ON THE TRIANGULATION OF THE REALIZATION OF A SEMISIMPLICIAL COMPLEX

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1.

In some mimeographed notes of Barratt [1], in which he shows that the geometric realization of a semisimplicial complex has a simplicial subdivision, there appear to be several errors, one in the statement of the subdivision theorem and two more in the proof. This note will give (we hope) a correct statement and proof of this theorem (Theorem 1.1), and will draw as a consequence the theorem of Milnor [4] that the homotopy groups of the realization $| S(X) |$ of the singular complex of a space are naturally isomorphic to those of the space itself.

We will write ssc for semisimplicial complex. Notation and terminology as in [4] or [5], except that we will denote the abstract n-simplex by $\Sigma(n)$, the geometric n-simplex by Δ^n.

The main result is the following theorem.

THEOREM 1.1. *Let X be a ssc and $| X |$ its realization* [4]. *Then there is a functor D from the category of ssc's to that of ordered simplicial complexes, a transformation of functors $\lambda : D \to 1$, and, for each X, a map $t_x : | DX | \to | X |$ such that*

(i) *t_x is a homeomorphism (and therefore a triangulation of $| X |$;*

(ii) *t_x defines a subdivision of the* CW *complex $| X |$; and*

(iii) *$| \lambda(X) |$ is homotopic to t_x by a homotopy F such that for each cell $| e |$ of $| DX |$, F maps $| e | \times | I$ into the smallest cell $| x |$ of $| X |$ which contains $t_x(| e |)$.*

We will give the proof later, in Sections 2, 3, and 4.

COROLLARY 1.2. (Simplicial approximation theorem). *If $f : | X | \to | Y |$ is any map, then f is homotopic to the realization of a ss map of subdivisions of $| X |$ and $| Y |$. If $| X |$, $| Y |$ are finite and therefore metrizable, then for any prescribed $\varepsilon > 0$, the homotopy between f and its ss approximation can be chosen so that it does not displace a point outside of an ε-disc.*

Proof. Apply the simplicial approximation theorem to the map

$$f' : | DX | \to | DY |, \quad \text{where} \quad f' = t_y^{-1} f t_x . \quad \blacksquare$$

Some remarks about the singular complex of a space. Let X be a space, and $S(X)$ its singular complex. Let $p_X : | S(X) | \to X$ be the map sending the point (P, x_n) of $| S(X) |$ into $x_n(P)$ [4].

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PROPOSITION 1.3. *Let K be any ordered simplicial complex and let $g : | K | \to X$ be any map. Then there is a unique ss map $g : K \to S(X)$ such that $g = p_x \mid g \mid$. Moreover, this universal condition uniquely determines $| S(X) |$ and p_X from among all maps $p' : | X' | \to X$ of ssc's into X up to equivalence.*

Proof. Given g, we define $\bar{g}$ as follows. Let σ be any n-simplex of $| K |$ and $\phi_\sigma : \Delta^n \to \sigma$ its characteristic map, monotone in terms of the orderings on the vertices of Δ^n and σ. If $P \in \sigma$, let

$$\bar{g} : (P) = (\phi_\sigma^{-1}(P), g \cdot \phi_\sigma) \in | S(X) |.$$

It is easy to check that $\bar{g} : | K | \to | S(X) |$ is well defined and it is clear from the definition that $g = p_X \bar{g}$.

To see that this map is unique, suppose g' were a second ss map from K to $S(X)$ such that $g = p_X \mid g' \mid$. Then for any σ, for any point P of σ,

$$\bar{g}(\sigma)(P) = g(P) = p_X(\phi_\sigma^{-1}(P), g'\sigma) = g'(\sigma)(P).$$

Thus $\bar{g}\sigma$ and $g'(\sigma)$ are the same singular n-simplex.

If X' were any other ssc and $p' : | X' | \to X$ a continuous map with the universal property above, then there is a homeomorphism $h : | S(X) | \to | X' |$ such that $p_X = p'h$. In fact, h is the realization of a ss isomorphism j. To construct h (or j), let x_n be any nondegenerate simplex of $S(X)$, and consider $| x_n | : \Delta^n \to X$. By the universal condition there is a unique $x_n' : (n) \to X'$ such that $p'x' = x_n$. Let $j(x) = x'$. It is easy to check that j defines a map of $S(X)$ into X' such that $| j | = p'$. Similarly, we can construct $j' : X' \to S(X)$ such that $p_X' = p_X \mid j \mid$. It follows that if $x : \Delta^n \to X$ is any singular simplex, that $p_X \mid j'\overline{jx} \mid = p' \mid j\bar{x} \mid = p_X \mid \bar{x} \mid$. Therefore, by the uniqueness of $\bar{x}$, $j'j\bar{x} = \bar{x}$, each $\bar{x}$, and therefore $j'j =$ identity. Similarly, $jj' =$ identity. ∎

