# AN EXTENSION OF RADON'S THEOREM[1]

BY

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## 1. introduction

Let "$m$-set" mean a set of $m$ points in the $d$-dimensional space $R^d$. An $m$-set is said to be $(r, k)$-divisible if it can be partitioned into $r$ pair-wise disjoint subsets in such a way that the intersection of the convex hulls of these $r$ subsets is at least $k$-dimensional. (We always assume $0 \leq k \leq d$. The empty set is $(-1)$-dimensional, while 0-dimensional sets are non-empty.) A classic theorem of J. Radon [5] asserts that each $(d + 2)$-set is $(2, 0)$-divisible. The first generalization for $r > 2$ was given by R. Rado [4]. B. Birch [1] conjectured (and proved for $d = 2$) that each $((d + 1)(r - 1) + 1)$-set is $(r, 0)$-divisible, while H. Tverberg [6] established this conjecture for all values of $d$. It is clear that if $k > 0$ then other conditions on a given $m$-set $S$ besides a lower bound on its cardinality are necessary if $S$ is to be $(r, k)$-divisible. For example, if all the points of $S$ were on a line in $R^d$, no subset would have a convex hull of dimension greater than one. The purpose of this paper is to consider various types of independence that may be imposed upon an $m$-set to insure $(r, k)$-divisibility (Section 3) and to prove the following theorem which extends the results mentioned above.

THEOREM 1. *Each $[(d + 1)(r - 1) + k + 1]$-set of strongly independent points in $R^d$ is $(r, k)$-divisible.*

A set $S$ in $R^d$ is said to be *strongly independent* provided that each finite family $\{S_1, \cdots, S_r\}$ of pair-wise disjoint subsets of $S$ has the following property:

$$\text{If } d_i = (\text{card } S_i) - 1 \leq d, \text{ then}$$

$$\dim \left(\textstyle\bigcap_{i=1}^{r} \operatorname{aff} S_i\right) = \max\left(-1, d - \textstyle\sum_{i=1}^{r} (d - d_i)\right). \tag{1}$$

(Condition (1) may be thought of as follows: Since $(d - d_i)$ is just the deficiency of aff $S_i$ when $S_i$ is in general position, condition (1) implies the general position of $S$ and its subsets. Thus the right side of the equation is essentially the dimension of the space reduced by the deficiencies of the flats aff $S_i$. This keeps the flats aff $S_i$ from forming "pencils of lines", "books of planes", etc.)

We will let lin $S$, aff $S$, card $S$, and conv $S$ denote respectively the linear

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span of $S$, the smallest flat containing $S$, the cardinality of $S$, and the convex hull of $S$.

## 2. Proof of the theorem

The proof of Theorem 1 follows directly from the following stronger result (which will be of independent interest in the next section).

THEOREM 2. *If* $S = S_1 \cup \cdots \cup S_r$ *is any partition of the* $[(d + 1)(r - 1) + k + 1]$*-set* $S$ *of strongly independent points of* $R^d$, *and if* $\operatorname{card} S_1 \leq d + 1$ *and* $\bigcap_{i=1}^r \operatorname{conv} S_i \neq \emptyset$, *then* $\dim(\bigcap_{i=1}^r \operatorname{conv} S_i) = k$.

Theorem 1 follows easily from Theorem 2. Suppose $S$ is any $[(d + 1)(r - 1) + k + 1]$-set of strongly independent points in $R^d$. The result of Tverberg asserts that $S$ is $(r, 0)$-divisible. Let $S = S_1 \cup \cdots \cup S_r$ be the corresponding partitioning of $S$. Choose $x \in \bigcap_{i=1}^r \operatorname{conv} S_i$. Then for each $i$, (by Carathéodory's theorem) $x$ is in the convex hull of some $d + 1$ or fewer points of $S_i$. Thus we may assume $\operatorname{card} S_i \leq d + 1$ for all $i$. Theorem 2 now implies that $S$ is $(r, k)$-divisible.

To prove Theorem 2, let $S$ be any $[(d + 1)(r - 1) + k + 1]$-set of strongly independent points in $R^d$ with a partitioning $S = S_1 \cup \cdots \cup S_r$ for which $\operatorname{card} S_i \leq d + 1$ and $\bigcap_{i=1}^r \operatorname{conv} S_i \neq \emptyset$. The strong independence of $S$ implies that $\dim(\bigcap_{i=1}^r \operatorname{aff} S_i) = k$. We want to show that

$$\dim(\bigcap_{i=1}^r \operatorname{conv} S_i) = k.$$

It clearly suffices to show that $\bigcap_{i=1}^r \operatorname{int} \operatorname{conv} S_i \neq \emptyset$, where $\operatorname{int} X$ denotes the relative interior of $X$. If $k = 0$, the result is clear, so assume $k \geq 1$ and proceed inductively. Let us assume the denial, that is assume

$$\bigcap_{i=1}^r \operatorname{int} \operatorname{conv} S_i = \emptyset. \tag{2}$$

