THE LOGARITHMIC POTENTIAL OPERATOR

BY
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Introduction

Let O be a bounded open plane set (not necessarily connected) with boundary components consisting of smooth Jordan curves. Let $\tilde{O}$ denote the open set complementary to $\bar{O}$.

The classical eigenvalue problems associated with the solutions of the differential equation

$$\frac{\lambda}{4} \nabla^2 \phi + \phi = 0$$

in O, arise from prescribing boundary values for ϕ or its normal derivative, or more generally, a relation between them [1].

However, it is also possible to ask for those solutions of the equation which are related to solutions of another equation in the complementary set $\tilde{O}$ through prescribed matching conditions at the common boundary.

Some properties of the solutions to the following problem of this type will be obtained as one of the results of the present paper:

Find those solutions of the given differential equation which admit a continuously differentiable extension to harmonic function(s) in the complementary set $\tilde{O}$, having the development

$$\phi(z) = k \log | z | + O\,(1/ | z |) \qquad \text{near infinity}.$$

Application of Green's identity to an annulus centered at any point z in the complex plane E, yields after a standard limiting argument, the equation

$$\lambda\phi(z) = -\frac{2}{\pi} \int_O \log | z - \zeta | \, \phi(\zeta) \, d\tau_\zeta$$

where τ denotes two-dimensional Lebesgue measure. Thus ϕ is seen to be an eigenfunction to an operator of integral type having as its kernel $-(2/\pi)\log | z - \zeta |$, and defined for sufficiently smooth functions.

More generally, if S is any plane set having positive measure, consider the associated logarithmic operator L defined by

$$(1) \qquad (Lf)(z) = -\frac{2}{\pi} \int_S \log | z - \zeta | f(\zeta) \, d\tau_\zeta , \qquad z \in S,$$

for any f which is square integrable over S.

The integral in (1) has received considerable attention in the case where $f \, d\tau$ is a regular Borel measure on S [2], [3] and is used to define the logarithmic capacity of S. The corresponding operators in higher dimensional space have

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been analyzed in detail by Cotlar [4] through the theory of singular integrals. However, the simpler operator presented above has several interesting properties which apparently have not been uncovered in previous investigations.

The foremost of these, is that L can have at most one negative eigenvalue, and its existence depends solely upon the magnitude of the transfinite diameter of the closure of the support of S (defined below). When suitably normalized, there is a unique real eigenfunction associated with the negative eigenvalue, and it has a continuously differentiable extension to the entire plane, E, which is everywhere positive and subharmonic.

The support of a plane set

If S is a Lebesgue measurable set in the plane, then S^*, the support of S, is defined as follows:

$$S^* = \{z : z \in S, \Delta r(z) \cap S \text{ has positive measure for any } r > 0\}$$

$$\Delta r(z) = \{\zeta : |z - \zeta| < r\}.$$

If S has positive measure then S^* is non-empty, and thus $S \sim S^*$ has measure zero. S^* is relatively closed in S, and may be related to the usual definition of the support of τ restricted to S, but the above definition is more useful for the present considerations.

Any function f which is integrable over S is integrable over S^* and

$$\int_S f\, d\tau = \int_{S^*} f\, d\tau.$$

Hence the operator L when defined as in (1) will have the same range for all sets S having a common support S^*, and so it will be assumed hereafter, whenever necessary, that S is itself a support set. (Observe that $(S^*)^* = S^*$.) It will also be assumed that S and $\bar{S}$ have the same measure.

The operator L

As is usual, let $L^2(S)$ be the space of measurable functions on S which are square integrable over S. For $f \in L^2(S)$, let $\|f\|_p = (\int_S |f|^p\, d\tau)^{1/p}$, $p = 1, 2$.