THEOREM 1.4 [4]. *The map p_X induces isomorphisms of homotopy groups in all dimensions.*

Proof. Let $[f] \in \pi_n(X, *)$ be the class of the map $f : (S^n, *) \to (X, *)$. We assume that S^n is the realization of an ordered simplicial complex K with vertex $*$. Then there is a unique ss map

$$K \xrightarrow{f'} S(X),$$

such that $f = p_X \mid f^* \mid$. Let the base point be chosen in $| S(X) |$ at the vertex lying in the preimage $p^{-1}(*)$. Then f' must map the vertex $*$ of $| K | = S^n$ onto $*$, since $p(| f' |(*)) =$ vertex $*$ of X. Hence

$$p_* : \pi_n(| S(X)|, *) \to \pi_n(X, *)$$

maps the class of $| f' |$ onto $[f]$. Therefore p_* is surjective.

Suppose that $g : (S^n, *) \to (X, *)$ composes with p to give a null homotopic class. Again we assume S^n is an ordered simplicial complex K with vertex at $*$. By Theorem 1.1, g is homotopic to a ss map $|g'| : |K| \to |S(X)|$, and thus $p \cdot |g'|$ is nullhomotopic. There is a map $H : |K| \times I \to X$ which reduces to $p \cdot |g'|$ on $|K| \times 0$, is constant at $*$ on $|K| \times (1)$ and sends $* \times I$ into $*$. $|K| \times I$ has a triangulation containing $|K|$ as the subset $|K| \times 0$, and $* \times I$ as a 1-simplex. Hence there is a factorization $H = p\,|H'|$, where H' maps the ssc $K \times I$ into $S(X)$. By the uniqueness of the factorization $|g'| = p \cdot |g'^*|$, so the map $|H'|$ must reduce on $|K| \times (0)$ to $|g'|$. It is thus a nullhomotopy of $|g'|$. Therefore, $|g'|$, and also g, must be nullhomotopic. Therefore p_* is injective, and so is an isomorphism. ∎

The technique of proof of Theorem 1.1 is the following. We use the barycentric subdivision functor on the category of ssc's [2], and a second functor, a starring subdivision functor.

We will prove that for any X, Sd X belongs to a class of ssc's (*regulated* ssc's) with the property that the realization is a regular CW complex. On this subcategory of ssc's, the starring functor $*$ is just the classical starring triangulation for regular CW complexes. Moreover, we will prove that if X is regulated, then there is a ss map $\lambda : {}^*X \to X$ (natural with respect to maps of regulated ssc's) which, upon realization, sends a simplex in the subdivision of the cell $|x_n|$ into a face of this simplex. This, together with some properties of Sd will be enough to establish Theorem 1.1.

2.

We will need the following facts about the barycentric subdivision functor Sd on the category of ssc's. The simplest way to define it is as the standard barycentric subdivision operator on the subcategory of ordered simplices and ordered simplicial maps, with vertex ordering in (decreasing) order of dimension of the face that the vertex represents, and then extending it as a direct limit functor over the category of all ssc's and maps. For our purposes, though, it is more convenient to define it by explicit analogy with that given by Milnor for the geometric realization, so that certain corresponding statements carry over to this case without further proof.

Let $\Sigma(n)$ be the abstract n-simplex, an ordered simplicial complex.

DEFINITION 2.1. Sd $\Sigma(n)$ is the ordered simplicial complex whose vertices are the faces of $\Sigma(n)$ partially ordered as follows: $\sigma < \tau$ if τ is a face of σ; and whose simplices are all nonincreasing sequences of vertices. If $\phi : \Sigma(n) \to \Sigma(m)$ is an ordered simplicial map (monotone on the vertices), then Sd$\,\phi$ maps $\langle \sigma_0, \cdots, \sigma_n \rangle$ into $\langle \phi\sigma_0, \cdots, \phi\sigma_n \rangle$. In particular, this defines $\mathrm{Sd}\, d_i^* : \mathrm{Sd}\, \Sigma(n) \to \mathrm{Sd}\, \Sigma(n+1)$, and $\mathrm{Sd}\, s_i^* : \mathrm{Sd}\, \Sigma(n+1) \to \mathrm{Sd}\, \Sigma(n)$.

If X is an ssc, let $M'(X)$ be the union $\bigcup \mathrm{Sd}\, \Sigma(n) \times x_n$ of one copy of Sd $\Sigma(n)$ for each n-simplex x_n of X, given the obvious ss structure. We define an equivalence relation $\mathfrak{R}$ on $M'(X)$ by setting $(\sigma, d_i x_n) \sim (\operatorname{Sd} d_i^* \sigma, x_n)$ for each simplex $\sigma \in \Sigma(n -)$, and $(\sigma, s_i x_n) \sim (\operatorname{Sd} s_i^* \sigma, x_n)$ for each $\sigma \in \Sigma(n + 1)$.

