ALGEBRAIC MODELS FOR MEASURES

BY

N. DINCULEANU AND C. FOIAŞ

1. Introduction

The purpose of this paper is to study probability measure spaces (X, Σ, μ) by means of algebraic models (Γ, φ) consisting of an abelian group Γ and a function of positive type φ on Γ (see Definitions 2 and 3). Algebraic models determine uniquely the measures, in the sense that two measures are essentially equal, or conjugate (see Definition 1) if and only if they possess isomorphic algebraic models (Theorem 2). Every algebraic measure system (Γ, φ) is an algebraic model for a certain measure (Theorem 3). In particular, we obtain a new reduction of a measure μ on an abstract set, to a regular Borel measure μ' on an abelian compact group (Theorem 4), and we give conditions in order that μ' should be a Haar measure (Theorem 5).

2. Conjugate measures

Let (X, Σ, μ) be a probability measure space. We denote by $\Gamma(\mu)$ the set of the (equivalence classes of) functions $f \in L^\infty(\mu)$ with $|f| \equiv 1$. Then $\Gamma(\mu)$ is a *multiplicative group* with the complex conjugate $\bar{f}$ as inverse of a function $f \in \Gamma(\mu)$. If we identify the circle group C with the constant functions of $\Gamma(\mu)$, we have $C \subset \Gamma(\mu)$.

Remark. Using the existence of a lifting (see [4]) we can consider that $\Gamma(\mu)$ is a group of μ-measurable functions $f : T \to E$ with $|f| \equiv 1$, such that $f, g \in \Gamma(\mu)$ and $f(x) = g(x)$ (μ-almost everywhere) imply $f(x) = g(x)$ for every $x \in X$.

We define the complex function φ_μ on $\Gamma(\mu)$ by

$$\varphi_\mu(f) = \int f \, d\mu \quad \text{for} \quad f \in \Gamma(\mu).$$

PROPOSITION 1. *φ_μ is a function of positive type on $\Gamma(\mu)$ and*

$$\varphi_\mu(f) = 1 \quad \textit{if and only if} \quad f = 1.$$

In fact, for every family $(f_i)_{1 \le i \le n}$ of functions of $\Gamma(\mu)$ and for every family $(\alpha_i)_{1 \le i \le n}$ of complex numbers we have

$$\sum_{i,j} \alpha_i \bar{\alpha}_j \varphi_\mu(f_i f_j^{-1}) = \sum_{i,j} \alpha_i \bar{\alpha}_j \int f_i \bar{f}_j \, d\mu = \int \Big| \sum_k \alpha_k f_k \Big|^2 d\mu \ge 0;$$

therefore φ_μ is of positive type.

Received March 16, 1967.

If $f = 1$, we have $\varphi_\mu(f) = \mu(X) = 1$. Conversely, suppose that $\varphi_\mu(f) = 1$. If we write $f = g + ih$ where g and h are real valued, then

$$\int g\, d\mu + i \int h\, d\mu = \int f\, d\mu = \varphi_\mu(f) = 1;$$

therefore $\int h\, d\mu = 0$. It follows that

$$1 = \varphi_\mu(f) = \int g\, d\mu = \int g^+\, d\mu - \int g^-\, d\mu \leq \int g^+\, d\mu$$

$$\leq \int |g|\, d\mu \leq \int |f|\, d\mu = 1;$$

consequently $\int g^-\, d\mu = 0$, hence $g^- = 0$ and $g = |g|$. Then $\int |g|\, d\mu = 1$; therefore $|g| = 1$, since $|g| \leq 1$. We deduce that $h = 0$ and $f = g = |g| = 1$.

DEFINITION 1. Let (X, Σ, μ) and (X', Σ', μ') be two probability measure spaces. We say that the measures μ and μ' are conjugate if there exists a linear isometry ϕ of $L^2(\mu)$ onto $L^2(\mu')$ such that $\phi L^\infty(\mu) \subset L^\infty(\mu')$ and

$$\phi(fg) = \phi f \cdot \phi g \qquad \text{for } f, g \in L^\infty(\mu).$$

It follows then (see Theorem 1 below) that $\phi L^\infty(\mu) = \phi L^\infty(\mu')$ and

$$\| \phi f \|_\infty = \| f \|_\infty \quad \text{for} \quad f \in L^\infty(\mu).$$

Remark 1°. Let (B, μ) and (B', μ') be the measure algebras associated with (X, Σ, μ) respectively (X', Σ', μ'). To say that μ and μ' are conjugate means that there exists a measure-preserving isomorphism S of the Boolean σ-algebra B onto the Boolean σ-algebra B' (see [3, p. 42–45]).

