A COMPLEX JUMP FORMULA FOR A CLASS OF CONVOLUTION TRANSFORMS

BY
Zeev Ditzian

1. Introduction

This paper is concerned with a class of convolution transforms defined as in [1, p. 210] by

$$(1.1)\qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)\,dt$$

where

$$(1.2)\qquad G(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\prod_{k=1}^{\infty}\left(1-\frac{s^2}{a_k^2}\right)\right]^{-1} e^{sz}\,ds \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} E(s)^{-1}e^{st}\,ds$$

and

$$(1.3)\qquad 0 < a_k \le a_{k+1} \text{ for } k \ge 1 \quad\text{and}\quad \lim_{k\to\infty} a_k\cdot k^{-1} = \Omega;\quad 0 < \Omega < \infty.$$

The Stieltjes transform

$$(1.4)\qquad F(x) = \int_0^{\infty}\frac{\Phi(t)}{x+t}\,dt$$

is, after a change of variables, a transform of the above mentioned class [1, p. 69]. A complex inversion formula for this class, which generalizes the classical one for the Stieltjes transform, was treated by I. I. Hirschman and D. V. Widder [1, Ch. IX]. In this paper the notation as well as many theorems of [1, Ch. IX] are frequently used.

A jump formula gives $\varphi(x+) - \varphi(x-)$ in terms of $f(x)$ where $f(x)$ and $\varphi(x)$ are related by (1.1). The jump formula of this paper is motivated by the following new jump formula for the Stieltjes transform that can also be proved independently.

Theorem 1.1. *Suppose*

(a) $\Phi(t) \in L_1(0, R)$ *for all* $R < \infty$,

(b) $F(x) = \int_0^{\infty}\frac{\Phi(t)}{x+t}\,dt$ *converges*,

(c) *there exist numbers* $\Phi(\xi \pm 0)$ *satisfying*

$$\int_0^h [\Phi(\xi \pm y) - \Phi(\xi \pm 0)]\,dy = o(h), \quad h \downarrow 0.$$

Then

$$(1.5)\qquad \lim_{\eta\to 0+} \tfrac{1}{2}i\eta(F'(-\xi - i\eta) - F'(-\xi + i\eta)) = \Phi(\xi+0) - \Phi(\xi-0).$$

We shall generalize (1.5) to a jump formula for the transforms defined by (1.1), (1.2) and (1.3) and the related Convolution-Stieltjes transform.

Received December 12, 1965.

2. Main results

In order to state the results of this paper we need the following definition of $c_\rho(\Omega)$ (as in [1, p. 226]).

DEFINITION 2.1. $c_\rho(\Omega)$ is a closed rectifiable curve going around the segment $[-i\pi\Omega, i\pi\Omega]$ in the positive direction and lying in the strip $|\operatorname{Im} z| \leq \pi\Omega/\rho$.

For the convolution Stieltjes case we shall get:

THEOREM 2.1. *If*
(1) $G(z)$ *is defined by* (1.2) *and* (1.3) *for some* Ω,
(2) $K(z) = \sum_{k=0}^{\infty} E^{(k)}(0)z^{-k-1}$ *where* $E(s)$ *is defined by* (1.2),
(3) $f(z) = \int_{-\infty}^{\infty} G(z-t)e^{ct}\,d\alpha(t)$ *converges,*
(4) $\alpha(t)$ *is of bounded variation in any finite interval,*
then
(A) $-a_1 < c < a_1$ *implies that*

$$\lim_{\rho\to 1-} \frac{\pi(1-\rho)}{2\Omega i} \int_{x_1}^{x_2} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz \tag{2.1}$$
$$= \alpha(x_2+) - \alpha(x_2-) - \alpha(x_1+) + \alpha(x_1-);$$

(B) $c > a_1$ *implies*

$$\lim_{\rho\to 1-} \frac{(\rho-1)\pi}{2\Omega i} \int_{x}^{\infty} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz \tag{2.2}$$
$$= \alpha(x+) - \alpha(x-);$$

(C) $c < -a_1$ *implies*

$$\lim_{\rho\to 1-} \frac{\pi(1-\rho)}{2\Omega i} \int_{-\infty}^{x} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz \tag{2.3}$$
$$= \alpha(x+) - \alpha(x-);$$

(D) *for any finite* c

$$\lim_{\rho\to 1-} \frac{\pi(1-\rho)}{2\Omega i} e^{-cx} \int_{c_\rho(\Omega)} f(x+\rho z)K(z)\,dz = \alpha(x+) - \alpha(x-). \tag{2.4}$$

For the convolution transform we get:

THEOREM 2.2. *If*
(1) *assumptions* (1) *and* (2) *of Theorem* 2.1 *are satisfied,*
(2) $f(z) = \int_{-\infty}^{\infty} G(z-t)\varphi(t)\,dt$ *converges,*
(3) $|\varphi(t)| \leq \operatorname{Cosh} pt$ *for some finite* p,
(4) $\varphi(x\pm) = \lim_{y\to 0+} \varphi(x \pm y)$ *exist,*
then

$$\lim_{\rho\to 1-} \frac{(1-\rho)\pi}{2\Omega i} \int_{c_\rho(\Omega)} f'(x+\rho z)K(z)\,dz = \varphi(x+) - \varphi(x-). \tag{2.5}$$

We can weaken assumptions (3) and (4) of the last theorem if we restrict the a_k's slightly more. Define $g(x \pm 0)$ as the numbers satisfying

$$\int_0^h [g(x \pm y) - g(x \pm 0)]\, dy = o(h), \quad h \downarrow 0 \tag{2.6}$$

if such numbers exist.

