A CLASS OF REPRODUCING KERNELS[1]

BY

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Let $P(x)$ be a homogeneous positive definite polynomial of order $2m$, $m > 0$ an integer, in $x \in E_n$ having constant coefficients. We will show that the kernel

$$K(x, y) = \int_{z \in E_n} e^{-2\pi i x \cdot z} \exp\left(2\pi i \beta \mid P(z) \mid^{1/2m} y\right) dz,$$

where $x \in E_n$, $y > 0$, $\beta^{2m} + 1 = 0$, the imaginary part of β is positive and the real part of β^2 is negative, satisfies the following five properties:

(1) $K(x, y) \in L^1(E_n)$, independently of y;

(2) $\int_{E_n} K(x, y)\, dx = 1$;

(3) $\int_{|x| \geq \delta > 0} \mid K(x, y) \mid^q dx \to 0$ as $y \to 0$, $1 \leq q$;

(4) $\mid K(x, y) \mid < A y^{-n}$, A independent of x, y;

(5) $\mid K(x, y) \mid < B y \mid x \mid^{-n-1}$, B independent of x, y.[2]

These are sufficient to guarantee that K is a reproducing kernel in the sense that, if we define

$$f(x, y) = f * K(x, y) = \int_{z \in E_n} f(z) K(x - z, y)\, dz$$

then $f(x, y) \to f(x)$ as $y \to 0$ in L^p norm and almost everywhere for any $f \in L^p(E_n)$, $1 \leq p < \infty$.

These kernels are of interest, since $K(x, y)$ and, hence, $f(x, y)$, as defined above will satisfy the elliptic equation

$$(\partial^{2m}/\partial y^{2m})u + P(D)u = 0$$

in $E_{n+1}^+ = \{(x, y) \mid x \in E_n, y > 0\}$, where $P(D)$ is the differential operator obtained from $P(x)$ by replacing each occurrence of x_i by $\partial/\partial x_i$, $i = 1, \cdots, n$.

Letting $x = \mid x \mid x'$, $\mid x \mid = \sum_{i=1}^n x_i^2$, from the homogeneity of P, we obtain by a simple change of variable the following identities for K:

$$K(x, y) = y^{-n} K(xy^{-1}, 1) = \mid x \mid^{-n} K(x', y \mid x \mid^{-1}) \quad \text{for all } x \in E_n,\ y > 0.$$

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[2] Throughout this paper the letters A, B, C will denote constants which are independent of x, y, z, t, or v. They may, however, have differing values in different parts of the same argument.

To prove that K has the indicated properties, we will first assume property 5 and prove the other four properties. Then, we will prove property 5.

To show property 1 we first observe that $|K(x, 1)| \leq A$ for $|x| \leq 1$ since $K(x, 1)$ is continuous. We then have

$$\begin{aligned}
&\int_{x \in E_n} |K(x\ y)|\, dx \\
&\qquad = \int_{x \in E_n} y^{-n} |K(xy^{-1}, 1)|\, dx \\
&\qquad = \int_{z \in E_n} |K(z, 1)|\, dz. \qquad (\text{letting } x_i = yz_i) \\
&\qquad \leq A\Omega_n + B \int_{|r| \geq 1} |z|^{-n-1}\, dz \qquad (\text{by property 5 and continuity})^3 \\
&\qquad \leq A\Omega_n + B\omega_n, \qquad \text{which is independent of } y^{\bullet}
\end{aligned}$$

Since we have shown that $K(x, y)$ is in $L^1(E_n)$, property 2 follows by the Fourier inversion formula, i.e.,

$$\int_{x \in E_n} K(x, y)\, dx = \exp\left[2\pi i\beta\, |P(0)|^{1/2m} y\right] = 1.$$

Using property 5, we obtain property 3 as follows:

$$\int_{|x| \geq \delta} |K(x, y)|^q\, dx \leq B^q y^q \int_{|x| \geq \delta} |x|^{-q(n+1)}\, dx \leq B^q y^q \omega_n \frac{\delta^{n-qn-q}}{-n + qn + q}$$

which tends to zero as $y \to 0$.

Property 4 is shown directly from the definition. Thus,

$$|K(x, y)| = y^{-n} |K(xy^{-1}, 1)| \leq y^{-n} \int_{z \in E_n} \exp\left[-2\pi I(\beta)\, |P(z)|^{1/2m}\right] dz \leq Ay^{-n},$$
$$I(\beta) = \text{imaginary part of } \beta.$$

We now must prove property 5, $|K(x, y)| \leq By\, |x|^{-n-1}$, to complete this derivation of the properties of K. This will be done through a sequence of lemmas. The heart of the argument will be found in the proofs of Lemmas 2 and 3.