If, for example, there exists an invertible measure-preserving transformation $T : X' \to X$, then μ and μ' are conjugate, by the isometry $\phi : L^2(\mu) \to L^2(\mu')$ defined by $\phi f = f \circ T$, for $f \in L^2(\mu)$.

Remark 2°. Consider the identity mappings I and I' of X and X' respectively. Then I and I' are measure-preserving transformations. To say that u and μ' are conjugate means that I and I' are conjugate (see [3, p. 44–45]).

PROPOSITION 2. *Let (X, Σ, μ) and (X', Σ', μ') be two probability measure spaces. If μ and μ' are conjugate, then there exists an isomorphism*

$$\phi : \Gamma(\mu) \to \Gamma(\mu')$$

such that

(i) $\phi\Gamma(\mu) = \Gamma(\mu')$;

(ii) $\phi c = c$ *for* $c \in C$;

(iii) if $\Gamma \subset \Gamma(\mu)$ *is a set generating* $L^2(\mu)$, *then* $\phi\Gamma$ *generates* $L^2(\mu')$;

(iv) *if* $\Gamma \subset \Gamma(\mu)$ *is an orthonormal system in* $L^2(\mu)$, *then* $\phi\Gamma$ *is orthonormal in* $L^2(\mu')$;

(v) $\varphi_\mu(f) = \varphi_{\mu'}(\phi f)$, *for* $f \in \Gamma(\mu)$.

Let $\phi : L^2(\mu) \to L^2(\mu')$ be a linear isometry realizing the conjugacy between μ and μ'.

Let $f \in \Gamma(\mu)$ and prove that $\phi f \in \Gamma(\mu')$. In fact $\| \phi f \|_\infty = \| f \|_\infty = 1$; therefore, $| \phi f | \leq 1$ (μ'-almost everywhere). If the set

$$A' = A'_n = \{x' \in X'; | (\phi f)(x') | < 1 - 1/n\}$$

were not μ'-negligible, for some n, then we would have

$$\int | \phi f |^2 \, d\mu' = \int_{A'} | \phi f |^2 \, d\mu' + \int_{X'-A'} | \phi f |^2 \, d\mu'$$

$$\leq \left(1 - \frac{1}{n}\right) \mu'(A') + \mu'(X' - A') < \mu'(X') = \mu(X) = \int | f |^2 \, d\mu$$

which contradicts $\| \phi f \|_2 = \| f \|_2$. It follows that A'_n is μ'-negligible for every n; therefore, $| \phi f | = 1$ (μ'-almost everywhere).

The restriction of ϕ to $\Gamma(\mu)$, still denoted by ϕ, is the required isomorphism.

Remark. We shall prove that, conversely, if there exists an isomorphism $\phi : \Gamma(\mu) \to \Gamma(\mu')$ satisfying conditions (i) and (v), then μ and μ' are conjugate (see corollary of Theorem 2).

3. Extension of linear isometries

Let (X, Σ, μ) and (X', Σ', μ') be finite measure spaces. In this section we give sufficient conditions in order that μ and μ' should be conjugate. We shall prove first some lemmas.

LEMMA 1. *Let $\phi : L^2(\mu) \to L^2(\mu')$ be a linear isometry. If $f \in L^\infty(\mu)$ and*

$$(\phi f)^n = \phi f^n \quad \textit{for every} \quad n,$$

then $\phi f \in L^\infty(\mu')$ and

$$\| \phi f \|_\infty = \| f \|_\infty .$$

Let $A = \{x; \phi f(x) \mid \geq \| f \|_\infty\}$. Then we have

$$\mu(A) \| f \|_\infty^{2n} \leq \int | \phi f |^{2n} \, d\mu' = \int (\phi f)^n \overline{(\phi f)^n} \, d\mu'$$

$$= \int f^n \cdot \overline{f^n} \, d\mu = \int | f |^{2n} \, d\mu \leq \| f \|_\infty^{2n} .$$

If $\| f \|_\infty = 0$, then $| \phi f | = 0$ (μ-almost everywhere); if $\| \phi f \|_\infty > 0$, then $\mu(A) = 0$. Therefore $\| \phi f \|_\infty \leq \| f \|_\infty$. We have also $\| \phi f \|_{2n} = \| f \|_{2n}$ for every n. Passing to the limit, as $n \to \infty$, we obtain $\| \phi f \|_\infty = \| f \|_\infty$.

LEMMA 2. *Let $\phi : L^2(\mu) \to L^2(\mu')$ be a linear isometry and let $A \subset L^2(\mu)$. If $f \in L^\infty(\mu)$ is such that $\phi f \in L^\infty(\mu')$ and*

$$\phi(fg) = \phi f \cdot \phi g \quad \textit{for every} \quad g \in A,$$

then this equality remains valid for every g in the closure $\bar{A}$ of A in $L^2(\mu)$.