DEFINITION 2.2. **Sd X** is the ss complex which we obtain by passing to the quotient. (The ss operations are thus given by: $d_i(\sigma, x_n) = (d_i \sigma, x_n)$ and $(s_j \sigma, x_n) = s_j(\sigma, x_n)$, where we have let (σ, x_n) stand for the equivalence class of the element $\sigma \times x_n$ of $\operatorname{Sd} \Sigma(n) \times x_n$.)

If $f : X \to Y$ is a ss map, then $\operatorname{Sd} f(\sigma, x_n) = (\sigma, fx_n)$.

It is easy to verify that these definitions define a functor from the category of ss complexes to itself.

To further describe the structure of Sd X, we want the analogue of Lemma 3 of [4] which describes a unique representative from each equivalence class of the quotient set Sd X. We note that any simplex σ of $\operatorname{Sd} \Sigma(n)$ has a unique representation $\operatorname{Sd} F^*\tau$, where $F^* : \Sigma(p) \to \Sigma(n)$ is a face operator embedding $\Sigma(p)$ as the F face of $\Sigma(n)$ and τ is an *interior* simplex of $\Sigma(p)$, that is, has for zero$^{\text{th}}$ vertex the vertex corresponding to the simplex $\Sigma(p)$ itself. (Upon realization, such a simplex of $\operatorname{Sd} \Delta^p$ would contain interior points of Δ^p.) This representation is found by finding the unique smallest face $F^*\Sigma(p)$ "containing σ", the one whose vertices are the distinct vertices from among the sequence of faces which give σ in $\operatorname{Sd} \Sigma(n)$. F^* is the face operator which embeds this face in $\Sigma(n)$. It follows then that the following lemma is proven exactly as Lemma 3 of [4]:

LEMMA 2.3. *In the equivalence class of every element $\sigma \times x_n$ of $M'(X)$ there is a unique irreducible representative $\tau \times y$, where τ is interior to* $\operatorname{Sd} \Sigma(p)$ *and y_p is nondegenerate. We define the selection function θ that picks this class out of the given class as follows: represent $\sigma \times x_n$ uniquely as* $\operatorname{Sd} F^*\tau_1 \times x_{n'}$, *where F^* is a face operator and τ_1 interior; let $Fx_n = Dy_p$, where D is a degeneracy operator and y_p nondegenerate, and let $\tau =$* $\operatorname{Sd} D^*\tau_1$. *Then $\theta(\sigma \times x_n) = \tau \times y_p$.*

Thus it follows that, as in [4]:

LEMMA 2.4. *The characteristic (ss) map ϕ of the simplex (σ, x_n) of* Sd X *is the composition of the inclusion map of σ into* $\operatorname{Sd} \Sigma(n)$ *with Sd ϕ_x, where ϕ_x is the characteristic map $\phi_x : \Sigma(n) \to X$ of x_n.* Sd ϕ_x *is bijective on the interior simplices on* $\operatorname{Sd} \Sigma(n)$.

PROPOSITION 2.5. *For any ssc X, there is a homeomorphism $t : |\operatorname{Sd} X| \to |X|$ identifying $|\operatorname{Sd} X|$ with a subdivision of the* CW *complex $|X|$.*

Proof. We will subdivide $|X|$ by a modified star-subdivision process. In each face of each simplex of $M(X)$ we will choose an interior point. We proceed inductively as follows. If $\sigma \times x_n$ is a vertex, we select it itself as the interior point. Suppose that we have selected one interior point of each simplex of dimension $>n$ in $M(X)$ so that whenever the face $\sigma \times x_n$ of $\Delta^n \times x_n$ is identified with the face $\tau \times x_m'$, the points chosen are identified with each other. Let $\sigma \times x_m$ be an n-dimensional face of $\Delta^m \times x_m$. If $\sigma = F^*\tau^n$ and Fx_m is a nondegenerate (n-)simplex of X, choose the barycenter of this face. If not, say $Fx_m = Dx_p'$, a degeneracy D of the nondegenerate p-simplex $x_p'(p < n)$, then choose any interior point of σ which is identified with the point chosen (inductively) in $\Delta^p \times x_p'$. It is easy to see that this extends the selection of interior points over dimension n, and thus we can select such "pseudo-barycenters" in each simplex of $M(X)$.