THEOREM 2.3. *Suppose*

(1) *assumption* (1) *and* (2) *of Theorem* 2.2 *are satisfied,*

(2) $\varphi(t) \in L_1(-R_1, R_2)$ *for every finite* R_i,

(3) *both* $\varphi(x \pm 0)$ *exist,*

(4) *for each* $\rho > 1$ *there exists an* n_ρ *such that* $\rho a_{n_\rho+k+1} < a_{n_\rho+k}$ *for any* $k \geq 0$ *and* $n_\rho \leq K/(1 - \rho)$ *where* K *is a constant independent of* ρ.

Then

$$\lim_{\rho \to 1-} \frac{\pi(1-\rho)}{2\Omega i} \int_{c_\rho(\Omega)} f'(x + \rho z) K(z)\, dz = \varphi(x + 0) - \varphi(x - 0). \tag{2.7}$$

Remark. The restriction $n_\rho \leq K/(1 - \rho)$ is essential in assumption (4) of Theorem 2.3 since without this restriction it can be proved that an integer n_ρ exists for which $\rho a_{n_\rho+k+1} < a_{n_\rho+k}$ for any $k > 0$. Assumption (4) is satisfied in many special cases so that Theorem 2.3 is applicable for instance for the Stieltjes, generalized Stieltjes and iterated Stieltjes transforms. (These are all the known classical transforms which are special cases of our class.) Assumption (4) is not always satisfied as the following example shows.

Let $G(t)$ be defined by $\{a_k\}$ as in (1.2), (1.3) and let $a_k = 2^n$ for $2^n \leq k \leq 2^n + [2^{n+1}/n]$ for all $n \geq 3$ and otherwise $a_k = k$. This $G(t)$ satisfy (1.3) with $\Omega = 1$ but as can be easily seen assumption (4) of Theorem 2.3 is not satisfied.

3. Some lemmas on $G(\rho, 0)$

Let $G(\rho, 0)$ be defined as in [1, p. 219] by

$$G(\rho, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[\prod_{k=1}^{\infty} \left(1 - \frac{\rho^2 s^2}{a_k^2}\right) \Big/ \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right) \right] e^{st}\, ds \tag{3.1}$$

$$\equiv \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(\rho s) \cdot E(s)^{-1} e^{st}\, ds \tag{3.2}$$

where $\{a_k\}$ satisfies (1.3).

LEMMA 3.1. *If* $G_j(\rho, t)$ *are defined by* (3.1) *with the appropriate sequences* $\{a_{k,j}\}$, $j = 1, 2$, *satisfying* $a_{k,1} \leq a_{k,2}$ *then*

$$0 < G_1(\rho, 0) \leq G_2(\rho, 0). \tag{3.3}$$

Proof. By definition and the change of variable $s = iy$ we have

$$G_j(\rho, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{\infty} [(1 + \rho^2 y^2 a_{k,j}^{-2}) \cdot (1 + y^2 a_{k,j}^{-2})^{-1}]\, dy, \qquad j = 1, 2.$$

Since $a_{k1} \leq a_{k,2}$ we obtain

$$(1 + \rho^2 y^2 a_{k1}^{-2})(1 + y^2 a_{k,1}^{-2})^{-1} \leq (1 + \rho^2 y^2 a_{k,2}^{-2})(1 + y^2 a_{k,2}^{-2})^{-1}$$

and therefore $(E_1(i\rho y)/E_1(iy)) \leqq (E_2(i\rho y)/E_2(iy))$ which completes the proof, Q. E. D.

LEMMA 3.2. *Suppose $G_1(\rho, t)$ and $G_2(\rho, t)$ are defined by $E_1(s)$ and $E_2(s)$ respectively where*

$$E_1(s) = E_2(s)(1 - s^2/a^2). \tag{3.4}$$

Then for $0 < \rho < 1$,

$$G_2(\rho, 0) - G_1(\rho, 0) \leqq (1 - \rho^2)G_2(\rho, 0) \leqq (1/\rho^2)(1 - \rho^2)G_1(\rho, 0). \tag{3.5}$$

Proof. By definition and using (3.4) we get

$$\begin{aligned} G_2(\rho, 0) - G_1(\rho, 0) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{E_2(\rho s)}{E_2(s)} - \frac{E_1(\rho s)}{E_2(s)}\right] ds \\ &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(1-\rho^2)s^2a^{-2}}{1 - s^2a^{-2}}\frac{E_2(\rho s)}{E_2(s)} ds \\ &= \frac{1-\rho^2}{2\pi}\int_{-\infty}^{\infty}\frac{y^2}{a^2+y^2}\frac{E_2(\rho i y)}{E_2(iy)} dy \\ &\leq (1-\rho^2)G_2(\rho, 0), \end{aligned}$$

(since $E_2(\rho iy)/E_2(iy) > 0$ and $0 < y^2/(y^2 + a^2) < 1$). From this result we get also

$$G_2(\rho, 0) \leqq (1/\rho^2)G_1(\rho, 0)$$

which concludes the proof of the lemma, Q.E.D.