In fact, let $g \in \bar{A}$ and let $g_n \in A$ be such that $g_n \to g$ in $L^2(\mu)$. Since f is bounded, we have $fg_n \to fg$ in $L^2(\mu)$. It follows that

$$\phi g_n \to \phi g \quad \text{and} \quad \phi(fg_n) \to \phi(fg) \quad \text{in} \quad L^2(\mu').$$

Since ϕf is bounded we have also

$$\phi f \cdot \phi g_n \to \phi f \cdot \phi g \quad \text{in} \quad L^2(\mu').$$

For every n we have

$$\phi(fg_n) = \phi f \cdot \phi g_n$$

whence, passing to the limit in $L^2(\mu')$, we obtain

$$\phi(fg) = \phi f \cdot \phi g.$$

LEMMA 3. *Let $\phi : L^2(\mu) \to L^2(\mu')$ be a linear isometry, let $A \subset L^\infty(\mu)$ be an additive group and let $g \in L^2(\mu)$. If*

$$\phi(fg) = \phi f \cdot \phi g$$

and

$$\| \phi f \|_\infty = \| f \|_\infty$$

for every $f \in A$, then these two equalities remain valid for every f in the closure $\tilde{A}$ of A in $L^\infty(\mu)$.

Let $f \in \tilde{A}$ and let $f_n \in A$ be such that $f_n \to f$ in $L^\infty(\mu)$. Then $f_n g \to fg$ in $L^2(\mu)$; therefore

$$\phi(f_n g) \to \phi(fg) \quad \text{in} \quad L^2(\mu').$$

We have also $f_n \to f$ in $L^2(\mu)$; therefore

$$\phi f_n \to \phi f \quad \text{in} \quad L^2(\mu').$$

From the equalities

$$\| \phi f_n - \phi f_m \|_\infty = \| f_n - f_m \|_\infty$$

we deduce that $\phi f_n \to \phi f$ in $L^\infty(\mu')$; therefore

$$\phi f_n \cdot \phi g \to \phi f \cdot \phi g \quad \text{in} \quad L^2(\mu').$$

For every n we have

$$\phi(f_n g) = \phi f_n \cdot \phi g \quad \text{and} \quad \| \phi f_n \|_\infty = \| f_n \|_\infty$$

whence, passing to the limit, as $n \to \infty$, we obtain

$$\phi(fg) = \phi f \cdot \phi g \quad \text{and} \quad \| \phi f \|_\infty = \| f \|_\infty .$$

THEOREM 1. *Let $A \subset L^\infty(\mu)$ be a set such that*

(1) *A is dense in $L^2(\mu)$;*

(2) *$\alpha f + \beta g \in A$, for $f, g \in A$ and α, β rational complex numbers;*

(3) *$\bar{f} \in A$, if $f \in A$;*

(4) *$fg \in A$, if $f, g \in A$;*

Let $\phi : A \to L^2(\mu')$ be a mapping such that

(5) *$\| \phi f \|_2 = \| f \|_2$;*

(6) $\phi(\alpha f + \beta g) = \alpha\phi f + \beta\phi g$ *for* $f, g \in A$ *and* α, β *rational;*
(7) $\phi\bar{f} = \overline{\phi f}$ *for* $f \in A$ (ϕ *is real*)*;*
(8) $\phi(fg) = \phi f \cdot \phi g$ *for* $f, g \in A$.

Then ϕ *can be extended to a linear isometry of* $L^2(\mu)$ *into* $L^2(\mu')$, *still denoted by* ϕ, *such that*

(j) $\phi(fg) = \phi f \cdot \phi g$ *for* $f, g \in L^\infty(\mu)$*;*
(jj) $\| \phi f \|_\infty = \| f \|_\infty$ *for* $f \in L^\infty(\mu)$.

Moreover, if ϕA *is dense in* $L^2(\mu')$, *then*

$$\phi L^2(\mu) = L^2(\mu') \quad \textit{and} \quad \phi L^\infty(\mu) = L^\infty(\mu').$$

Proof. The fact that ϕ can be extended to a linear isometry of $L^2(\mu)$ into $L^2(\mu')$ follows immediately from conditions (1), (2) and (5), (6). From conditions (3) and (7) we deduce also that ϕ is real. Applying Lemma 1, it follows that

$$\| \phi f \|_\infty = \| f \|_\infty \quad \text{for} \quad f \in A.$$

Since $\bar{A} = L^2(\mu)$, from Lemma 2 we deduce that

$$\phi(fg) = \phi f \cdot \phi g \quad \text{for} \quad f \in A \quad \text{and} \quad g \in L^2(\mu).$$

By Lemma 3 we have

$$(*) \qquad \phi(fg) = \phi f \cdot \phi g \quad \text{for} \quad f \in \tilde{A} \quad \text{and} \quad g \in L^2(\mu)$$

and

$$(**) \qquad \| \phi f \|_\infty = \| f \|_\infty \quad \text{for} \quad f \in \tilde{A},$$

where $\tilde{A}$ is the closure of A in $L^\infty(\mu)$. The rest of the proof is divided into several parts.