Then we subdivide each cell $|x_n|$ of $|X|$ by starring $\Delta^n \times x_n$ for each simplex x_n of X, joining the already defined subdivision on the boundary of each face to the chosen interior point by straight lines, segments of planes, etc. Because the maps d^* and s^* which identify the simplices of $M(X)$ to give $|X|$ are linear, they will define a consistent subdivision of $|X|$. Clearly this subdivision is the homeomorphic image of $|\mathrm{Sd}\, X|$. We define such a homeomorphism as follows. Take any simplex $\sigma \times x_n$ (irreducible representative) and map it into $\Delta^n \times x$ in $M(X)$ by mapping its vertices onto the "pseudo-barycenters" to which they correspond in $\Delta^n \times x_n$, the points already chosen, and extending linearly. It is easy to check that this is indeed a homeomorphism. ∎

PROPOSITION 2.6. *There is a natural* ss *map* $\lambda(x) : \mathrm{Sd}\, X \to X$ *and a homotopy* $F : |\mathrm{Sd}\, X| \times I \to |X|$ *between* $|\lambda_x|$ *and t, for any map t as in the preceding proposition, with the further property that F maps* $|x_n| \times I$, *for any n-cell* $|x_n|$ *of* $|\mathrm{Sd}\, X|$, *into the cell whose interior contains* $t(|x_n|)$.

Proof. The map λ is the one which sends the simplex $\sigma \times x_n$ into $\phi_x(\tau)$, where τ is the simplex of $\Sigma(n)$ whose vertices are the last vertices of σ. It is easy to check that this is well defined, and it maps a simplex $\sigma \times x_n$ of $\mathrm{Sd}\, X$ into a face of x_n. The homotopy F is defined in the simplices of $M(X) = \bigcup_{n>0} \Delta^n xX_n$ and then projected onto $|X|$; it is the linear homotopy which moves each "pseudo-barycenter" towards the last vertex of the simplex it corresponds to. ∎

3. Regulated ssc's

DEFINITION 3.1. Let x be a simplex of the ssc X and let $\phi : \Sigma(n) \to X$ be its characteristic map. We say that x is regulated if for each pair of faces $y \neq y'$ which both contain the zero$^{\text{th}}$ vertex, $\phi(y) \neq \phi(y')$ and on vertices $k \neq 0$, $\phi(k) \neq \phi(0)$. If each nondegenerate simplex of X is regulated, we say that X is regulated.

PROPOSITION 3.2. *For any ssc X, the ssc* $\mathrm{Sd}\, X$ *is regulated.*

Proof. This follows from the fact that for any nondegenerate x in X, with characteristic map ϕ, the map $\mathrm{Sd}\, \phi : \mathrm{Sd}\, \Sigma(n) \to \mathrm{Sd}\, X$ is bijective on the "interior simplices", those simplices of Sd $\Sigma(n)$ which have for zero$^{\text{th}}$ vertex the vertex corresponding to Sd $\Sigma(n)$. ▮

The property of being regulated is the only property of Sd X we will need.

Let x be a regulated n-simplex of X, let $\phi_x : \Sigma(n) \to X$ be its characteristic map, and $|\phi_x|$ the corresponding realization, a characteristic map for the cell $|x_n|$ of $|X|$. We will first analyze the identifications of points of Δ^n under the map $|\phi_x|$.

LEMMA 3.3. *There is an integer p and a face operator F such that $|\phi|$ is bijective on all open cells outside of the p-dimensional face $F^*\Delta^p$ of Δ^n. On this face, there is a face $F'^*\Delta^q$ such that the restriction of $|\phi_x|$ to $F^*\Delta^p$ is $D^*|\phi_{F'x}|$, where D is a suitable degeneracy map and $F'x$ a nondegenerate face of x.*

Proof. We note that if $|\phi|$ identifies interior points of two open cells, then it identifies all interior points of one with the other. Thus no interior point of the n-dimensional simplex Δ^n is identified with any other point. Let p be the highest dimension in which an interior point of a p-dimensional cell, say $F^*\Delta^p$ is identified with an interior point of another cell, say $F''^*\Delta^r$. Then, clearly ϕ_x identifies $F^*\Sigma(p)$ with a degeneracy of $F''^*\Sigma(r)$. Let $F^{(\text{iii})*}\Sigma(s)$ be the smallest face of $\Sigma(n)$ containing both $F^*(\Sigma(p))$ and $F''^*\Sigma(n)$, the face whose set of vertices is the union of the sets of vertices of the two faces. If we assume that $F''^*\Sigma(n)$ is not a face of $F^*\Sigma(p)$, then the set of vertices is strictly bigger than that of $F^*\Sigma(p)$ and hence $F^{(\text{iii})*}\Sigma(s)$ is a strictly larger face of $F^*\Sigma(n)$ than $F^*\Sigma(p)$. This means in particular that $F^{(\text{iii})}x = \phi_x(F^{(\text{iii})*}\Sigma(s))$ is nondegenerate since its dimension is larger than p. Because we are assuming x is a regulated simplex, and because $F^*\Sigma(p)$ and $F''^*(r)$ are simplices which are identified (or whose degeneracies are identified) by ϕ_x, it follows that neither of them can have the zero$^{\text{th}}$ vertex of $F^{(\text{iii})*}\Sigma(s)$ for a vertex. This is impossible; by hypothesis, $F^{(\text{iii})*}\Sigma(s)$ is the smallest simplex whose vertices contain those of the two given faces. Hence it is not possible, if x is regulated, for $F''^*\Delta^r$ to lie outside of $F^*\Delta^p$.