LEMMA 3.3. *Let $G(\rho, M, t)$ be defined by*

$$G(\rho, M, t) = \frac{1}{2\pi i}\int_{-i\infty}^{\infty}\frac{\sin \pi\rho s M}{\sin \pi s M} e^{st} ds \tag{3.6}$$

where $0 < M$ is some real constant; then

$$G(\rho, M, 0) = \frac{1}{\pi^2(1-\rho)M} + o\left(\frac{1}{1-\rho}\right), \quad \rho \uparrow 1. \tag{3.7}$$

Proof. It is enough to prove our theorem for $M = 1$ since one can easily show that

$$G(\rho, M, t) = M^{-1}G(\rho, 1, t\cdot M^{-1}). \tag{3.8}$$

By definition of $\sin \pi s$ and by setting $s = iy$ we get

$$G(\rho, 1, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\pi y\rho} - e^{+\pi y\rho}}{e^{-\pi y} - e^{+\pi y}} dy$$

$$= \frac{1}{\pi}\int_0^\infty e^{-\pi(1-\rho)y}\frac{1-e^{-2\pi y\rho}}{1-e^{-2\pi y}}\,dy$$

$$= \frac{1}{\pi}\int_0^\infty e^{-\pi(1-\rho)y}\,dy - \frac{1}{\pi}\int_0^\infty e^{-\pi(1-\rho)y}\cdot e^{-2\pi y\rho}\frac{1-e^{-2\pi y(1-\rho)}}{1-e^{-2\pi y}}\,dy.$$

By Cauchy's theorem

$$\frac{1-e^{-2\pi y(1-\rho)}}{1-e^{-2\pi y}} = \frac{-2\pi(1-\rho)e^{-2\pi\eta(1-\rho)}}{-2\pi e^{-2\pi\eta}} = (1-\rho)e^{2\pi\eta\rho}$$

for some η, $0 < \eta < y$ and so we get

$$\left| e^{-2\pi y\rho}\frac{1-e^{-2\pi y(1-\rho)}}{1-e^{-2\pi y}}\right| \leq 1-\rho.$$

This implies

$$G(\rho, 1, 0) = \frac{1}{\pi^2(1-\rho)} + O(1) = \frac{1}{\pi^2(1-\rho)} + o\left(\frac{1}{1-\rho}\right), \quad \rho \uparrow 1,$$

Remark. The zeros of $E(s) = (1/\pi sM)\sin \pi sM$ are $a_k = k/M$.

THEOREM 3.1. *Suppose $G(\rho, t)$ be defined by (3.1) and let*

$$\lim_{k\to\infty} a_k\cdot k^{-1} = \Omega.$$

Then

$$G(\rho, 0) = \frac{\Omega}{\pi^2(1-\rho)} + o\left(\frac{1}{1-\rho}\right), \quad \rho\uparrow 1. \tag{3.9}$$

Proof. Since $\lim_{k\to\infty} a_k\cdot k^{-1} = \Omega$ we have $k_0 = k_0(\varepsilon)$ such that for $k \geq k_0$, $\Omega - \varepsilon \leq a_k\cdot k^{-1} \leq \Omega + \varepsilon$. Applying Lemma 3.2 to both $G(\rho, t)$ and $G(\rho, M, t)$ k_0 times we get

$$G_{k_0}(\rho, 0) - G(\rho, 0) \leq (1-\rho^2)k_0 G_{k_0}(\rho, 0), \tag{3.10}$$

$$G_{k_0}(\rho, M, 0) - G(\rho, M, 0) \leqq (1-\rho^2)k_0 G_{k_0}(\rho, M, 0) \tag{3.11}$$

where

$$G_{k_0}(\rho, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\prod_{k=k_0+1}^{\infty}\left(1-\frac{\rho^2 s^2}{a_k^2}\right)\Big/\prod_{k=k_0+1}^{\infty}\left(1-\frac{s^2}{a_k^2}\right)\right]e^{st}\,ds$$

and

$$G_{k_0}(\rho, M, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\prod_{k=k_0+1}^{\infty}\left(1-\frac{\rho^2 s^2 M^2}{k^2}\right)\Big/\prod_{k=k_0+1}^{\infty}\left(1-\frac{s^2M^2}{k^2}\right)\right]e^{st}\,ds.$$

By Lemma 3.1 we get

$$G_{k_0}\left(\rho, \frac{1}{\Omega-\varepsilon}, 0\right) \leq G_{k_0}(\rho, 0) \leq G_{k_0}\left(\rho, \frac{1}{\Omega+\varepsilon}, 0\right).$$

From (3.10) and (3.11) we get for $\rho < 1$ great enough

$$G(\rho, 0) < G_{k_0}(\rho, 0) \leq \frac{1}{1-k_0(1-\rho^2)}G(\rho, 0),$$

$$G(\rho M, 0) \leq G_{k_0}(\rho, M, 0) \leq \frac{1}{1-k_0(1-\rho^2)}G(\rho, M, 0).$$

Therefore Lemma 3.3 implies

$$\frac{(\Omega-\varepsilon)}{\pi^2(1-\rho)}+o\left(\frac{1}{1-\rho}\right)\le G(\rho,0)(1+o(1))$$
$$\le\frac{(\Omega+\varepsilon)}{\pi^2(1-\rho)}+o\left(\frac{1}{1-\rho}\right),\quad \rho\uparrow 1;$$

since ϵ is arbitrary,

$$G(\rho,0)=\frac{\Omega}{\pi^2(1-\rho)}+o\left(\frac{1}{1-\rho}\right),\quad \rho\uparrow 1.$$

4. Some preliminary results

The results of this section will be used in proving the theorems of §2.

THEOREM 4.1. *Suppose $G(t)$ is defined by* (1.2) *and* (1.3) *and*

$$f(z)=\int_{-\infty}^{\infty}G(z-t)e^{ct}\,d\alpha(t)$$

converges for any $z=x+iy$ in the strip $(-\infty<x<\infty;\ |y|<\pi\Omega)$; then

$$f'(z)=\int_{-\infty}^{\infty}G'(z-t)e^{ct}\,d\alpha(t)$$

converges uniformly in any compact subset of $(-\infty<x<\infty;\ |y|<\pi\Omega)$

Proof. By Theorem 2.2a of [1, pp. 213–214] we get

$$G'(z-t)/G(z_0-t)=O(1),\quad t\to\pm\infty$$

and

$$\frac{d}{dt}[G'(z-t)/G(z_0-t)]=O(1/t^2),\quad t\to\pm\infty$$

uniformly in $|y|<\pi(\Omega-\eta)$.