(a) For every $f \in \tilde{A}$ and for every continuous complex function φ defined on the complex plane we have $\varphi \circ f \in \tilde{A}$.

In fact, we remark first that $\tilde{A}$ is a subalgebra of $L^\infty(\mu)$ such that $\bar{g} \in \tilde{A}$ for $g \in A$. Let $a > \| f \|_\infty$. There exists a sequence $p_n(z, \bar{z})$ of polynomials in z and $\bar{z}$ converging uniformly to φ on the disc $| z | \leq a$. Then $p_n(f(x), \bar{f}(x))$ converge uniformly to $\varphi(f(x))$ on X. Since $p_n(f, \bar{f}) \in \tilde{A}$, it follows that $\varphi \circ f \in \tilde{A}$.

(b) We have $\phi L^\infty(\mu) \subset L^\infty(\mu')$.

Let $f \in L^\infty(\mu)$ and $a > \| f \|_\infty$; let φ be a continuous complex function defined on the complex plane such that $\varphi(z) = z$ for $| z | \leq a$ and $| \varphi(z) | \leq a$ for every z. Then $\varphi \circ f = f$ (μ-almost everywhere).

Let further $f_n \in \tilde{A}$ be a sequence such that $f_n \to f$ in $L^2(\mu)$ and μ-almost everywhere. Then the functions $h_n = \varphi \circ f_n$ belong to $\tilde{A}$ and $h_n \to \varphi \circ f = f$ μ-almost everywhere. Since

$$| h_n | \leq | \varphi | \leq a \quad \text{for every} \quad n,$$

it follows that $h_n \to f$ in $L^2(\mu)$. Then

$$\phi h_n \to \phi f \quad \text{in} \quad L^2(\mu')$$

and we may assume that $\phi h_n \to \phi f$ (μ'-almost everywhere). Since, by (**) we have

$$\| \phi h_n \|_\infty = \| h_n \|_\infty \leq a \quad \text{for every} \quad n$$

it follows that

$$\| \phi f \|_\infty \leq a;$$

therefore

$$\phi f \in L^\infty(\mu').$$

(c) Taking into account (*) and applying Lemma 2 for each $f \in L^\infty(\mu)$, we deduce that

$$\phi(fg) = \phi f \cdot \phi g \quad \text{for} \quad f \in L^\infty(\mu) \quad \text{and} \quad g \in L^2(\mu).$$

From Lemma 1 it follows then that

$$\| \phi f \|_\infty = \| f \|_\infty \quad \text{for} \quad f \in L^\infty(\mu)$$

(d) If ϕA is dense in $L^2(\mu')$, then clearly $\phi L^2(\mu) = L^2(\mu')$. Considering now the inverse isometry $\phi^{-1} : L^2(\mu') \to L^2(\mu)$ and applying the part of the theorem already proved, we deduce that $\phi^{-1} L^\infty(\mu') \subset L^\infty(\mu)$; therefore $L^\infty(\mu') \subset \phi L^\infty(\mu)$, consequently $\phi L^\infty(\mu) = L^\infty(\mu')$. The theorem is completely proved.

Remark. This theorem was not stated separately so far, but was used to prove the main theorems in [1] and [2], and will be used also to prove Theorem 2 of this paper.

4. Algebraic measure systems

The considerations of the preceding sections lead to the following

DEFINITION 2. A system (Γ, φ) consisting of an abelian group Γ and a complex function of positive type φ on Γ such that

$$\varphi(\gamma) = 1 \quad \text{if and only if} \quad \gamma = 1,$$

is called an algebraic measure system (or a measure system).

Two measure systems (Γ, φ) and (Γ', φ') are said to be isomorphic if there exists an isomorphism ϕ of Γ onto Γ' such that

$$\varphi(\gamma) = \varphi'(\phi\gamma) \quad \text{for} \quad \gamma \in \Gamma.$$

Example. If (X, Σ, μ) is a probability measure space and if $\Gamma \subset \Gamma(\mu)$ is a group, then (Γ, φ_μ) is a measure system. In particular $(1, \varphi_\mu)$, (C, φ_μ) and $(\Gamma(\mu), \varphi_\mu)$ are measure systems.

Remarks. Let Γ be an abelian group and $\varphi \neq 0$ a function of positive type on Γ. We can always consider that $\varphi(1) = 1$, replacing, if necessary, φ by the function of positive type $\varphi/\varphi(1)$.