Suppose that P is an interior point of Δ^p, and (F^*P, x) the corresponding point of the simplex $\Delta^n \times x_n$ in the union $M(X)$, and suppose that $|\phi_x|$ identifies this point with (F'^*Q, x_n), where $F'x$ is a nondegenerate face of x and Q an interior point of Δ^q. That is $(F^*P, x) \sim (Q, F'x)$, the latter an irreducible point of $M(X)$. Then the selection function θ sends (F^*P, x) into $(Q, F'x)$ so that $Fx = DF'x$ and $D^*P = Q$, where D is a suitable degeneracy operator, by definition of θ. It thus follows that for any other point of $F^*\Delta^p$, that $|\phi_x|(F^*P', x) = |\phi_{F'x}|(D^*P', F'x)$, and that therefore the restriction of $|\phi_x|$ to the face $F^*\Delta^p$ is the composition $D^*|\phi_{F'x}|$. ▮

To find the remaining identifications produced by $|\phi_x|$, repeat this analysis on (Δ^q, F'^*x). Since $F'x$ is a nondegenerate face of a regular simplex, it is regular. Thus there is a dimension p_2 of a face $F_2'^*\Delta^{p_2}$ such that on each open cell of dimension larger $\phi_{F'x}$ is bijective, as it is also on all open cells of dimension p_2 outside of $F_2'^*\Delta$, etc. Iteration of the preceding lemma gives the following:

COROLLARY 3.4. *If x is a regulated n-simplex of X, then $|\phi_x| : \Delta^n \to |X|$ makes the following identifications (ana no others). There is a descending sequence of faces of Δ^n,*

$$\Delta^n = \tau_0 \supset \sigma_1 \supset \tau_1 \supset \sigma_2 \supset \tau_2 \supset \cdots \supset \sigma_r \supset \tau_r,$$

of dimensions $\dim \sigma_i = p_i$, $\dim \tau_i = q_i$, *and degeneracy operators D_i such that*

(i) $|\phi_x| \mid \tau_i$ *is bijective on all open cells outside of* σ_{i+1},

(ii) $|\phi_x| \mid \sigma_{i+1} = D_{i+1}^*(|\phi_x| \mid \tau_{i+1})$;

(iii) $|\phi_x|$ *is bijective on the interior of* τ_{i+1}. ▌

We will now prove that a regulated n-simplex realizes to a regular n-cell in $|X|$.

LEMMA 3.5. *Let $\Delta^n \supset \sigma \supset \tau$ be proper faces, let $D^* : \sigma \to \tau$ be a degeneration map, let L be the quotient of Δ^n by the identifications of D^*, and let $\phi : \Delta^n \to L$ be the quotient map. Then there is a homeomorphism $h : \Delta^n \to L$ such that if $r \leq q =$ dim τ, for any r-dimensional face ξ, with inclusion map $i : \xi \to \tau$, we have $hi = \phi \mid \xi$.*

Proof. Let σ' be the face of Δ^n opposite σ. Each point P of Δ^n has a unique representation $P = (1 - t)Q + tQ'$, where $Q \in \sigma$, $Q' \in \sigma'$. Define

$$\begin{aligned} \rho(P) &= P && \text{if } \tfrac{1}{2} \leq t \leq 1; \\ &= t(Q + Q') + (1 - 2t)D^*(Q) && \text{if } 0 \leq t \leq \tfrac{1}{2}. \end{aligned}$$

Then

(i) $\rho(P) = \rho(P')$ if and only if $\phi(P) = \phi(P')$;

(ii) Im ρ is a compact convex subset of Δ^n; and

(iii) contains the simplex ω of Δ^n spanned by σ' and τ, and the subset of Δ^n consisting of points with $t \geq \frac{1}{2}$.

