ON THE NUMBER OF CO-MULTIPLICATIONS OF A SUSPENSION[1]

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In [2], Arkowitz and Curjel established a criterion for determining when an associative H-space possesses only a finite number of multiplications. That their result can be dualized is the subject of the present note.

An H'-structure, or co-multiplication, on a space Z is a based map

$$\varphi : Z \to Z \vee Z$$

which has the property that the compositions $\pi_1 \circ \varphi$ and $\pi_2 \circ \varphi$ are both homotopic to the identity map of Z, where π_1 and π_2 are the obvious projections.

Let X be a CW-complex of locally-finite type. Then, by the Hilton-Milnor Theorem, $\Omega\Sigma(X \vee X)$ is homotopy equivalent to $\prod_k \Omega\Sigma P_k$ where k runs through a set of basic products for the set $\{1, 2\}$. To each basic product k there is associated a positive integer $\omega(k)$, the *weight* of k, and P_k has the homotopy type of

$$\overbrace{X \wedge X \wedge \cdots \wedge X}^{\omega(k)}.$$

Moreover, the homotopy equivalence is given by a map of the form $\prod_k \Omega g_k$, where $g_k : \Sigma P_k \to \Sigma(X \vee X)$ is an iterated generalized Whitehead product which is associated with the basic product k. In particular $P_1 = P_2 = X$ and the maps $g_i : \Sigma X \to \Sigma(X \vee X)$ $(i = 1, 2)$ are the inclusions. All g_k with $\omega(k) \geq 2$ are Whitehead products involving both the first and second factors of $\Sigma(X \vee X)$. For more details see [3] or [7].

If $f : X \to \Omega\Sigma(X \vee X)$ is any map, then there is a map $\bar{f} : X \to \prod_k \Omega\Sigma P_k$ with $\prod_k \Omega g_k \circ \bar{f} \sim f$. Let $p_k : \prod \Omega\Sigma P_k \to \Omega\Sigma P_k$ denote the projection, and let $\pi_i : \Sigma(X \vee X) \to \Sigma X$ $(i = 1, 2)$ denote the projections.

THEOREM 1. $\Omega\pi_i \circ f \sim p_i \circ \bar{f}$ $(i = 1, 2)$.

Proof. By the above, $\Omega\pi_1 \circ f \sim \Omega\pi_1 \circ \prod_k \Omega g_k \circ \bar{f}$. Since $\Omega\pi_1$ is a homomorphism, $\Omega\pi \circ \prod_k \Omega g_k = \prod \Omega(\pi_1 \circ g_k)$. But every basic product k with $\omega(k) \geq 2$ involves both 1 and 2, thus $\pi_i \circ g_k \sim *$ $(i = 1, 2$ and $\omega(k) \geq 2)$. Since also $\pi_1 \circ g_2 \sim *$ and $\pi_2 \circ g_1 \sim *$, Theorem 1 is proved.

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COROLLARY 2. *H'-structures on ΣX are in* 1-1 *correspondence with elements of the group* $\oplus_{\omega(k)\geq 2}[\Sigma X, \Sigma P_k]$.[2]

As an immediate consequence we have that if G is finitely generated and $n \geq 4$ then the Moore space $K'(G, n)$ has a unique H'-structure.

Let X be a finite-dimensional CW-complex and let $I = \{i_1, i_2, \cdots i_K\}$ be the set of integers for which $\tilde{H}_{i_j}(X; Q)$ is non-trivial ($i_j > 0, j = 1, \cdots K$). Let N be the set

$$\{\textstyle\sum_{j=1} c_j i_j\,;\ c_j \ \text{ integers, } \ c_j \geq 0, \ \text{ and } \ \Sigma c_j \geq 2\}.^3$$

THEOREM 3. *The number of co-multiplications on ΣX is finite if and only if $N \cap I = \Phi$.*