Application of the Schwarz inequality to (1) yields the following estimates for $f \in L^2(S)$, and for z, z_0 in any compact set K:

$$|(Lf)(z)| \leq \frac{2}{\pi} \sup_{z \in K} \left(\int_S (\log |z - \zeta|)^2\, d\tau_\zeta \right)^{1/2} \|f\|_2 \leq M_K \|f\|_2$$

$$|(Lf)(z) - (Lf)(z_0)| \leq \frac{2}{\pi} \left(\int_S \left(\log \frac{|z - \zeta|}{|z_0 - \zeta|} \right)^2 d\tau_\zeta \right)^{1/2} \|f\|_2$$

$$\leq |z - z_0|^\alpha \frac{M_K}{\alpha} \|f\|_2 \qquad \text{each } \alpha \in (0, 1)$$

(utilizing the inequality $\log(1 + X) \leq X^\alpha/\alpha$ for $X > 0$, $\alpha \in (0, 1)$). M_K is used generically to denote a positive constant which depends on the compact set K.

Hence L maps bounded sets of $L^2(S)$ into equibounded equicontinuous sets of $C^\alpha(K)$ for each $\alpha \in (0, 1)$ and each compact set $K \subset E$, the complex plane. (For each $\alpha \in (0, 1)$ and each integer $n = 0, 1, 2, \cdots, C^{n+\alpha}(S)$ is the space of continuous functions on a set S, having n^{th} order partial derivatives in S^0, which are Hölder continuous on S with exponent α.) From the Ascoli theorem, it follows that L is a compact operator from $L^2(S) \to C^\alpha(K)$ when the latter space is supplied with the topology of uniform convergence.

Since L is compact and has a real symmetric kernel, it is well known that its spectrum consists of at most a bounded sequence of real numbers having zero as the only possible limit point. Each non-zero point in the spectrum is an eigenvalue of finite multiplicity, and there is at least one such eigenvalue [5].

Let λ be an eigenvalue to L, and ϕ an associated eigenfunction which may be assumed to be real. Then, as a function in the range of L, ϕ may be extended continuously to E through the defining equation

$$(2) \qquad \lambda\phi(z) = -\frac{2}{\pi}\int_S \log|z-\zeta|\,\phi(\zeta)\,d\tau_\zeta, \qquad \text{all } z \in E.$$

LEMMA. *Let λ be an eigenvalue of L, and ϕ an associated real eigenfunction. Then for each $z \in E$ and each $r > 0$, the following formula holds:*

$$(3) \qquad \lambda\phi(z) = \frac{\lambda}{2\pi}\int_0^{2\pi} \phi(z+re^{i\theta})\,d\theta - \frac{2}{\pi}\int_{S\cap\Delta_r} \log\frac{|z-\zeta|}{r}\,\phi(\zeta)\,d\tau_\zeta$$

where $\Delta_r = \{\zeta \mid |z-\zeta| < r\}$.

Proof. From (2),

$$\begin{aligned}\lambda\phi(z+re^{i\theta}) &= -\frac{2}{\pi}\int_S \log|z+re^{i\theta}-\zeta|\,\phi(\zeta)\,d\tau_\zeta \\ &= -\frac{2}{\pi}\left\{\int_{S\cap\Delta_r} \log|z+re^{i\theta}-\zeta|\,\phi(\zeta)\,d\tau_\zeta \right.\\ &\qquad \left. + \int_{S\cap\{\zeta:|z-\zeta|>r\}} \log|z+re^{i\theta}-\zeta|\,\phi(\zeta)\,d\tau_\zeta\right\}.\end{aligned}$$

Integrating over θ, interchanging the orders of integration (which is permissible) and subsequently utilizing the well-known formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1-ae^{i\theta}|\,d\theta = \min(0, \log|a|)$$

yields

$$\begin{aligned}&\lambda\frac{1}{2\pi}\int_0^{2\pi} \phi(z+re^{i\theta})\,d\theta \\ &\qquad = -\frac{2}{\pi}\int_S \log|z-\zeta|\,\phi(\zeta)\,d\tau_\zeta + \frac{2}{\pi}\int_{S\cap\Delta_r} \log\frac{|z-\zeta|}{r}\,\phi(\zeta)\,d\tau_\zeta\end{aligned}$$

which gives the desired result.