The rest of the proof is similar to that of Theorem 2.2b of [1, pp. 214–215], Q. E. D.

THEOREM 4.2. *Suppose $G(z)$ is defined by* (1.2) *and* (1.3), *$G(\rho,t)$ by* (3.1), *$C_\rho(\Omega)$ by Definition* 2.1 *and $K(z)$ by*

$$K(z)=\textstyle\sum_{k=0}^{\infty}E^{(k)}(0)z^{-k-1}\tag{4.1}$$

and let

$$f(z)=\int_{-\infty}^{\infty}G(z-t)e^{ct}\,d\alpha(t)\tag{4.2}$$

converge for some real c; then

$$\frac{1}{2\pi i}\int_{c_\rho(\Omega)}f'(u+\rho z)K(z)\,dz=\int_{-\infty}^{\infty}G'(\rho,u-t)e^{ct}\,d\alpha(t).\tag{4.3}$$

Proof. By (4.2) we have

$$\frac{1}{2\pi i}\int_{c_\rho(\Omega)} K(z)f'(u+\rho z)\,dz = \frac{1}{2\pi i}\int_{c_\rho(\Omega)} K(z)\,dz \int_{-\infty}^{\infty} G'(z-t)e^{ct}\,d\alpha(t)$$

(by Theorem 4.1 we may interchange the order of the integrations)

$$= \int_{-\infty}^{\infty} e^{ct}\,d\alpha(t)\,\frac{1}{2\pi i}\int_{c_\rho(\Omega)} K(z)\,dz \cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s[E(s)]^{-1}e^{s(u+\rho z-t)}\,ds$$

(we may interchange the order of the integrations since $|E(iy|^{-1} = O(e^{-\pi(\Omega-\varepsilon)|y|}$, $y \to \pm\infty$ [1, p. 213] imply the uniform convergence of the inner integral in $|\,\mathrm{Imz}\,| \leq \pi(\Omega-\varepsilon)$)

$$= \int_{-\infty}^{\infty} e^{ct}\,d\alpha(t)\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{se^{-s(u-t)}}{E(s)}\,ds\,\frac{1}{2\pi i}\int_{c_\rho(\Omega)} K(z)e^{sz\rho}\,dz$$

$$= \int_{-\infty}^{\infty} e^{-ct}\,d\alpha(t)\cdot\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s\,\frac{E(\rho s)}{E(s)}\,e^{-s(u-t)}\,ds$$

$$= \int_{-\infty}^{\infty} G'(\rho, u-t)e^{ct}\,d\alpha(t), \qquad \text{Q. E. D.}$$

THEOREM 4.3. *Under the assumption of Theorem* 4.2

(A) $-a_1 < c < a_1$ *implies*

$$\frac{1}{2\pi i}\int_{x_1}^{x_2} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dx$$
$$= \int_{-\infty}^{\infty} G'(\rho, t)e^{-ct}\alpha(x_2-t)\,dt - \int_{-\infty}^{\infty} G'(\rho, t)e^{-ct}\alpha(x_1-t)\,dt;$$

(B) $c = a_1, -a_1$ *implies*

$$\frac{1}{2\pi i}\int_{x_1}^{x_2} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz$$
$$= \int_{-\infty}^{\infty} G'(\rho, t)e^{-ct}[\alpha(x_2-t)-\alpha(x_1-t)]\,dt,$$

(C) $c > a_1$ *implies that* $\alpha(+\infty)$ *exists and*

$$\frac{1}{2\pi i}\int_{x}^{\infty} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz$$
$$= \int_{-\infty}^{\infty} G'(\rho, t)e^{-ct}[\alpha(+\infty)-\alpha(x_1-t)]\,dt,$$

(D) $c < -a_1$ *implies that* $\alpha(-\infty)$ *exists and*

$$\frac{1}{2\pi i}\int_{-\infty}^{x_2} e^{-cu}\,du \int_{c_\rho(\Omega)} f'(u+\rho z)K(z)\,dz$$
$$= \int_{-\infty}^{\infty} G'(\rho, t)e^{-ct}[\alpha(x_2-t)-\alpha(-\infty)]\,dt.$$

Proof. Using the asymptotic estimations [1, p. 232]

$$(4.4)\quad G^{(n)}(\rho, \pm t) = \frac{d^n}{dt^n}\,(p(\rho, \pm t)e^{-a_1|t|}) + o(e^{-(a_1+\varepsilon)|t|}),\quad t \to \infty,$$

where $p(\rho, t) \neq 0$ is a polynomial, we get for $M(\rho, n)$ large enough that $G^{(n)}(\rho, t)$ does not change sign for $t > M(\rho, n)$ and $t < -M(\rho, n)$. Therefore $G^{(n-1)}(\rho, t)$ is monotonic in $(-\infty, -M(\rho, \eta))$ and in $(M(\rho, \eta), \infty)$. Applying the above consideration for $n = 2$ the proof of the theorem follows by arguments similar to those of the proof of Theorem 6.1b in [1, pp. 227–230], Q. E. D.

5. The proof of the main results

We shall prove Theorem 2.2 first and use it in the proof of Theorem 2.1.