Proof. H'-structures on ΣX are in 1-1 correspondence with the group $\oplus_{\omega(k)\geq 2}[X, \Omega\Sigma P_k]$. Since X is finite-dimensional, only finitely many $[X, \Omega\Sigma P_k]$ are non-zero. By the results of Arkowitz and Curjel [1], $\oplus_k[X, \Omega\Sigma P_k]$ is finite if and only if $\rho(\oplus_k[X, \Omega\Sigma P_k]) = \Sigma_k\, \rho[X, \Omega\Sigma P_k] = 0$, where ρ denotes the rank in the sense of [1]. By Corollary 3.4 of [1],

$$\rho[X, \Omega\Sigma P_i] = \textstyle\sum_m \beta_m(X)\cdot\rho(\pi_m(\Omega\Sigma P_i))$$

where $\beta_m(X) = \rho(H_m(X))$. Theorem 3 is therefore equivalent to the following.

LEMMA 4. (i) *If $\pi_j(\Omega\Sigma P_k)$ has an infinite cyclic direct summand ($\omega(k) \geq 2$) then $j \in N$.*

(ii) *If $j \in N$ then there exists a k ($\omega(k) \geq 2$) for which $\pi_j(\Omega\Sigma P_k)$ has an infinite cyclic direct summand.*

The proof of 4 requires the following special case of a result due to Berstein [4]. F denotes the class of finite groups in the sense of Serre.

LEMMA 5. *Let Z be a finite* CW-*complex and let $\{u_\alpha^i\}_{i=1}^N$ be the generators of $H_+(Z; Q)$. Then there exists a map $f : \bigvee_{i,\alpha} S_\alpha^{i+1} \to \Sigma Z$ with the property that $f_* : H_i(\bigvee_{i,\alpha} S_\alpha^{i+1}) \to H_i(\Sigma Z)$ (and thus $f_* : \pi_i(\bigvee_{i,\alpha} S_\alpha^{i+1}) \to \pi_i(\Sigma Z)$) is an F-isomorphism for all i.*

Proof of 4. (i)

$$P_k = \overbrace{X \wedge X \wedge \cdots \wedge X}^{r}$$

for some r, consequently $H_{i+1}(\Sigma P_k\,; Q)$ is non-trivial only if $i \in N$. By 5

[2] It can be shown that $\prod_{\omega(k)\geq 2} \Omega P_k$ is homotopy-equivalent to $(\Omega\Sigma X)\ b\ (\Omega\Sigma X)$, the "flat" product of [5].

[3] The elements of N are dual to the cup numbers of [3].

there is a map

$$\bigvee_{\alpha,i} S_\alpha^{i_n+1} \xrightarrow{f} \Sigma P_k$$

such that f_* is an F-isomorphism; by the above remark the i_n which appear are all elements of N. It follows that $\pi_j(\Omega\Sigma P_k)$ has an infinite cyclic component if and only if $\pi_k(\Omega(\bigvee_{\alpha,i_n} S^{i_n+1}))$ does. From Hilton's result [6] we have that

$$\pi_j(\Omega(\bigvee_{\alpha,i_n} S_\alpha^{i_n+1})) \cong \oplus_m \pi_j(\Omega S^{l_m+1});$$

moreover, the definition of the basic products and the fact that $i_n \in N$ for all n implies that $l_m \in N$ for all m. Since $\pi_j(\Omega S^{l_m+1})$ has an infinite cyclic component only if $k = l_m$ or $k = 2\cdot l_m$, part (i) is proved. To prove part (ii) let $n = \sum_{j=1}^K c_j i_j$ and let $m = \sum_{j=1}^K c_j$. There is a basic product P_k such that

$$P_k = \overbrace{X \wedge X \wedge \cdots \wedge X}^{m}$$

$(\omega(k) \geq 2$ because $m \geq 2)$. Since the element $u_{i_1}^{c_1} \otimes u_{i_2}^{c_2} \otimes \cdots \otimes u_{i_K}^{c_K}$ in $H_n(P_k; Q)$ is non-trivial, Lemma 5 shows that $\pi_n(\Omega\Sigma P_k)$ has an infinite cyclic direct summand. This completes the proof.

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