These facts, easy to verify, are left to the reader. From (i) and the compactness of Δ^n, it follows that Im ρ is homeomorphic to L. From (ii) and (iii), Im ρ contains a small copy of Δ^n, the convex set with vertices $\{v_i'\}$, corresponding to the vertices of σ', and $\{(v_i + b')\frac{1}{2}\}$ where b' is the centroid of σ', and v_i is a vertex of σ'. By radial projection from the centroid of the face of this simplex determined by the simplices corresponding to σ and τ, we get a homeomorphism $h : \Delta^n \to$ Im ρ which restricts to a linear homeomorphism of the simplex corresponding to ω onto ω. ▌

PROPOSITION 3.6. *Let x be a regulated n-simplex of the ssc X. Then $|x|$ is a regular n-cell of the* CW *complex $|X|$. Hence if X is a regulated ssc, then $|X|$ is a regular* CW *complex.*

Proof. Suppose x is the simplex for which ϕ_x has the form described in Corollary 3.4. Let L_i be the quotient space of Δ^n after identifying points in the faces $\sigma_1, \cdots, \sigma_i$ by $D_1^*, \cdots, D_i^*$. Thus, $L_r = |x|$. Applying the preceding lemma inductively to L_i, we have a homeomorphism of L_{i+1} with Δ^n in which the points of faces of dimension $\leq$ dimension τ_{i+1} are identified with their images under the quotient map with the corresponding point of τ_{i+1}. Thus L_{i+2} is obtained from L_{i+1} by collapsing the points of σ_{i+1} by the degeneration map D_{i+1^*}, and the inductive argument can proceed until we reach $L_r = |x|$. ∎

4. The Functor *

DEFINITION 4.1. Let X be a ssc. We define $^*X$, an ordered simplicial complex as follows. The *vertices* of $^*X$ are the nondegenerate simplices of X ordered as follows: $x < y$ if y is a face of x. A sequence of vertices $\langle x_{(0)}, \cdots, x_{(n)}\rangle$ for which $x_{(0)} \leq \cdots \leq x_{(n)}$ is a simplex of $^*X$. If $f : X \to Y$ is a ss map, we define an order preserving simplicial map $^*f : {}^*X \to {}^*Y$ by setting $^*f(\langle x\rangle)$ = the nondegenerate simplex of Y for which $f(x)$ is a degeneracy, and extending to a simplicial map.

This defines * as a functor from the category of ssc's to that of ordered simplicial complexes, a subcategory. On the simplicial subcategory, this is just barycentric subdivision: on the general ssc, looking back at the description of star subdivision of regular CW complexes, we see that if $|X|$ is a regular CW complex, this is the starring simplicial subdivision. In this case, we note that the simplex $\langle x_{(0)}, \cdots, x_{(n)}\rangle$ corresponds to a simplex lying in the cell $|x_{(0)}|$ of $|X|$; and if $n = \dim x_{(0)}$, this will correspond to a simplex whose interior lies in the interior of $|x_{(0)}|$.

In fact, let $\tau_x : |{}^*X| \to |X|$ be a triangulation by star subdivision of the *regular* CW complex $|X|$.

PROPOSITION 4.2. *Let $|X|$ be a regular* CW *complex, and let $\phi : {}^*X \to X$ be a ss map such that $\phi(\langle x_{(0)}, \cdots, x_{(n)}\rangle)$ is a face of $x_{(0)}$ for each simplex nondegenerate in $^*X$. Then $|\phi|$ is homotopic to τ_x by a homotopy which maps $|\langle x_{(0)}, \cdots, x_{(n)}\rangle| \times I$ into the cell whose interior contains the interior of $\tau_x(\langle x_{(0)}, \cdots, x_{(n)}\rangle)$.*

Proof. Obvious, for $|\phi|$ and τ_x images of each simplex of $|*|$ are contained in the same contractible subset of $|X|$, the cell whose interior contains the interior of the τ image. Thus the assignment of this cell is an aspherical carrier carrying both t_x and $|\phi|$. ∎

PROPOSITION 4.3. *Let X be a regulated ssc. Then there is an ss map (natural with respect to maps of regulated ssc's) $^*\lambda : {}^*X \to X$ such that $^*\lambda(\langle x_{(0)}, \cdots, x_{(n)}\rangle)$ is a face of $x_{(0)}$ for each nondegenerate simplex of $^*X$.*

Proof. Let $v = \langle x_{(0)}, \cdots, x_{(n)}\rangle$ be an arbitrary n-simplex of $^*X$; let $\phi : \Sigma(m) \to X$ be the characteristic map of a simplex of X which is nondegenerate and which has $x_{(0)}$ for a face. Let $\sigma^m = \langle 0, 1, \cdots, m\rangle$ be the fundamental m-simplex of $\Sigma(m)$ and let $\{F_i\,\sigma^m\}$ be a descending sequence of faces of σ^m for which $\phi(F_i\,\sigma^m) = F_i\,\phi(\sigma^m) = x_{(i)}$. Let L_i be the last vertex of $F_i\,\sigma^m$, and let $L = \langle L_n, \cdots, L_0\rangle$ be the corresponding face of σ^m. Define ${}^*\lambda(v) = \phi(L)$. Since L is a face of $F_0\,\sigma^m$, $\phi(L)$ will be a face of $x_{(0)}$. If we can prove that ${}^*\lambda(v)$ is independent of the choice of the particular descending family of faces $F_i\,\sigma^m$ (because of the identifications under the characteristic map of the simplex $\phi(\sigma^m)$ this is not unique), then clearly the map ${}^*\lambda$ will satisfy our conditions.