Proof of Theorem 2.2. Using Theorem 4.2 and some substitutions it is obvious that we have only to prove

$$(5.1)\quad \lim_{\rho\to 1-} I_\rho = \lim_{\rho\to 1-} \frac{(1-\rho)\pi^2}{\Omega}\int_{-\infty}^{\infty} G'(\rho, t)\varphi(x-t)\,dt = \varphi(x+) - \varphi(x-).$$

We shall show first

$$(5.2)\quad \lim_{\rho\to 1-} I_\rho = \lim_{\rho\to 1-} \frac{(1-\rho)\pi^2}{\Omega}\int_{-\infty}^{\infty} G_m'(\rho, t)\varphi(x-t)\,dt$$

where

$$(5.3)\quad G_m(\rho, t) = \int_{-\infty}^{\infty}\left[\prod_{k=m+1}^{\infty}\left(1-\frac{\rho^2 s^2}{a_k^2}\right)\Big/\prod_{k=m+1}^{\infty}\left(1-\frac{s^2}{a_k^2}\right)\right]e^{st}\,dt$$

and $a_m \neq a_{m+1}$.

Let μ be the multiplicity of a_1 as a zero of $E(s)$. We write, as was done in [1, p. 232],

$$(1-\rho^2 s^2/a_1^2)^\mu/(1-s^2/a_1^2)^\mu = \textstyle\sum_{k=0}^{\mu} u_k(\rho)(1-s^2/a_1^2)^{-k}$$

where $\lim_{\rho\to 1-} u_k(\rho) = 0$ for $k > 0$ and $\lim_{\rho\to 1-} u_0(\rho) = 1$. $G(\rho, t)$ can be written as

$$G(\rho, t) = \textstyle\sum_{k=0}^{\mu} u_k(\rho)G_{1k}(\rho, t)$$

where $G_{10}(\rho, t) = G_\mu(\rho, t)$, $G_{1k}(\rho, t) = G_\mu(\rho, t) * H_k(t)$ and

$$H_k(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(1-\frac{s^2}{a_1^2}\right)^{-k} e^{st}\,ds.$$

Since $G_\mu(\rho, t) \in C^\infty(-\infty, \infty)$ also $G_{1k}(\rho, t) \in C^\infty(-\infty, \infty)$, $G'(\rho, t) = \sum_{k=0}^{\mu} u_k(\rho)G_{1k}'(\rho, t)$, and $G_{1k}'(\rho, t) = G_\mu'(\rho, t) * H_k(t)$.

We can write I_ρ in the following manner:

$$I_\rho = \frac{\pi^2}{\Omega}\sum_{k=0}^{\mu} u_k(\rho)\,(1-\rho)\int_{-\infty}^{\infty} G_{1k}'(\rho, t)\varphi(u-t)\,dt$$

since the integrals in the sum converge. For $k > 0$

$$I_{\rho,k} \equiv (1-\rho)\int_{-\infty}^{\infty} G'_{1k}(\rho, t)\varphi(u-t)\,dt = (1-\rho)\int_{-\infty}^{\infty} G'_{\mu}(\rho, t)B_k(t)\,dt$$

where

$$B_k(t) = \int_{-\infty}^{\infty} H_k(t)\varphi(u-t)\,dt = o(e^{a_1|t|}), \quad t \to \pm\infty,$$

$$|I_{\rho,k}| \le (1-\rho)M\int_{-\infty}^{\infty} |G'_{\mu}(\rho, t)| \operatorname{Cosh} a_1 t\,dt \le 2(1-\rho)G_{\mu}(\rho, 0)M +$$

$$+ (1-\rho)2M\int_{-\infty}^{\infty} G_{\mu}(\rho, t) \operatorname{Cosh} a_1 t\,dt \le L.$$

Therefore (5.2) is valid for $m = \mu$. Iteration of the same process gives (5.2) for any finite m. Choosing such m that $a_{m+1} > p$ we proceed by dividing (5.2)

$$\frac{(1-\rho)\pi^2}{\Omega}\left\{\int_{-\infty}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty}\right\} G'_m(\rho, t)\varphi(x-t)\,dt \tag{5.4}$$

$$\equiv I_1 + I_2 + I_3 + I_4 .$$

For any δ using $|\varphi(t)| \le M \operatorname{Cosh} pt$ we obtain

$$|I_4| \le \frac{(1-\rho)\,\pi^2}{\Omega} M\int_{\delta}^{\infty} G'_m(\rho, t) \operatorname{Cosh} p(x-t)\,dt$$

$$\le M_1\,(1-\rho)G_m(\rho, \delta) + M_2\,(1-\rho)\int_{-\infty}^{\infty} G_m(\rho, t) \operatorname{Cosh} p(x-t)\,dt$$

$$= o(1), \quad \rho \uparrow 1.$$

By the same method $I_1 = o(1)$, $\rho \uparrow 1$. By Theorem 3.1 we have for all $\delta > 0$,

$$\lim_{\rho\to1-} \frac{(1-\rho)\pi^2}{\Omega}\int_{-\delta}^{0} G'_m(\rho, t)\,dt = \lim_{\rho\to1-} \frac{(1-\rho)\pi^2}{\Omega}\int_{0}^{\delta} G'_m(\rho, t)\,dt = 1.$$

Choosing δ so that both $|\varphi(x+t) - \varphi(x+)| < \varepsilon$ and

$$|\varphi(x-t) - \varphi(x-)| < \varepsilon$$

for $0 < t < \delta$, we obtain

$$|I_2 - \varphi(x+)| \le \frac{(1-\rho)\pi^2}{\Omega}\int_{-\delta}^{0} |G'_m(\rho, t)|\,|\varphi(x-t) - \varphi(x+)|\,dt$$

$$+ o(1), \quad \rho \uparrow 1$$

$$\le \varepsilon\frac{(1-\rho)\pi^2}{\Omega}\int_{-\delta}^{0} G'_m(\rho, t)\,dt + o(1) \le \varepsilon\cdot 2 + o(1), \quad \rho \uparrow 1.$$

By the same method also $|I_3 + \varphi(x-)| < 2\varepsilon + o(1)$, $\rho \uparrow 1$. Since ε is arbitrary this concludes the proof of Theorem 2.2, Q. E. D.