Note that strong independence implies that each set $S_i$ is in general position and therefore is the set of vertices of a $(\operatorname{card} S_i - 1)$-dimensional simplex. Since $\bigcap_{i=1}^r \operatorname{conv} S_i \neq \emptyset$ there exists some $S_i$, say $S_1$, and some point of $S_1$, say $s_1 \in S_1$, for which (letting $T_1 = S_1 \sim \{s_1\}$) it is true that $\operatorname{conv} T_1 \cap (\bigcap_{i=2}^r \operatorname{conv} S_i) \neq \emptyset$. If we apply the induction hypothesis to the

$$[(d + 1)(r - 1) + (k - 1) + 1]\text{-set } T = T_1 \cup (\bigcup_{i=2}^r S_i)$$

it follows that

$$\dim(\operatorname{conv} T_1 \cap (\bigcap_{i=2}^r \operatorname{conv} S_i)) = k - 1$$

while the strong independence implies that

$$\dim(\operatorname{aff} T_1 \cap (\bigcap_{i=2}^r \operatorname{aff} S_i)) = k - 1.$$

These equations imply that

$$\operatorname{int} \operatorname{conv} T_1 \cap (\bigcap_{i=2}^r \operatorname{int} \operatorname{conv} S_i) \neq \emptyset. \tag{3}$$

But (2) and (3) imply that $\bigcap_{i=2}^{r} \text{aff } S_i \subseteq \text{aff } T_1$. Thus

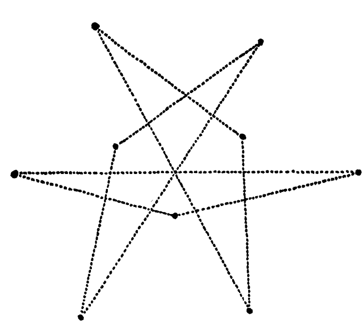
$$\begin{aligned}(4) \qquad \bigcap_{i=1}^{r} \text{aff } S_i &= \text{aff } S_1 \cap (\bigcap_{i=2}^{r} \text{aff } S_i) \cap \text{aff } T_1 \\ &= \text{aff } T_1 \cap (\bigcap_{i=2}^{r} \text{aff } S_i).\end{aligned}$$

But the left side of (4) is a $k$-dimensional flat while the right side of (4) is a $(k - 1)$-dimensional flat. This contradiction establishes the induction and thus the theorem.

## 3. Strong independence and general position

The essential difference between Theorems 1 and 2 is that Theorem 1 asserts that for a particular set $S$ *there exists* at least one partition with a certain property, while Theorem 2 asserts that *every* (suitably restricted) partition of $S$ has this property. The induction proof of Theorem 2 makes use of this stronger statement and the strong independence in a way that appears to be essential. It would be interesting to know what is the weakest condition on the set $S$ of Theorem 1 to assure $(r, k)$-divisibility. Tverberg made strong use of sets $S$ which are *algebraically independent* (i.e., the $md$ real coordinates of the points of the $m$-set $S$ are algebraically independent over the field of rationals.) We will say that the set $S \subset R^d$ is *weakly independent* provided $S$ is in general position and "no subspaces are parallel," that is, if $L_i$ is a line in aff $S_i$, $S_i \subset S$, and $L_1$ and $L_2$ are parallel, then there exists a line $L$ parallel to $L_1$ and $L_2$ in the space (aff $S_1$) $\cap$ (aff $S_2$). Clearly algebraic independence of a set implies strong independence, which implies weak independence, which implies general position, and no implication may be reversed. The following example shows the necessity of strong independence in Theorem 2, and suggests that a weaker condition might suffice for Theorem 1.

*Example.* Let $S$ be the 9 points as shown in $R^2$. These points are in general position since each 3 determine a non-degenerate simplex (they are also weakly independent), but they are not strongly independent since 6 of



them determine a pencil of 3 lines meeting at a single point. If $d = 2$, $r = 3, k = 2$, then $[(d + 1)(r - 1) + k + 1] = 9$. Yet these 9 points have been partitioned to form 3 intersecting triangles (as shown) for which $\bigcap_{i=1}^{3} \operatorname{conv} S_i$ is 0-dimensional. This shows that Theorem 2 fails if strong independence is replaced by weak independence or general position. However the 9 points are still (3,2)-divisible as Theorem 5 below shows. That is, there exists a different way to partition these into 3 sets $S_i$, each with 3 points, and have $\dim (\bigcap_{i=1}^{3} \operatorname{conv} S_i) = 2$.

The proofs of Theorems 1 and 2 do not actually use the full power of strong independence. It would be sufficient to require that if $\{S_1, \cdots, S_j\}$ is a family of disjoint subsets of $S$ and $j \leq r$, then the family satisfies (1). It may be shown that if $S \subset R^d$ is weakly independent then each family $\{S_1, S_2\}$ of disjoint subsets of $S$ satisfies the property (1). Hence if $r = 2$, the strong independence of Theorem 2 may be reduced to weak independence. However, even a stronger result is possible when $r = 2$, as the next theorem shows. We first state an obvious lemma.