We first need to introduce some notation. For α an n-tuple of non-negative integers, i.e., $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_i \geq 0$, $i = 1, \cdots, n$, we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad D = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \text{and} \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

LEMMA 1. *$D^\alpha(|P(z)|^{j/2m}) = C_{\alpha,j}\, |P(z)|^{(j/2m)-k} Q(z)$, $j = 1, 2$, where m is any integer greater than or equal to 1 if $j = 1$ and m is any integer greater*

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[3] Ω_n is the volume of the unit sphere in E_n, $|x| \leq 1$. ω_n is the area of the surface, $|x| = 1$, of the unit sphere in E_n.

than or equal to 2 if $j = 2$, $k = |\alpha|$, and Q is either a homogeneous polynomial of degree $k(2m-1)$ or Q is identically zero, $(Q(z) \equiv 0)$.

Proof. $(\partial/\partial z_i)(|P(z)|^{(j/2m)}) = (j/2m)|P(z)|^{(j/2m)-1}\,\partial P/\partial z_i$. Since $\partial P/\partial z_i$ is either homogeneous of degree $2m-1$ or identically zero, the lemma is true for $k = 1$.

We proceed by induction on k. Suppose the lemma is true for $k = q$, i.e.,

$$D^\alpha(|P(z)|^{1/2m)} = C_{\alpha,j}|P(z)|^{(j/2m)-q}Q(z)$$

for all α such that $|\alpha| = q$, and the order of $Q(z)$ is $q(2m-1)$ or else $Q(z) \equiv 0$. Then

$$\begin{aligned}
&\frac{\partial}{\partial z_i}[D^\alpha(|P(z)|^{j/2m})] \\
&\quad = C_{\alpha,j}((j/2m)-q)|P(z)|^{(j/2m)-q-1}Q(z)\frac{\partial P}{\partial z_i} + C_{\alpha,j}|P(z)|^{(j/2m)-q}\frac{\partial Q}{\partial z_i} \\
&\quad = C_{\alpha,j}|P(z)|^{(j/2m)-q-1}[((j/2m)-q)Q(z)\frac{\partial P}{\partial z_i} + \frac{\partial Q}{\partial z_i}P(z)].
\end{aligned}$$

The polynomial in brackets is either zero or homogeneous of degree $(q+1)(2m-1)$. Thus, the lemma is true for all k.

LEMMA 2. (a) *$|K(x, 1)| \leq B|x|^{-n-1}$ for n even.*

(b) *Let $K^*(x) = \int_{z \in E_n} \exp[-2\pi i x\cdot z + \pi^2\beta^2|P(z)|^{2/2m}]\,dz$. Then, for n odd, $|K^*(x)| \leq B|x|^{-n-2}$.*

Proof. (a) Since

$$|x|^{n+1}|K(x, 1)| \leq (|x_i| + |x_2| + \cdots + |x_n|)^{n+1}|K(x, 1)|,$$

it suffices to show that $|x^\alpha||K(x, 1)|$ is bounded for all α such that $|\alpha| = n+1$.

Now

$$\begin{aligned}
|x^\alpha K(x, 1)| &= \left|x^\alpha \int_{E_n} \exp[-2\pi i x\cdot z + 2\pi i\beta|P(z)|^{1/2m}]\,dz\right| \\
&= \left|C_\alpha \int_{E_n} \exp[-2\pi i x\cdot z]D^\alpha(\exp[2\pi i\beta|P(z)|^{1/2m}])\,dz\right|^4
\end{aligned}$$

where C_α is a constant depending only on α.

Let $\phi(z)$ be a C^∞ function with support in $|z| \leq 2$ such that $\phi(z) \equiv 1$ for $|z| \leq 1$. Then

$$\begin{aligned}
&\int_{E_n} e^{-2\pi i x\cdot z}D^\alpha(\exp[2\pi i\beta|P(z)|^{1/2m}])\,dz \\
&\quad = \int_{E_n} e^{-2\pi i x\cdot z}D^\alpha(-2\pi i\beta|P(z)|^{1/2m}\phi(z) + \exp[2\pi i\beta|P(z)|^{1/2m}])\,dz \\
&\qquad\qquad + 2\pi i\beta\int_{E_n} e^{-2\pi i x\cdot z}D^\alpha(|P(z)|^{1/2m}\phi(z))\,dz.
\end{aligned}$$

[4] This and all future integrals in this lemma will be taken in the Cauchy principle value sense at the origin.

The first of these integrals is bounded since all of the derivatives of order $n + 1$ of the function

$$-2\pi i\beta \mid P(z) \mid^{1/2m} \phi(z) + \exp [2\pi i\beta \mid P(z) \mid^{1/2m}]$$

are in $L^1(E_n)$. It is enough to show that

$$\int_{E_n} e^{-2\pi i x\cdot z} D^\alpha(\mid P(z) \mid^{1/2m} \phi(z))\, dz$$

is bounded. Indeed, we need only show that

$$\int_{\mid z \mid \leq 1} e^{-2\pi i x\cdot z} D^\alpha(\mid P(z) \mid^{1/2m})\, dz$$

is bounded, $\mid \alpha \mid = n + 1$, since the function

$$\begin{aligned} f(z) &= D^\alpha(\mid P(z) \mid^{1/2m} \phi(z)), & \mid z \mid > 1, \\ &= 0, & \mid z \mid \leq 1, \end{aligned}$$

is in $L^1(E_n)$.