COROLLARY 1. $\{0\}$ *is not an eigenvalue* (*not in the point spectrum*) *of* L.

Proof. If $\lambda = 0$, then (3) shows that for any possible associated eigenfunction ϕ,

$$\int_{S\cap\Delta_r} \log \frac{|z - \zeta|}{r} \phi(\zeta)\, d\tau_\zeta = 0 \qquad \text{for all } z, r.$$

From continuity of ϕ, it follows that $\phi = 0$ a.e. on S, and hence ϕ is not an eigenfunction.

COROLLARY 2. *If an eigenfunction of* L *is constant in a neighborhood of a point of* S, *then that constant value is zero. In particular, the eigenfunctions of* L *cannot be constant on* S.

Proof. (Recall that S is assumed to coincide with its support.) The hypotheses are exactly those needed to guarantee that the eigenfunction ϕ has its constant value, say a, in a set $S \cap \Delta_r$ of positive measure. Since the associated eigenvalue cannot be zero, (3) may again be applied to yield

$$a \int_{S\cap\Delta_r} \log \frac{|z - \zeta|}{r}\, d\tau_\zeta = 0$$

and this gives the desired result.

COROLLARY 3. *Each eigenfunction to* L *is harmonic in* $\tilde{S}$, *the open set complementary to* $\bar{S}$, *and near infinity has the development*

$$\phi(z) = \frac{2}{\pi\lambda} \log |z| \int_S \phi\, d\tau + O\left(\frac{1}{|x|}\right) \tag{4}$$

Proof. This well-known property of logarithmic potentials follows from (3) and the fact that for $z \in \tilde{S}$, the area integral vanishes for sufficiently small r. Thus ϕ has the Gauss mean value property in $\tilde{S}$. The development is a trivial consequence of definition (2).

Negative eigenvalue(s) and associated eigenfunctions

More interesting properties of the operator L emerge when attention is restricted to its negative eigenvalues.

THEOREM 1. *If* λ *is a negative eigenvalue to* L, *and* ϕ *is an associated eigenfunction so normalized that* $\int_S \phi\, d\tau \geq 0$, *then* ϕ *is positive and subharmonic everywhere in* E.

Proof. The development (4) together with the hypothesized normalization $\int_S \phi\, d\tau \geq 0$ imply that ϕ cannot have a negative limiting value at infinity. Thus if ϕ assumes a negative value in E, it must have a strict negative minimum at some point $z_0 \in E$. However, ϕ remains negative in a neighborhood of z_0, and so by (3)

$$\phi(z_0) \geq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta})\, d\theta,$$

for sufficiently small $r > 0$.

Hence ϕ cannot have a strict negative minimum at z_0, and so $\phi \geq 0$ everywhere in E.

To show that ϕ must be positive everywhere, consider (3) for any fixed $z \in E$, and for $r > 0$, let

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(z + re^{i\theta})\, d\theta$$

$$= \phi(z) + \frac{2}{\pi\lambda} \int_{S \cap \Delta_r} \log \frac{|z - \zeta|}{r} \phi(\zeta)\, d\tau_\zeta .$$

For $r > r_0 > 0$,

$$\frac{M(r) - M(r_0)}{r - r_0} = \frac{2}{\pi\lambda(r - r_0)} \log \frac{r_0}{r} \int_{S \cap \Delta_{r_0}} \phi(\zeta)\, d_\zeta$$

$$+ \frac{2}{\pi\lambda(r - r_0)} \int_{S \cap [\Delta_r \sim \Delta_{r_0}]} \log \frac{r}{|z - \zeta|} \phi(\zeta)\, d\tau_\zeta .$$