Proof of Theorem 2.1. By Theorem 4.3, since $\alpha(t\pm)$ exists for every t and using (5.1) with

$$e^{-ct}[\alpha(x_2 - t) - \alpha(x_1 - t)], \qquad e^{-ct}[\alpha(\infty) - \alpha(x_1 - t)]$$

and

$$e^{-ct}[\alpha(x_2 - t) - \alpha(-\infty)]$$

instead of $\varphi(t)$ in cases (A), (B), and (C) respectively, we conclude the proof of the first three cases. In case (D) using Theorem 6.1a of [1, pp. 226–227]

$$\begin{aligned}
\lim_{\rho\to1-} \frac{\pi(1-\rho)}{2\Omega i} e^{-cx} \int_{c_\rho(\Omega)} f(x+\rho z)K(z)\,dz
&= \lim_{\rho\to1-} \frac{\pi^2(1-\rho)}{\Omega} \int_{-\infty}^{\infty} G(\rho, x-t)e^{-c(x-t)}\,d\alpha(t) \\
&= \lim_{\rho\to1-} -\frac{\pi^2(1-\rho)}{\Omega} \int_{-\infty}^{\infty} G'(\rho, x-t)e^{-c(x-t)}\alpha_1(t)\,dt \\
&\quad + \lim_{\rho\to1-} \frac{\pi^2(1-\rho)}{\Omega}\, c \int_{-\infty}^{\infty} G(\rho, x-t)e^{-c(x-t)}\alpha_1(t)\,dt \\
&\equiv I_1 + I_2 .
\end{aligned}$$

By Theorem 7.1b of [1, p. 231], $I_2 = 0$. $\alpha_1(t) = \alpha(t)$ for $a_1 < c < a_2$, $\alpha_1(t) = \alpha(\infty) - \alpha(t)$ for $c \geq a_1$ and $\alpha_1(t) = \alpha(t) - \alpha(-\infty)$ for $c \leq a_1$. I_1 can be estimated as cases (A), (B) and (C) of this theorem, Q. E. D.

Proof of Theorem 2.3. As in the proof of Theorem 2.2 it will be enough to prove, for μ the multiplicity of a_1 as a zero of $E(s)$,

$$\begin{aligned}
\lim_{\rho\to1-} I_\rho &= \lim_{\rho\to1-} \frac{(1-\rho)\pi^2}{\Omega} \int_{-\infty}^{\infty} G_\mu'(\rho, t)\varphi(x-t)\,dt \\
&= \varphi(x+0) - \varphi(x-0), \\
I_\rho &= \frac{(1-\rho)\pi^2}{\Omega}\left\{\int_{-\infty}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty}\right\} G_\mu'(\rho, t)\,\varphi(x-t)\,dt. \\
&\equiv I_1 + I_2 + I_3 + I_4 .
\end{aligned} \tag{5.5}$$

Evaluating I_1 we set $\alpha_1(t) = -\int_t^{-\delta} \varphi(x-y)\,dy = o(e^{-a_1 t}),\ t\to\infty$,

$$\begin{aligned}
|I_1| &= \left|\frac{(1-\rho)\pi^2}{\Omega} \int_{-\infty}^{-\delta} G_\mu''(\rho, t)\alpha_1(t)\,dt\right| \\
&\leq M\, \frac{(1-\rho)\pi^2}{\Omega} \int_{-\infty}^{-\delta} |\, G_\mu''(\rho, t) \,|\, \exp(-a_1\cdot t)\,dt.
\end{aligned} \tag{5.6}$$

Similarly

$$|I_3| \leq M\, \frac{(1-\rho)\pi^2}{\Omega} \int_{+\delta}^{\infty} |\, G_\mu''(\rho, t)\,|\, \exp(a_1\cdot t)\,dt. \tag{5.7}$$

We shall bring now more estimations for $G''_\mu(\rho, t)$ and $G'_\mu(\rho, t)$. We define $E_\rho(s)$ with the help of n_ρ defined by assumption (4).

(5.8) $E_\rho(s) = E_\mu(s)\cdot(1 - s^2/a^2_{n_\rho})^{-1}, \qquad E_\mu(s) = \prod_{k=\mu+1}^{\infty}(1 - s^2/a_k^2).$

$$(5.9) \qquad G_{*,\rho}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{E_\mu(\rho s)}{E_\rho(s)} e^{st}\, ds.$$

Hence

$$(5.10) \qquad G_{*,\rho}(t) = (1 - a_{n_\rho}^{-2} D^2)G_\mu(\rho, t).$$

Tanno proved [3] that $G_*(t)$ defined by

$$(5.11) \qquad G_*(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\prod_{k=1}^{\infty}(1 - s^2/c_k^2)\Big/\prod_{k=1}^{\infty}(1 - s^2/a_n^2)\right)e^{st}\, ds$$

where $0 < a_k < c_k$ and $\lim_{k\to\infty} a_k k^{-1} = \Omega < \lim_{k\to\infty} c_k k^{-1}$ is a density function satisfying

$$tG'_*(t) \geq 0 \quad \text{and} \quad G_*(t) \leq \frac{16}{|t|^3}\sum_{k=1}^{\infty}(a_k^{-2} - c_k^{-2}).$$