We proceed by induction on n. If $n = 0$, there is nothing to prove. Consider the case $n = 1$. Here $v = \langle x_{(0)}, x_{(1)}\rangle$. Let

$$\phi(F_0\,\sigma^m) = \phi(F_0'\,\sigma^m) = x_{(0)} \quad \text{and} \quad \phi(F_1\,\sigma^m) = \phi(F_1'\,\sigma^m) = x_{(1)},$$

where F_1 deletes whichever vertices of σ^m that F_0 does and some others besides, and similarly for F_1' and F_0'. Since they both have the same image in X, the dimensions of $F_0\,\sigma^m$ and $F_0'\,\sigma^m$ are the same, as are the dimensions of $F_1\,\sigma^m$ and $F_1'\,\sigma^m$. Since $\phi(F_0\,\sigma^m) = \phi(F_0'\,\sigma^m)$, the map $h : F_0\,\sigma^m \to F_0'\,\sigma^m$ defined by mapping the i^{th} vertex of $F_0\,\sigma^m$ onto the i^{th} vertex of $F_0'\,\sigma^m$ satisfies $\phi \mid F_0\,\sigma^m = (\phi \mid F_0'\,\sigma^m)h$. Thus

$$\phi(\langle L_1, L_0\rangle) = \phi(\langle h(L_1), h(L_0)\rangle) = \phi(\langle h(L_1), L_0'\rangle),$$

where L_0' is the last vertex of F_0', and it suffices to prove that

$$\phi(\langle h(L_1), L_0'\rangle) = \phi(\langle L_1', L_0'\rangle),$$

where L_1' is the last vertex of $F_1'\,\sigma^m$. Either $h(L_1) \leq L_1'$ or $h(L_1) \geq L_1'$; assume the former. Then $\langle h(L_1), L_1', L_0'\rangle = w$ is a face of $\Sigma(m)$, and $\phi(w)$ a face of a regulated simplex. It must be degenerate because $\phi(h(L_1)) = \phi(L_1')$ and thus its zero$^{\text{th}}$ and first vertices are identified. If $\phi(w) = s_0\,z$, then $d_0\,\phi(w) = d_1\,\phi(w)$ and we are done. If $\phi(w) = s_1\,z$, then z is a 1-simplex of X which has its first and second vertices equal (since they are $\phi(h(L_1'))$ and $\phi(L_1')$) and therefore is degenerate. Thus $\phi(w) = s_1\,s_0\,z'$ and therefore $d_0\,\phi(w) = d_1\,\phi(w)$ and the proposition is true in this case too.

Assume inductively that ${}^*\lambda$ is well defined whenever v has dimension $\leq n - 1$, for $n \geq 2$, and consider the case that v, as above, has $n + 1$ vertices. Let $\{F_i'\,\sigma^m\}$ be a second descending sequence of faces of σ^m for which $\phi(F_i'\,\sigma^m) = x_{(i)}$, let L_i' be the last vertices and L' the simplex they form. Let $z = \phi(L)$, $z' = \phi(L')$. We will prove $z = z'$.

Our induction hypothesis implies that all corresponding faces of z and z' are equal.

Suppose first that z and z' are degenerate, say $z = Dz_1$, $z' = D'z_1'$, where D and D' are degeneracy operators and z_1, z_1' are nondegenerate. Let F and F' be face operators such that $FD = 1$ and $F'D' = 1$. Then $z_1 = FD'z_1' = FD'F'Dz_1$ so that $FD'F'D = 1$. Similarly $F'DFD' = 1$ so that $FD' = F'D = 1$ and $z_1 = z_1'$. It is then easy to see that $z = z'$.

Next, suppose that z is degenerate, say $z = s_i z_1$. Then $d_i z = d_{i+1} z$ and thus the corresponding faces of z' are equal. Because this implies that a face of z' containing its zero$^{\text{th}}$ vertex is identified to another one, and because z' is a face of a regulated simplex, it follows that z' must also be degenerate and therefore we are back in the case just above.

Finally, suppose that z (and therefore also z') is nondegenerate. Then the following lemma implies that $z = z'$.