It is easily shown that the second term on the right approaches zero as $r \rightarrow r_0$. Hence

$$0 < M'(r) = -\frac{2}{\pi\lambda r} \int_{S \cap \Delta_r} \phi(\zeta)\, d\tau_\zeta , \qquad \text{for any } r > 0$$

$$\leq -\frac{2}{\pi\lambda r} \int_{\Delta_r} \phi(\zeta)\, d\tau_\zeta = -\frac{4}{\lambda r} \int_0^r \rho M(\rho)\, d\rho$$

or

$$M'(r) \leq -(2/\lambda) r M(r).$$

Integrating this inequality gives

$$M(r) \leq M(r_0) \exp\ (-(r^2 - r_0^2)/\lambda) \qquad \text{for } r > r_0 > 0$$

and hence

$$M(r) \leq \phi(z)\ e^{-r^2/\lambda} \qquad \text{for any } r \geq 0.$$

If $\phi(z) = 0$ for any $z \in E$, then $M(r) \equiv 0$ which implies that $\phi \equiv 0$, a contradiction.

Therefore $\phi > 0$ everywhere, and again from (3)

$$\phi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z + re^{i\theta})\, d\theta$$

for any $z \in E$, $r > 0$. This proves that ϕ is subharmonic everywhere [6].

Finally, from the development (4), it follows that ϕ becomes logarithmically infinite at infinity.

THEOREM 2 (uniqueness). *L can have at most one negative eigenvalue, and the associated eigenfunction when suitably normalized is unique.*

Proof. Suppose there were two linearly independent real eigenfunctions ϕ_1, ϕ_2 associated with negative eigenvalues of L (not necessarily distinct). Then it may be assumed that they are orthogonal so that $\int_S \phi_1 \phi_2\, d\tau = 0$ and, from the preceding theorem, that each is positive. The contradiction is apparent.

Hence there can be within a constant factor at most one eigenfunction associated with the negative eigenvalues of L. In particular, there can be at most one negative eigenvalue, since the eigenfunctions associated with distinct eigenvalues are always linearly independent.

Hereafter, the unique negative eigenvalue of L will be denoted by μ when it exists, and the unique associated positive normalized eigenfunction by ψ, i.e.

$$\int_S \psi^2 \, d\tau = 1. \tag{5}$$

Existence and relation to transfinite diameter

From Hilbert space theory, it is known that if μ exists, then it satisfies the relations

$$\begin{aligned}\mu &= \inf_{f \in L_1^2(D)} \left\{ -\frac{2}{\pi} \int_S \int_S \log | z - \zeta | f(z) f(\zeta) \, d\tau_z \, d\tau_\zeta \right\} \\ &= -\frac{2}{\pi} \int_S \int_S \log | z - \zeta | \psi(z) \psi(\zeta) \, d\tau_z \, d\tau_\zeta\end{aligned}$$

where $L_1^2(S) = \{f : f \in L^2(S) \text{ with } \| f \|_2 \leq 1\}$.

From the studies of Pólya and Szegö [7], [2] it is known that the transfinite diameter d of $\bar{S}$ satisfies a similar inequality, viz.,

$$-\log d \leq \inf_{\nu \in \mathfrak{B}_1} \left\{ -\int_{\bar{S}} \int_{\bar{S}} \log | z - \zeta | \, d\nu(z) \, d\nu(\zeta) \right\}$$

where $\mathfrak{B}_1$ is the class of regular Borel measures ν on $\bar{S}$ normalized so that $\nu(\bar{S}) = 1$.