It is not hard to see that $G_{*,\rho}(t)$ satisfies the assumption on $G_*(t)$. Therefore

$$(5.12) \qquad tG'_{*,\rho}(t) \geq 0, \qquad \int_{-\infty}^{\infty} G_{*,\rho}(t)dt = 1$$

and

$$(5.13) \qquad G_{*,\rho}(t) \leq \frac{16}{|t|^3}\left\{(1 - \rho^2)\sum_{k=\mu}^{\infty} a_k^{-2} - 2a_{n_s}^{-2}\right\} \leq \frac{16}{|t|^3}(1 - \rho^2)\sum_{k=\mu}^{\infty} a_k^{-2}.$$

We define also

$$(5.14) \qquad G_{\pm,\rho}(t) = (1 \mp D/a_{n_\rho})G_\mu(\rho, t) = H_{\rho,\pm}(t)^* G_{*,\rho}(t)$$

where $H_{\rho,-}(-t) = H_{\rho,+}(t) = 0, \qquad t < 0$

$= \frac{1}{2}a_{n_\rho}, \qquad t = 0,$

$= a_{n_\rho}\exp(-t^{a_{n_\rho}}), \quad t > 0.$

Since (5.14)

$$G_{+,\rho}(u) = a_{n_\rho}\int_u^{\infty} G_{*,\rho}(t)\exp(-a_{n_\rho}(u - t))\, dt$$

(5.15) $|G_{+,\rho}(u)| \leq a_{n_\rho}$

and for $u > 0$, $0 < G_{+,\rho}(u) < \max_{t>u} G_{*,\rho}(t) = G_{*,\rho}(u)$.

Similarly

(5.16) $|G_{-,\rho}(u)| \leq a_{n_\rho}$ and for $u < 0$, $0 < G_{-,\rho}(u) < G_{*,\rho}(u)$.

Using again (5.14) and since $G_{+,\rho}(t)$ and $G_{*,\rho}(t) \in C^\infty(-\infty, \infty)$ we obtain

$$G'_{+,\rho}(u) = a_{n_\rho}\int_u^{\infty} G'_{*,\rho}(t)\exp(-a_{n_\rho}(t - u))\, dt$$

by (4.12), $G'_{+,\rho}(u) < 0$ for $u > 0$; similarly $G'_{-,\rho}(u) > 0$ for $u < 0$.

In order to estimate I_1 and I_4 write by (5.14)

$$G''_\mu(\rho, t) = a_{n_\rho} G'_\mu(\rho, t) - a_{n_\rho} G'_{+,\rho}(t) = -a_{r_\rho} G'_\mu(\rho, t) + a_{n_\rho} G'_{-,\rho}(t)$$

$$\begin{aligned}
| I_1 | &\leq \frac{M\pi^2}{\Omega}(1-\rho)\int_{-\infty}^{-\delta} | G''_\mu(\rho, t) | \exp(-a_1 t)\, dt \\
&\leq M_1 (1-\rho)a_{n_\rho}\left\{\int_{-\infty}^{-\delta} G'_\mu(\rho, t)e^{-a_1 t}\, dt + \int_{-\infty}^{-\delta} G'_{-,\rho}(t)e^{-a_1 t}\, dt\right\} \\
&\leq M_2\left\{o(1) + a_1\int_{-\infty}^{-\delta} G_{-,\rho}(t)e^{-a_1 t}\, dt + e^{-a_1\delta}G_{-,\rho}(-\delta)\right\} \\
&\leq M_2\left\{o(1) + a_1\int_{-\infty}^{-\delta} G_{*,\rho}(t)e^{-a_1 t}\, dt + e^{-a_1\delta}G_{*,\rho}(-\delta)\right\} \\
&\leq M_2\left\{o(1) + \frac{e^{+a_1\delta}}{\left(\sinh\frac{a_1\delta}{2}\right)^2}\int_{-\infty}^{-\delta} G_{*,\rho}(t)\left(\sinh\frac{a_1\delta}{2}\right)^2 dt \right. \\
&\qquad\qquad\left. + e^{-a_1\delta}\cdot\frac{16}{\delta^3}(1-\rho^2)\sum_{k=\mu}^{\infty} a_k^{-2}\right\} \\
&\leq M_2\left\{o(1) + \frac{e^{+a_1\delta}}{\left(\sinh\frac{a_1\delta}{2}\right)^2}\int_{-\infty}^{\infty} G_{*,\rho}(t)\left(\sinh\frac{a_1 t}{2}\right)^2 dt + o(1)\frac{1}{\delta^3}\right\} \\
&\leq M_2\left\{o(1) + \frac{e^{a_1\delta}}{\left(\sinh\frac{a_1\delta}{2}\right)^2}\frac{1}{4}\left[\frac{E_\mu(\rho a_1)}{E_\rho(a_1)} - 2 + \frac{E_\mu(-\rho a_1)}{E_\rho(-a_1)}\right] + o(1)\frac{1}{\delta^3}\right\} \\
&= o(1), \quad \rho\uparrow 1.
\end{aligned}$$

By the same method but using $G_{+,\rho}(t)$ instead of $G_{-,\rho}(t)$ we get $I_4 = o(1)$, $\rho \uparrow 1$ for any $\delta > 0$.

