$ *order Whitehead products* $[\iota, \cdots, \iota]$ *is a single element which is equal to* $(n+1)!\,\sigma$, *where*

$$\iota \in \pi_2(P^n(C)) \quad and \quad \sigma \in \pi_{2n+1}(P^n(C))$$

are generators of their respective groups.

Proof. It is well known that if $X = P^n(C)$ then

$$X_{2n} = X_{2n-1} = \cdots = X_2 = K(Z, 2)$$

and $k_{2n+2} = \alpha^{n+1}$ where $\alpha \in H^2(Z, 2, Z)$ is the fundamental class. Since X_{2n} is an H-space, the map $S^2 \vee \cdots \vee S^2 \to X_{2n}$, which, restricted to each S^2, represents a generator of $\pi_2(K(Z, 2))$, can be extended to h, mapping the cartesian product of $(n+1)$ copies of S^2 to X_{2n}. Moreover $h \mid T$ can be lifted to $g : T \to P^n(C)$. Clearly, $W(g) \in [\iota, \cdots, \iota]$.

A straightforward calculation such as [3, 3.15][2] shows that for all such extensions, $h^*(\alpha^{n+1}) = (n+1)!\,s$ where s is a generator of $H^N(\times S^{n_i}; Z)$. Thus by Corollary 1

$$(q_{2n+1})_* W(g) = \langle j_{\#} k^*(\alpha^{n+1}), \mu\rangle = (n+1)!\,\langle j_{\#}\, s, \mu\rangle.$$

Since $\langle j_{\#}\, s, \mu\rangle$ is a generator of $\pi_{2n+1}(X_{2n+1})$ it follows that $(q_{2n+1})_*^{-1}\langle j_{\#}\, s, \mu\rangle$ is a generator of $\pi_{2n+1}(P^n(C))$.

We close by noting that our characterization of $W(g)$ differs in spirit from that given by J-P. Meyer [1] in the case of the usual Whitehead product. One wonders if his characterization can in some way be extended to the general case.

BIBLIOGRAPHY

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[2] We take this opportunity to note that [3, 3.15] requires the additional hypothesis that each k_i is even. This does not affect the validity of the results of [3].