LEMMA 3. *If $S_1 \cup S_2$ is a set of exactly $d + k + 2$ points which are in general position in $R^d$, if* card $S_i \leq d + 1$ *and $S_1$ and $S_2$ are disjoint, and if* aff $S_1 \cap$ aff $S_2 \neq \emptyset$, *then* dim (aff $S_1 \cap$ aff $S_2$) $= k$.

*Proof.* Since the two flats aff $S_1$ and aff $S_2$ intersect, and the sum of their codimensions is $d - k$, it is clear that the dimension of their intersection must be at least $k$. If dim (aff $S_1 \cap$ aff $S_2$) $> k$ it is clear that the flat spanned by $S_1 \cup S_2$ has dimension less than $d$, and therefore is contained in a hyperplane. But then the general position of $S_1 \cup S_2$ would imply that card $(S_1 \cup S_2) \leq d$. Thus dim (aff $S_1 \cap$ aff $S_2$) $= k$.

THEOREM 4. *Each $(d + k + 2)$-set $S$ in general position in $R^d$ is $(2, k)$-divisible. Furthermore, if $S = S_1 \cup S_2$ is any partition of $S$ for which* card $S_i \leq d + 1$ *and* conv $S_1 \cap$ conv $S_2 \neq \emptyset$, *then* dim (conv $S_1 \cap$ conv $S_2$) $= k$.

*Proof.* $S$ is $(2, 0)$-divisible by Radon's theorem, and $0 \leq k \leq d$, so there exist partitions $S = S_1 \cup S_2$ with card $S_i \leq d + 1$ and conv $S_1 \cap$ conv $S_2 \neq \emptyset$. The lemma implies that dim (aff $S_1 \cap$ aff $S_2$) $= k$, and with this fact the proof now proceeds in a way similar to the proof of Theorem 2. The details are omitted.

Note that the example given above shows that Lemma 3 and Theorem 4 cannot be extended to values of $r \geq 2$. It is also interesting to note that a direct proof of Theorem 4 may be given using the machinery of Gale transformations as recently developed by M. Perles [3]. If $\psi$ is the transformation taking the $d + k + 2$ points of $S \subset R^d$ onto the image set $\bar{S}$ in the associated $(k + 1)$-dimensional Gale space, then it may be shown that the condition of general independence of $S$ is equivalent to the condition that at most $k$ points of $\bar{S}$ lie on any hyperplane through the origin in the Gale space $R^{k+1}$. This fact allows us to directly choose $k + 1$ affinely independent points in $R^d$ which are in common to conv $S_1$ and conv $S_2$, where $S_1$ and $S_2$ are the appropriate subsets of $S$ from Theorem 4.

We now consider the special case $d = 2$ where $k$ assumes only the values 0, 1, and 2. The conditions placed on $S$ in this case are much weaker than general position.

THEOREM 5. *Let $S$ be a set of $3(r - 1) + k + 1$ points in the plane $R^2$. Then $S$ is $(r,k)$-divisible provided*

(a) *if $k = 0$, no additional requirement is made;*
(b) *if $k = 1$, $S$ is not contained in the union of two lines in $R^2$;*
(c) *if $k = 2$, there does not exist any line in $R^2$ which contains more than one third of the points of $S$.*

*Proof.* Define the *center* of $S$ (denoted $C(S)$) to be the set of all points $x$ of $R^2$ such that each closed half plane which contains $x$ also contains at least one third of the points of $S$. Equivalently, $C(S)$ may be defined as the intersection of all closed half planes in $R^2$ which contain more than two thirds of the points of $S$. It is well known that $C(S)$ is non-empty. (See Theorem 2.8 of Danzer-Grunbaum-Klee [2] for a more general statement and historical background.) Birch [1] proved that each vertex $v$ of the convex polygon $C(S)$ is a *point of divisibility*, that is, $S$ may be partitioned into $r$ subsets $S_1, \cdots, S_r$ such that $v \in \bigcap_{i=1}^{r} \operatorname{conv} S_i$. With reasoning similar to that used by Birch it is easy to show that if $k = 1$, then each boundary point of $C(S)$ is a point of divisibility, and if $k = 2$ (so that card $S = 3r$) then each interior point of $C(S)$ is a point of divisibility. It follows that if $k = 1$, then $S$ is $(r,1)$-divisible provided that $C(S)$ is not just a point, and if $k = 2$, then $S$ is $(r,2)$-divisible provided $C(S)$ has interior points. But it may be shown that if $k = 1$ and $S$ does not lie on the union of two lines then $C(S)$ has more than one point, and if $k = 2$ and no line contains more than one third of the points of $S$ then $C(S)$ has non-empty interior. This completes the proof of the theorem.

COROLLARY 6. *Let $S$ be a set of $3(r - 1) + k + 1$ points in general position in $R^2$. Then $S$ is $(r,k)$-divisible.*

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