The relation implied by the strong similarity of these definitions manifests itself in the following:

THEOREM 3. *The operator L possesses a negative eigenvalue iff the transfinite diameter d of $\bar{S}$ exceeds one, in which case $-\mu < (2/\pi) A \log d$, where A is the area of S.*

Proof. Assume μ exists and let ψ be the unique associated non-negative eigenfunction. Then the restriction of the measure $\psi \, d\tau / \| \psi \|_1$ to S is in $\mathfrak{B}_1$. Hence

$$\begin{aligned}-\log d &\leq -\int_S \int_S \log | z - \zeta | \frac{\psi(z)\psi(\zeta) \, d\tau_z \, d\tau_\zeta}{(\| \psi \|_1)^2} \\ &= \frac{\pi}{2} \frac{\mu}{(\| \psi \|_1)^2}.\end{aligned}$$

By the Schwarz inequality and assumed normalization of ψ,

$$(\| \psi \|_1)^2 < \int_S d\tau_\zeta \int_S \psi^2(z) \, d\tau_z = A$$

with strict inequality since ψ is not constant. Therefore, $-\mu < (2/\pi) A \log d$

as asserted. Moreover, $\log d > 0 \Rightarrow d > 1$ proving the necessity of this condition.

To prove its sufficiency, it is necessary to resort to the general theory of the transfinite diameter as initiated by Fekete. It is known [2] that for each positive integer N there exist distinct points $z_1, z_2, \cdots, z_N$ in S for which the positive numbers

$$d_N = \prod_{i,j=1;\, i<j}^{N} | z_i - z_j |^{2/N(N-1)}$$

define a sequence coverging to d monotonically from above.

For a given N define

$$r_n = (1/4n) \min_{1 \le i < j \le N}(| z_i - z_j |, \tfrac{1}{2}), \quad \text{for } n = 1, 2, \cdots$$

and

$$\begin{aligned} U_{jn} &= S \cap \{z : | z - z_j | \le r_n\}, & j &= 1, 2, \cdots, N \\ m_{jn} &= \tau(U_{jn}), & n &= 1, 2, \cdots \end{aligned}$$

where τ is plane Lebesgue measure. Since S is assumed to be a support set, each $m_{jn} > 0$ and the Borel measures ν_n are defined through the simple functions

$$\begin{aligned} f_n(z) &= 1/Nm_{jn}, \quad z \in U_{jn}, & j &= 1, 2, \cdots, N; n = 1, 2, \cdots \\ &= 0, \quad \text{otherwise} \end{aligned}$$

by

$$d\nu_n = f_n \, d\tau, \qquad n = 1, 2, \cdots.$$

The measures ν_n are absolutely continuous with respect to τ by definition and are normalized by construction. Hence each $\nu_n \in \mathcal{B}_1$. For each $n = 1, 2, \cdots$, consider the integral

$$I[\nu_n] = -\int_S \int_S \log | z - \zeta | \, d\nu_n(z) \, d\nu_n(\zeta)$$

which may be split as follows:

$$I[\nu_n] = -\int_{S \times S} \log | z - \zeta | \, d\lambda_n \tag{6}$$
$$- \sum_{j=1}^{N} \frac{1}{(Nm_{jn})^2} \int_{U_{jn}} \int_{U_{jn}} \log | z - \zeta | \, d\tau_z \, d\tau_\zeta$$

where λ_n is the Borel measure on the product space $\bar{S} \times \bar{S}$ defined through the simple function

$$\begin{aligned} \Lambda_n(z, \zeta) &= f_n(z) f_n(\zeta), \quad (z, \zeta) \in U_{in} \times U_{jn}, & i &\ne j; i, j = 1, 2, \cdots, N \\ &= 0, \quad \text{otherwise} \end{aligned}$$

by

$$\lambda_n = \Lambda_n \, d\tau \times d\tau.$$

Note that

$$\int_{S \times S} d\lambda_n \le \int_S \int_S d\nu_n(z) \, d\nu_n(\zeta) = 1$$

By Alaoglu's theorem [8], there exists a subsequence $\{\lambda_{n_k}\}$ and a limit measure λ_N for which

$$\lim_{k\to\infty} \int_{S\times S} g \, d\lambda_{n_k} = \int_{S\times S} g \, d\lambda_N$$

for each function $g \in C(\bar{S} \times \bar{S})$. It is simple to verify that the limit measure λ_N is the atomic measure which assigns to each point (z_i, z_j), $i \neq j$; $i, j = 1, 2, \cdots, N$, the mass N^{-2}.