By Lemma 1, we see that we need only show that

$$\int_{\mid z \mid \leq 1} e^{-2\pi i x\cdot z} Q(z) \mid P(z) \mid^{(1/2m)-n-1}\, dz$$

is bounded, where $Q(z)$ is homogeneous of degree $(n + 1)\ (2m - 1)$. The case where $Q(z) \equiv 0$ is clearly bounded.

Letting $z' = z/\mid z \mid$, we can write this integral as

$$\int_{\mid z \mid \leq 1} e^{-2\pi i x\cdot z} [Q(z') \mid P(z') \mid^{(1/2)-n-1}] \mid z \mid^{-n}\, dz.$$

From the order of homogeneity, for n even, we have $Q(z') = -Q(-z')$. Hence,

$$[Q(z') \mid P(z') \mid^{(1/2m)-n-1}] \mid z \mid^{-n}$$

satisfies the conditions for a singular integral kernel, and, therefore, this last integral is bounded as a principle value integral.[5] This completes the proof of (a).

(b) is trivial in the case $m = 1$ since $\exp (\pi^2\beta^2 P(z))$, $R(\beta^2) < 0$ has rapidly decreasing derivatives of all orders, and thus, its Fourier transform multiplied by any polynomial is bounded. In the case $m > 1$, (b) can be proved by the same technique used to prove (a), using Lemma 1 with $j = 2$.

LEMMA 3. $\mid K(x', y) \mid \leq Ay$.

Proof. We use essentially different techniques to prove this lemma in the cases n even and n odd. It is interesting that this difference depends only on

[5] See A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., vol. 88 (1952), pp. 85–139.

the dimension of the space and not at all on the degree of the polynomial P. I have been unable to discover a way of avoiding this.

For n even, using Lemma 2(a), we have

$$| K(x', y) | = y^{-n} | K((x'/y), 1) | \leq y^{-n} B | (x'/y) |^{-n-1} = By.$$

For n odd, we use the fact that for $R(\beta) < 0$, $R(\beta^2) > 0$, we have

$$e^{\beta} = \pi^{-1/2} \int_0^\infty u^{-1/2} e^{-u} e^{-(\beta^2/4u)} \, du.$$

Thus

$$\begin{aligned}
&| K(x', y) | \\
&= \pi^{-1/2} \left| \int_{E_n} e^{-2\pi i x' \cdot z} \left(\int_0^\infty u^{-1/2} e^{-u} \exp [\pi^2 \beta^2 | P(z) |^{2/2m} y^2 u^{-1}] \, du \right) dz \right| \\
&= \frac{y}{\sqrt{\pi}} \left| \int_{E_n} e^{-2\pi i x' \cdot z} \left(\int_0^\infty \frac{e^{-ty^2}}{\sqrt{t}} \exp [\pi^2 \beta^2 | P(z) |^{2/2m} t^{-1}] \, dt \right) dz \right|, \quad \text{letting } u = ty^2, \\
&= \frac{y}{\sqrt{\pi}} \left| \int_0^\infty e^{-ty^2} t^{-1/2} \left(\int_{E_n} \exp [-2\pi i x' \cdot z + \pi^2 \beta^2 | (z) |^{2/2m} t^{-1}] \, dz \right) dt \right|, \\
&\qquad \text{by Fubini's theorem,} \\
&= \frac{y}{\sqrt{\pi}} \left| \int_0^\infty e^{-ty^2} t^{(n-1)/2} \left(\int_{E_n} \exp [-2\pi i x' \eta \cdot t^{1/2} + \pi^2 \beta^2 | P(\eta) |^{2/2m}] \, d\eta \right) dt \right|, \\
&\qquad \text{letting } z_i = t^{1/2} \eta_i, \\
&= \frac{y}{\sqrt{\pi}} \left| \int_0^\infty e^{-ty^2} t^{(n-1)/2} K^*(t^{1/2} x') \, dt \right| \\
&= \frac{2y}{\sqrt{\pi}} \left| \int_0^\infty e^{-v^2 y^2} v^n K^*(vx') \, dv \right|, \qquad \text{letting } t = v^2, \\
&\leq y \left| \frac{2}{\sqrt{\pi}} \int_0^\infty v^n | K^*(vx') | \, dv + \frac{2}{\sqrt{\pi}} \int_0^\infty v^n | K^*(vx') | \, dv \right|.
\end{aligned}$$

The first integral is clearly bounded since K^* is a continuous function. The second integral is dominated by $\int_1^\infty v^n B v^{-n-2} \, dv = B$ by Lemma 2(b).

We are now ready to show property 5 of our kernel, i.e.,

$$| K(x, y) | \leq By | x |^{-n-1}.$$

Thus,

$$K(x, y) | = | x |^{-n} | K(x', y | x |^{-1}) | \leq | x |^{-n} Ay | x |^{-1} = Ay | x |^{-n-1}$$

by Lemma 3. This completes the derivation of the properties of K.

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