We shall show now that choosing δ so that

$$\left|\int_x^{x+h} [\varphi(x \pm t) - \varphi(x \pm 0)]\, dt\right| \leq \varepsilon h \quad \text{for all} \quad h \leq \delta$$

we get

$$| I_2 - \varphi(x+0) | < \varepsilon M \quad \text{and} \quad | I_3 + \varphi(x-0) | < \varepsilon M,$$

$$\begin{aligned}
| I_2 &- \varphi(x+0) | \\
&= \left|\frac{(1-\rho)\pi^2}{\Omega}\int_{-\delta}^{0} G'_\mu(\rho, t)[\varphi(x-t) - \varphi(x+0)]\, dt\right| + o(1), \quad \rho\uparrow 1 \\
&\leq \frac{(1-\rho)\pi^2}{\Omega}\left\{\varepsilon\int_{-\delta}^{0} | G''_\mu(\rho, t) | \, |t|\, dt + G'_\mu(\rho, -\delta)\varepsilon\delta\right\} \\
&\leq \frac{a_{n_\rho}(1-\rho)\pi^2}{\Omega}\,\varepsilon\left\{-\int_{-\delta}^{0} G'_\mu(\rho, t)t\, dt - \int_{-\delta}^{0} G'_{-,\rho}(t)t\, dt \right. \\
&\qquad\qquad\left. + G_\mu(\rho, -\delta)\delta + G_{-,\rho}(-\delta)\delta\right\}
\end{aligned}$$

$$\leq M_1 \varepsilon \left\{ \int_\delta^0 G_\mu(\rho, t)\, dt + \int_{-\delta}^0 G_{-,\rho}(t)\, dt + 2G_\mu(\rho, -\delta)\delta + 2G_{*,\rho}(-\delta)\delta \right\}$$

$$\leq M_1 \varepsilon \left\{ 2 + 4^3 \frac{(1-\rho^2) \sum a_k^{-2}}{\delta^2} \right\} \leq \varepsilon M.$$

The same method but taking $G_{+,\rho}(t)$ instead of $G_{-,\rho}(t)$ yields

$$| I_3 + \varphi(x - 0) | < \varepsilon M + o(1), \quad \rho \uparrow 1.$$

The fact that ε is arbitrary completes the proof of the theorem, Q. E. D.

6. Application to jump formula

Theorems 2.1–2.3 enable us to find a new jump formula for the Stieltjes transform, its iterates and its generalization mentioned by Sumner [2].

The result for Stieltjes transform was already state in Theorem 1.1 which is a particular case of Theorem 2.3.

Proof of Theorem 1.1. Using the transformation from (1.4) to (1.1) given in [1, p. 69] namely

$$f(x) = F(e^x)e^{x/2}, \qquad \varphi(x) = \pi e^{x/2}\Phi(e^x), \qquad G(t) = (1/2\pi) \operatorname{Sech}(t/2)$$

and by [1, p. 225], $K(z) = z/(\pi^2 + z^2)$, we get

$$\varphi(x+0) - \varphi(x-0) = \lim_{\rho \to 1-} -\tfrac{1}{2}\pi^2(1-\rho)[f'(x - i\pi\rho) + f'(x + i\pi\rho)],$$

$$\Phi(\xi + 0) - \Phi(\xi - 0) = \lim_{\rho \to 1-} \tfrac{1}{2}\pi(1-\rho)[F'(\xi e^{-i\pi\rho})\xi e^{-3i\pi\rho/2}$$

$$+ F'(\xi e^{i\pi\rho})\xi e^{3i\pi\rho/2}] - \lim_{\rho \to 1-} \tfrac{1}{2}\pi(1-\rho)[F(\xi e^{-i\pi\rho})e^{-i\pi\rho/2} + F(\xi e^{+i\pi\rho})e^{i\pi\rho/2}.$$

Since the second part is zero by the known complex inversion formula we get for $\eta = \pi(1-\rho)\xi$,

$$\begin{aligned} &\Phi(\xi + 0) - \Phi(\xi - 0) \\ &\qquad = \lim_{\eta \to 0+} \tfrac{1}{2}\eta i[F'(-\xi - i\eta) - F'(-\xi + i\eta)], \quad \text{Q. E. D.} \end{aligned} \tag{6.1}$$

Remark. Under the assumptions of Theorem 1.1 one can prove directly the more general formula

$$\begin{aligned} &\Phi(\xi + 0) - \Phi(\xi - 0) \\ &\qquad = \lim_{\eta \to 0+} i\eta F'(-\xi - i\eta) = \lim_{\eta \to 0+} i\eta F'(-\xi + i\eta). \end{aligned} \tag{6.2}$$

Similarly we obtain, using Theorem 2.1.(D) with $c = -\frac{1}{2}$ for $F(x)$ defined by

$$F(x) = \int_0^\infty \frac{d\alpha(t)}{x + t}, \tag{6.3}$$

the following result:

$$\alpha(\xi+) - \alpha(\xi-) = \lim_{\eta \to 0+} + \tfrac{1}{2}\eta i[F(-\xi + i\eta) - F(-\xi - i\eta)]. \tag{6.4}$$

However one can prove directly that

$$\begin{aligned}\alpha(\xi+) - \alpha(\xi-) &= \lim_{\eta\to 0+} i\eta F(-\xi + i\eta) \\ &= \lim_{\eta\to 0+} - i\eta F(-\xi - i\eta).\end{aligned} \tag{6.5}$$

References

1. I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton Univ. Press, Princeton, 1955.
2. D. B. Sumner, *An inversion formula for the generalized Stieltjes transform*, Bull. Amer. Math. Soc., vol. 55 (1949) pp. 174–192.
3. Y. Tanno, *On the convolution transform*, Kōdai Math. Sem. Rep., vol. 11 (1959) pp. 40–50.

Washington University
St. Louis, Missouri
Michigan State University
East Lansing, Michigan