We might have $\varphi(\gamma) = 1$ for some $\gamma \neq 1$, so that, in general, (Γ, φ) is not a measure system, in the sense of Definition 2. However, the following proposition states that we can replace (Γ, φ) by a measure system $(\tilde{\Gamma}, \tilde{\varphi})$

which is not essentially different from (Γ, φ). We shall call $(\tilde{\Gamma}, \tilde{\varphi})$ the measure system associated with (Γ, φ).

PROPOSITION 3. *Let Γ be an abelian group and φ a function of positive type on Γ, such that $\varphi(1) = 1$.*

(1) *The set $C' = \{\gamma \in \Gamma; |\varphi(\gamma)| = 1\}$ is a group and*

$$\varphi(\gamma'\gamma) = \varphi(\gamma')\varphi(\gamma) \quad \textit{for } \gamma' \in C' \quad \textit{and} \quad \gamma \in \Gamma.$$

(2) *The set $C_1 = \{\gamma \in \Gamma; \varphi(\gamma) = 1\}$ is a group and*

$$\varphi(\gamma_1) = \varphi(\gamma_2) \quad \textit{if } \gamma_1 \gamma_2^{-1} \in C_1 .$$

(3) *The function $\tilde{\varphi}$ defined on $\tilde{\Gamma} = \Gamma/C_1$ by*

$$\tilde{\varphi}(\tilde{\gamma}) = \varphi(\gamma)$$

is of positive type, $\tilde{\varphi}$ is injective on $\tilde{C}'$ and $(\tilde{\Gamma}, \tilde{\varphi})$ is a measure system.

Consider Γ equipped with the discrete topology. Then the dual $G = \Gamma^{\wedge}$ is an abelian compact group. For every $\gamma \in \Gamma$ and $x \in G$ we denote by $\langle x, \gamma\rangle$ the value of the character x in γ. By Bochner's theorem there exists a positive regular Borel measure μ on G such that

$$\varphi(\gamma) = \int \langle x, \gamma\rangle \, d\mu(x) \quad \text{for} \quad \gamma \in \Gamma.$$

(a) If $\gamma \in C'$ and $\varphi(\gamma) = c$, then $\langle x, \gamma\rangle = c$ (μ-almost everywhere). In fact

$$\int \langle x, \gamma\rangle \, d\mu(x) = \varphi(\gamma) = c;$$

therefore

$$\int |\langle x, \gamma\rangle - c|^2 \, d\mu(x) = \int (1 - c\,\overline{\langle x, \gamma\rangle} - \bar{c}\,\langle x, \gamma\rangle + 1) \, d\mu$$
$$= 0$$

whence $\langle x, \gamma\rangle = c$ (μ-almost everywhere).

(b) If $\gamma' \in C'$ and $\gamma \in \Gamma$, then $\varphi(\gamma'\gamma) = \varphi(\gamma')\varphi(\gamma)$. In fact $\varphi(\gamma') = c$ with $|c| = 1$; therefore $\langle x, \gamma'\rangle = c$ (μ-almost everywhere). It follows that

$$\varphi(\gamma'\gamma) = \int \langle x, \gamma'\gamma\rangle \, d\mu(x) = \int \langle x, \gamma'\rangle\langle x, \gamma\rangle \, d\mu(x)$$
$$= c \int \langle x, \gamma\rangle \, d\mu(x) = \varphi(\gamma')\varphi(\gamma).$$

(c) C' is a group and φ is a homomorphism on C'. In fact, $\varphi(1) = 1$, hence $1 \in C'$. If $\gamma, \gamma' \in C'$, then

$$\varphi(\gamma\gamma') = \varphi(\gamma)\varphi(\gamma') \quad \text{and} \quad |\varphi(\gamma\gamma')| = |\varphi(\gamma)|\,|\varphi(\gamma')| = 1,$$

hence $\gamma\gamma' \in C'$. Finally, if $\gamma \in C'$, then

$$\varphi(\gamma^{-1}) = \int \langle x, \gamma^{-1}\rangle \, d\mu(x) = \int \overline{\langle x, \gamma\rangle} \, d\mu(x) = \overline{\varphi(\gamma)};$$

therefore $|\varphi(\gamma^{-1})| = 1$, hence $\gamma^{-1} \in C'$ and $\varphi(\gamma^{-1}) = [\varphi(\gamma)]^{-1}$.

(d) We have $C_1 \subset C'$ and $C_1 = \varphi^{-1}(1)$; since φ is a homomorphism on C', it follows that C_1 is a group.

If now $\gamma_1 \gamma_2^{-1} \in C_1$, then $\varphi(\gamma_1 \gamma_2^{-1}) = 1$; therefore

$$\varphi(\gamma_1) = \varphi(\gamma_1 \gamma_2^{-1}\gamma_2) = \varphi(\gamma_1 \gamma_2^{-1})\varphi(\gamma_2) = \varphi(\gamma_2).$$

The statement (3) is evident.