LEMMA 4.4. *Let X be a regulated ssc and z and z' nondegenerate faces of a simplex v of X. If $d_n z = d_n z'$, where $n = \dim z = \dim z'$, then either $z = z'$ or there is an $n + 1$-simplex u of X such that $z = d_n u$, $z' = d_{n+1} u$, or $z' = d_n u$, $z = d_{n+1} u$. If in addition, $n \geq 2$ and $d_{n-1} z = d_{n-1} z'$, then $z = z'$.*

Proof. Let $\phi : \Sigma(m) \to X$ be the characteristic map of v, and let U be a minimal face of $\Sigma(m)$ such that $u = \phi(U)$ has both z and z' as faces. Since z and z' are nondegenerate, u must be nondegenerate. If $Fu = z$ and $F'u = z'$, then because of the minimality of U the zero$^{\text{th}}$ vertex of U must lie in either $d_n FU$ or in $d_n F'U$. Since $d_n Fu = d_n F'u$, then, $d_n FU = d_n F'U$. Thus again because of the minimality of U, either U is an $(n + 1)$-simplex and F and F' delete the nth and $(n + 1)^{\text{st}}$ vertices, or U is an n-simplex and $z = z'$. (Otherwise we could replace U by the strictly smaller simplex whose vertices are the n vertices common to $d_n FU$ and $d_n F'U$ and throw in the last two vertices which are cancelled by d_n from FU and $F'U$.)

If in addition, $n \geq 2$ and $d_{n-1} z = d_{n-1} z'$, then this is a relation identifying two faces of u one of which contains the zero$^{\text{th}}$ vertex and thus the corresponding faces of U must be equal, in particular the last vertices must be identical. Thus $z = z'$. ∎

It follows from 2.6, 4.2, and 4.3 that the following statement is true:

PROPOSITION 4.5. *Let $\mathfrak{C}$ be the category of regulated ssc's and ss maps. Then there is a natural transformation ${}^*\lambda : * \to 1$ on $\mathfrak{C}$. Moreover, if X is any regulated ssc and $t_X : | {}^*X | \to | X |$ a starring triangulation, then there is a homotopy $F : | {}^*X | \times I \to | X |$ such that*

$$F \mid | {}^*X | \times (0) = | {}^*\lambda_X | \quad \text{and} \quad F \mid | {}^*X | \times I = t_X,$$

and for each simplex σ of ${}^*X$, $F \mid \sigma \times I$ has its image in the cell whose interior contains $t(\sigma)$. ∎

Proposition 4.5 together with the corresponding proposition about Sd implies Theorem 1.1. The subdivision D is the composition of $*$ with Sd. The composition of the maps λ for the two functors Sd and $*$ give the map λ called for in the theorem, and similarly the composition of the two homotopies gives the desired homotopy.

5.

In closing we would like to discuss the relation between the functors Sd and $*$, and the transformations associated with them. The difference between them is that Sd is a direct limit functor, while $*$ is not. In fact, if we start with $*$, restrict it to the model subcategory of abstract n-simplices and ordered simplicial maps, and propagate this restriction over all ssc's as a direct limit functor, we get Sd. Thus on general grounds, there is a transformation of functors $\Psi : \mathrm{Sd} \to *$. Let $^*\lambda : * \to 1$ be the transformation we have constructed on the subcategory of regulated ssc's, which includes the model subcategory. If we restrict this to the model subcategory and extend over all ssc's by direct limits, the result is just the natural transformation $\lambda : \mathrm{Sd} \to 1$. With the definition given in 4.2 for the map $^*\lambda$, we have proven the following:

PROPOSITION. *On the category of regulated ssc's,* $\lambda = {}^*\lambda \cdot \Psi$. ▌

The question arises whether we can define $^*\lambda$ over all ssc's so that this is true, and the following example shows that this is not so. Let X be the ssc with one nondegenerate simplex in dimension two and one vertex. $^*X$ is 1-cell, and it is clearly impossible to have the map $| \lambda(X) | : | \mathrm{Sd}\, X | \to | X |$, a map of S^2 to itself of degree 1, factor through a map into I. If we define a ssc to be regular if in each nondegenerate n-simplex there is a vertex with the property that no face of dimension $\leq n$ which has this distinguished vertex for a vertex is nondegenerate [1], then it is still impossible to define $^*\lambda$ for this subcategory of the ss category. For example, if X is the ssc obtained from the ordered 2-simplex by identifying vertices (1) and (2) (but not identifying the 1-simplex they determine) then a similar argument shows that $| \lambda(X) | : | \mathrm{Sd}\, X | \to | X |$, a map of spaces of the homotopy type of S^1 and of degree 1, cannot factor through $| {}^*X |$, a contractible space. (This by the way, shows that a "regular" simplex in terms of [1] need not give a regular cell upon realization, as is asserted in [1].) It is impossible to find any ss map in these cases from $^*X$ to X which maps onto X.

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