Following Hille [2], introduce for each $M > 0$ the continuous function

$$[\log |z - \zeta|]_M = \min (\log M, -\log |z - \zeta|)$$

and note that for $M > r^{-1}$

$$\begin{aligned}\lim_{k\to\infty} - \int_{S\times S} \log |z - \zeta| \, d\lambda_{n_k} &= \lim_{k\to\infty} \int_{S\times S} [\log |z - \zeta|]_M \, d\lambda_{n_k} \\ &= \int_{S\times S} [\log |z - \zeta|]_M \, d\lambda_N \\ &= -\int_{S\times S} \log |z - \zeta| \, d\lambda_N \\ &= -\left(\frac{1}{N}\right)^2 \sum_{i,j=1}^{N} \log |z_i - z_j| \\ &= -\frac{N-1}{N} \log d_N .\end{aligned}$$

The remaining term in (6) approaches zero as $n \to \infty$, since for $z, \zeta \in U_{jn}$, it follows that $|z - \zeta| \leq 2r_n < 1/n$. Thus

$$\begin{aligned}0 \leq \int_{U_{jn}} (-\log |z - \zeta|) \, d\tau \leq K \int_{U_{jn}} \frac{d\tau_\zeta}{|z - \zeta|^{1/2}} \\ \leq 2\pi K \int_0^{2r_n} \frac{r \, dr}{r^{1/2}} = K'(r_n)^{3/2}\end{aligned}$$

and so

$$\begin{aligned}0 &\leq \sum_{j=1}^{N} \frac{1}{(Nm_{jn})^2} \int_{U_{jn}} d\tau_z \int_{U_{jn}} (-\log |z - \zeta|) \, d\tau_\zeta \\ &\leq \frac{K'(r_n)^{3/2}}{N} \sum_{j=1}^{N} \frac{1}{Nm_{jn}} \leq \frac{K'}{N} \left(\frac{1}{n}\right)^{3/2}\end{aligned}$$

where K, K' are positive constants. Therefore there exists an n_k so large that

$$-\log d \leq I[\nu_{n_k}] \leq -\frac{N-1}{N} \log d_N + \frac{1}{N}$$

If $d > 1$, then the right-hand side of the inequality is negative for N sufficiently large. Hence there exist functions $f_n \in L^2(S)$ for which

$$-\int_S \int_S \log |z - \zeta| f_n(z) f_n(\zeta) \, d\tau_z \, d\tau_\zeta < 0$$

which by Hilbert space theory guarantees the existence of a negative eigenvalue for L.

As a result of the previous theorem, it follows that the L operator is positive for support sets of transfinite diameter not exceeding one, while for the other support sets, the operator of integral type with kernel

$$-(2/\pi)\log|z - \zeta| + \mu\psi(z)\psi(\zeta) \qquad (z \neq \zeta)$$

is positive [5].

Of more immediate significance is the fact that for support sets of transfinite diameter exceeding one, the eigenvalue μ constitutes a new and well-defined functional, which is related to the transfinite diameter through the inequality

$$-\mu < (2/\pi)A \log d \qquad \text{(Theorem 3).}$$

The dependence of this functional on the support set has also been investigated, and the results will be presented in a subsequent paper.

Differentiability

The logarithmic operator was introduced through an eigenvalue problem related to the solutions of the classical differential equation $(\lambda/4)\nabla^2\phi + \phi = 0$ in a bounded open set. We conclude this paper with a brief study of the differentiability of the eigenfunctions of the logarithmic operator.