COROLLARY. *If (Γ, φ) is a measure system, then φ is an injective homomorphism of C' into the circle group C. Moreover, identifying an element $\gamma \in C'$ with the number $\varphi(\gamma) = c \in C$, we have*

$$\varphi(c\gamma) = c\varphi(\gamma) \quad \text{for} \quad c \in C' \quad \text{and} \quad \gamma \in \Gamma.$$

In fact, in this case $C_1 = \{1\}$.

Remark. If C' is divisible, then there exists a group $\Gamma' \subset \Gamma$ such that $\Gamma = C' \cdot \Gamma'$ (direct product).

We shall see that we can always imbed Γ in an abelian group Γ_1 containing the whole circle group C (which is divisible) and write $\Gamma_1 = C \cdot \Gamma_1'$ (direct product). The function φ can also be extended to a function of positive type φ_1 on Γ_1 such that

$$\{\gamma \in \Gamma_1 ; \ |\varphi_1(\gamma)| = 1\} = C.$$

In this case, (Γ_1', φ_1) is again a measure system such that $\gamma \in \Gamma_1'$ and $\gamma \neq 1$ imply $|\varphi(\gamma)| < 1$.

5. Algebraic models of measures

DEFINITION 3. Let (X, Σ, μ) be a probability measure space. We say that a measure system (Γ, φ) is an algebraic model of the measure μ, if there exists an injective homomorphism $J : \Gamma \rightarrow \Gamma(\mu)$ such that

(a) $J\Gamma$ generates $L^2(\mu)$

and

(b) $\varphi(\gamma) = \varphi_\mu(J\gamma)$ for $\gamma \in \Gamma$.

It follows that if $\Gamma \subset \Gamma(\mu)$ is a group generating $L^2(\mu)$, then (Γ, φ) is an algebraic model for μ. In particular, $(\Gamma(\mu), \varphi_\mu)$ is an algebraic model for μ.

If, in Definition 3 we identify Γ and $J\Gamma$, we can always consider that an algebraic model (Γ, φ) of μ is such that $\Gamma \subset \Gamma(\mu)$ and $\varphi = \varphi_\mu$.

Algebraic models determine measures uniquely up to a conjugacy:

THEOREM 2. *Two probability measures are conjugate if and only if they possess isomorphic algebraic models.*

Let (X, Σ, μ) and (X, Σ, μ') be two probability measure spaces.

If μ and μ' are conjugate, then, by Proposition 2, the algebraic models $(\Gamma(\mu), \varphi_\mu)$ and $(\Gamma(\mu'), \varphi_{\mu'})$ are isomorphic.

Conversely, suppose that μ and μ' possess isomorphic algebraic models (Γ, φ) and (Γ', φ'). We may consider $\Gamma \subset \Gamma(\mu), \varphi = \varphi_\mu$ and $\Gamma' \subset \Gamma(\mu'), \varphi' = \varphi_{\mu'}$. Let ϕ be the isomorphism of Γ onto Γ' such that

$$\varphi_\mu(f) = \varphi_{\mu'}(\phi f) \quad \text{for} \quad f \in \Gamma.$$

Consider the space $A \subset L^2(\mu)$ of the linear combinations

$$f = \sum_{i=1}^n \alpha_i f_i \quad \text{with} \quad f_i \in \Gamma \quad \text{and } \alpha_i \quad \text{scalars.}$$

For such a function we have

$$\int |f|^2 d\mu = \int |\sum_k \alpha_k f_k|^2 d\mu = \int \sum_{i,j} \alpha_i \overline{\alpha_j} f_i \overline{f_j}\, d\mu$$

$$= \sum_{i,j} \alpha_i \overline{\alpha_j} \varphi_\mu(f_i \overline{f_j}) = \sum_{i,j} \alpha_i \overline{\alpha_j} \varphi_{\mu'}(\phi f_i \cdot \overline{\phi f_j}) = \int |\sum_k \alpha_k \phi f_k|^2 d\mu'.$$

It follows that if $f = \sum \alpha_i f_i = 0$ (μ-almost everywhere) then $\sum \alpha_i \phi f_i = 0$ (μ'-almost everywhere), so that we may define unambiguously

$$\phi f = \sum_{i=1}^n \alpha_i \phi f_i .$$

Then $\phi : A \to L^\infty(\mu')$ is a linear multiplicative mapping such that

$$\| \phi f \|_2 = \| f \|_2 \quad \text{for } f \in A.$$

By Theorem 1, ϕ may be extended to a linear isometry of $L^2(\mu)$ onto $L^2(\mu')$, still denoted by ϕ, such that $\phi L^\infty(\mu) = \phi L^\infty(\mu')$ and

$$\phi(fg) = \phi f \cdot \phi g \quad \text{for} \quad f, g \in L^\infty(\mu)$$

so that μ and μ' are conjugate.

COROLLARY. *Two measures μ and μ' are conjugate if and only if the measure systems $(\Gamma(\mu), \varphi_\mu)$ and $(\Gamma(\mu'), \varphi_{\mu'})$ are isomorphic.*

The following theorem states that every measure system is a model for a certain measure.

THEOREM 3. *Every measure system (Γ, φ) is a model for a regular Borel probability measure μ on an abelian compact group G.*

Moreover, if $\varphi(\gamma) = 0$ for $\gamma \neq 1$, then μ is the Haar measure on G.

Consider on Γ the discrete topology and take $G = \Gamma^\wedge$, the group of characters of Γ. Let μ be the unique positive regular Borel measure on G such that

$$\varphi(\gamma) = \int \langle x, \gamma \rangle \, d\mu(x) \quad \text{for} \quad \gamma \in \Gamma.$$

For every $\gamma \in \Gamma$, denote by $\langle \cdot, \gamma \rangle$ both the function $x \to \langle x, \gamma \rangle$ on G and the equivalence class of this function in $L^2(\mu)$ and put

$$J\gamma = \langle \cdot, \gamma \rangle.$$

It is clear that J is a homomorphism of Γ into $\Gamma(\mu)$, that $J\Gamma$ generates $L^2(\mu)$ (since the continuous functions on G are uniform limits of linear combinations of characters of G) and that

$$\varphi(\gamma) = \varphi_\mu(J\gamma) \quad \text{for} \quad \gamma \in \Gamma.$$

To prove that J is injective, suppose that

$$\langle x, \gamma \rangle = 1 \quad (\mu\text{-almost everywhere}).$$

Then

$$\varphi(\gamma) = \int \langle x, \gamma \rangle \, d\mu(x) = 1;$$

therefore $\gamma = 1$.

The statement concerning Haar measure is evident.

Remark. If Γ is countable, then G is metrisable [2].

As a corollary we have the following theorem which reduces integration on abstract sets to integration on a compact group, with respect to a regular Borel measure.

THEOREM 4. *Every probability measure is conjugate to a regular Borel measure on an abelian compact group.*

In fact, $(\Gamma(\mu), \varphi_\mu)$ is an algebraic model for μ, and, by the preceding theorem, it is an algebraic model for a regular Borel measure μ' on an abelian compact group. We use then Theorem 2 to deduce that μ and μ' are conjugate.

Remark. If in Theorem 3, we identify an element $\gamma \in \Gamma$ with the character $\langle \cdot, \gamma \rangle$, we can consider $\Gamma \subset \Gamma(\mu)$ and $\varphi = \varphi_\mu$. Then the group

$$\Gamma_1 = \{c\gamma; \ c \in C, \ \gamma \in \Gamma\}$$

contains Γ and the whole circle group C, and φ_μ is an extension of φ from Γ to Γ_1. We can write now $\Gamma_1 = C \cdot \Gamma_1'$ (direct product).

Evidently, (Γ_1, φ_μ) and (Γ_1', φ_μ) are algebraic models of μ as well, and are not isomorphic.

In case Γ is a direct product, the measure μ in Theorem 3 can be made more precise.

PROPOSITION 5. *Let (Γ, φ) be a measure system such that $\Gamma = C' \cdot \Gamma'$ (direct product) and*

$$\varphi(c) = c \quad \text{for} \quad c \in C'$$

where C' is a subgroup of the circle group (equipped with the discrete topology).

Let μ and μ' be the measures on $G = \Gamma^\wedge$ respectively on $G' = \Gamma'^\wedge$ such that

$$\varphi(\gamma) = \int_G \langle x, \gamma\rangle \, d\mu(x) \quad \textit{for} \quad \gamma \in \Gamma$$

and

$$\varphi(\gamma) = \int_{G'} \langle x, \gamma\rangle \, d\mu'(x) \quad \textit{for} \quad \gamma \in \Gamma'.$$

Then

$$G = C'^\wedge \times G' \quad \textit{and} \quad \mu = \varepsilon_z \otimes \mu'$$

where $z \in C'^\wedge$ is the character defined by $\langle z, c\rangle = c$ for $c \in C'$, and ε_z is the measure on $C'^\wedge$ defined by

$$\int f(u) \, d\varepsilon_z(u) = f(z) \quad \textit{for continuous} \quad f : C'^\wedge \to C.$$

In particular, if $\varphi(\gamma) = 0$ for $\gamma \notin C'$, then μ' is the Haar measure on G'.