Let S be a bounded support set, and L, the logarithmic operator on $L^2(S)$. Suppose that λ is an eigenvalue of L and ϕ is an associated (real) eigenfunction. It has already been proven that $\phi \in C^\alpha(K)$ for each compact set $K \subset E$ and for each $\alpha \in (0, 1)$, and that ϕ is harmonic in $\tilde{S}$.

Following Vekua [9], we introduce the operator T by

$$(Tf)(z) = -\frac{1}{\pi}\int_S \frac{f(\zeta)\, d\tau_\zeta}{\zeta - z}$$

and observe that from the estimate

$$\int_K |Tf|^2\, d\tau \leq \frac{1}{\pi^2}\int_K d\tau_z \int_S \frac{d\tau_\eta}{|\eta - z|}\int_S \frac{|f(\zeta)|^2\, d\tau_\zeta}{|\zeta - z|} \leq M_K(\|f\|_2)^2$$

it follows that T is also a bounded operator from $L^2(S)$ into $L^2(K)$ for each compact set $K \subset E$.

Inasmuch as $(Tf)(z)$ may be obtained by formally differentiating $(-Lf(z))$ with respect to z (in the complex sense), it is not difficult to show that Tf is indeed the weak z derivative of $-Lf$ relative to $L^2(K)$ for any compact set $K \subset E$. Moreover, when $f \in C^\alpha(\bar{S})$, then defining $f = 0$ in $\tilde{S}$ gives

$$\begin{aligned} |Tf(z) - Tf(z_0)| &\leq \frac{1}{\pi}\int_E \frac{|f(\zeta + z) - f(\zeta + z_0)|}{|\zeta|}\, d\tau_\zeta \\ &\leq M_K|z - z_0|^\alpha \qquad \text{for } z, z_0 \in K. \end{aligned}$$

Hence $Tf \in C^\alpha(K)$ also. In particular then, each eigenfunction $\phi \in C^1(E)$ and is in $C^{1+\alpha}(K)$ for each compact set $K \subset E$ and each $\alpha \in (0, 1)$.

By resorting to the theory of singular integrals, Vekua establishes much stronger results, one of which implies that if D is a disc and $f \in C^\alpha(\bar{D})$, then

$Tf \in C^{1+\alpha}(D)$, and

$$\frac{\partial}{\partial z}(Tf)(z) = f(z), \quad z \in D$$
$$= 0, \quad z \notin \bar{D}.$$

Using this fact with a partition of unity yields the desired

THEOREM 4. *Let S be a bounded support set. Then any eigenfunction ϕ to the logarithmic operator L on $L^2(S)$ has a continuously differentiable extension to E. Moreover ϕ is twice continuously differentiable in S^0 (the interior of S) and there satisfies the equation*:

$$(\lambda/4)\nabla^2\phi + \phi = 0.$$

Proof. Since $\phi \in C^\alpha(\bar{S})$, then for each disc $D \subset S^0$,

$$\lambda\phi(z) = -\frac{2}{\pi}\int_{D_1} \log|z-\zeta|\, p(\zeta)\phi(\zeta)\, d\tau_\zeta$$
$$-\frac{2}{\pi}\int_{S\sim D_1} \log|z-\zeta|\,[1-p(\zeta)]\phi(\zeta)\, d\tau_\zeta$$

where $p(\zeta)$ is a C^∞ function which is radially symmetric with respect to the center of the disc D, is unity on a concentric subdisc D_1 and vanishes identically outside D. Vekua's theorem applied to the first integral gives for $z \in D_1^0$,

$$\lambda\frac{\partial^2\phi}{\partial\bar{z}\partial z} = \frac{\lambda}{4}\nabla^2\phi = -p\phi + 0 = -\phi$$

since the second integral is harmonic for $z \in D_1^0$. We remark that arbitrary support sets may have empty interiors and hence the major assertion of the theorem may be vacuuous. However, if S is open, then $S^* = S = S^0$ and we have identified the eigenfunctions and eigenvalues of L with those of the problem posed in the introduction.

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