We remark that $C' \cdot \Gamma'$ is isomorphic (and homeomorphic) with $C' \times \Gamma'$ so that $\Gamma^\wedge = C'^\wedge \times \Gamma'^\wedge$.

By Proposition 3 we have $\varphi(c\gamma) = c\varphi(\gamma)$, for $c \in C'$ and $\gamma \in \Gamma'$.

If we write an element $c\gamma \in C' \cdot \Gamma'$ as a pair $(c, \gamma) \in C' \times \Gamma'$ we have

$$\varphi(c\gamma) = \int \langle (u, v), (c, \gamma)\rangle \, d\mu(u, v).$$

On the other hand

$$\int \langle (u, v), (c, \gamma)\rangle \, d\varepsilon_z(u) \, d\mu'(v)$$

$$= \int \langle u, c\rangle \, d\varepsilon_z(u) \int \langle v, \gamma\rangle \, d\mu'(v) = \langle z, c\rangle \varphi(\gamma) = c\varphi(\gamma) = \varphi(c\gamma).$$

From the uniqueness of the measure μ in Bochner's theorem, we deduce that $\mu = \varepsilon_z \otimes \mu'$.

The statement concerning the Haar measure is evident.

THEOREM 5. *A probability measure μ on a measurable space (X, Σ) is conjugate to a Haar measure on an abelian compact group G if and only if there exists a group $\Gamma' \subset \Gamma(\mu)$ which is an orthonormal basis of $L^2(\mu)$.*

Suppose first that μ is conjugate to the Haar measure ν on G and let $\phi : L^2(\nu) \to L^2(\mu)$ be the linear isometry such that $\phi L^2(\nu) = L^2(\mu)$, $\phi L^\infty(\nu) = L^\infty(\mu)$ and

$$\phi(fg) = \phi f \cdot \phi g \quad \text{for} \quad f, g \in L^\infty(\nu).$$

The group $G^\wedge$ of the characters of G is an orthonormal basis of $L^2(\nu)$, therefore, $\Gamma' = \phi G^\wedge$ is an orthonormal basis of $L^2(\mu)$.

Conversely, suppose that there exists a group $\Gamma' \subset \Gamma(\mu)$ which is an orthonormal basis of $L^2(\mu)$. Then (Γ, φ_μ) is an algebraic model of μ. Moreover

$$\varphi_\mu(\gamma) = 1 \quad \text{if and only if} \quad \gamma = 1;$$

therefore, by Theorem 3, (Γ, φ_μ) is an algebraic model of the Haar measure ν on the abelian compact group $G = \Gamma^\wedge$. By Theorem 2, μ and ν are conjugate.

Remark. The following two conditions are equivalent:

(a) there exists a group $\Gamma' \subset \Gamma(\mu)$ which is an orthonormal basis of $L^2(\mu)$;

(b) there exists an orthonormal basis $\Gamma'' \subset \Gamma(\mu)$ of $L^2(\mu)$ such that the set $\Gamma = C\Gamma'' = \{c\gamma; c \in C, \gamma \in \Gamma''\}$ is a group.

Evidently condition (a) implies condition (b). Conversely, suppose condition (b) satisfied. Then there exists a group $\Gamma' \subset \Gamma$ such that $\Gamma = C \cdot \Gamma'$ (direct product). If $\gamma \in \Gamma'$ and $\gamma \neq 1$, then $\gamma = c\gamma''$ with $\gamma'' \in \Gamma''$ and $\gamma'' \neq 1$, therefore,

$$\int \gamma \, d\mu = c \int \gamma'' \, d\mu = 0.$$

It follows that if $\gamma_1 \neq \gamma_2$ are two elements of Γ', then $\gamma_1 \bar{\gamma}_2 = \gamma_1 \gamma_2^{-1} \neq 1$; therefore, $\int \gamma_1 \bar{\gamma}_2 \, d\mu = 0$, consequently Γ' is an orthonormal basis of $L^2(\mu)$.

References

1. N. Dinculeanu and C. Foiaş, *A universal model for ergodic transformations on separable measure spaces*, Michigan Math. J., vol. 13 (1966), pp. 109–117.
2. C. Foiaş, *Automorphisms of compact abelian groups, as models for measure-preserving invertible transformations*, Michigan Math. J., vol. 13 (1966), pp. 349–352.
3. P. R. Halmos, *Lectures on ergodic theory*, Math. Soc. of Japan, Tokyo, 1956.
4. A. Ionescu Tulcea and C. Ionescu Tulcea, *On the lifting property*, *I*. J. Math. Analysis and Appl., vol. 3 (1961), pp. 537–546.
5. I. Kaplansky, *Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, 1954.

Queen's University
Kingston, Ontario, Canada
University of Bucarest
Romania