INVARIANT IDEALS OF POSITIVE OPERATORS IN $C(X)$. II

BY

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The present paper constitutes the second part of a study of ideals in $C(X)$ invariant under a given positive linear operator. While it will be necessary to have part I[2] on hand for an understanding of certain details, we shall briefly recall some basic definitions, notations, and results of part I.

Throughout the paper, T denotes a positive linear operator on the complex Banach algebra $C(X)$, where X is compact Hausdorff. A T-ideal (Def. 1) is a closed proper ideal $J \subset C(X)$ such that $T(J) \subset J$. Every T-ideal J gives rise to a positive operator T_J on $C(X)/J$. In general, $C(X)/J$ is identified with $C(S_J)$, where S_J (called the support of J) is the unique closed subset of X such that $J = \{f : f(S_J) = (0)\}$. T is called irreducible (Def. 2) if (0) is the only T-ideal; a T-ideal J is maximal if and only if T_J is irreducible. T is called ergodic (Def. 3) if for each $f \in C(X)$, the convex closure of the orbit $\{f, Tf, T^2f, \cdots\}$ contains a fixed vector of T; if the semigroup $\{T^n\}$ is bounded, ergodicity of T is equivalent with the strong convergence (for $n \to \infty$) of the averages

$$M_n f = n^{-1}(f + Tf + \cdots + T^{n-1}f) \to Pf,$$

P being a positive projection onto the fixed space of T. If $M_n \to P$ norm converges, T is called uniformly ergodic (Def. 3a). If $Te = e$ where $e(s) = 1$ for all $s \in X$, T is called a Markov operator.

THEOREM 1 (§2). *For each maximal T-ideal J, there exists an eigenvector (measure) $\phi \geq 0$ of the adjoint operator T' such that*

$$J = \{f : \phi(|f|) = 0\}.$$

The corresponding eigenvalue $\rho \geq 0$ is zero iff ϕ is supported by a single point $s \in X$ for which $Te(s) = 0$.

THEOREM 2 (§3). *If T is an ergodic Markov operator and Φ denotes the (weak* compact) set of all positive, normalized T-invariant measures on X, the mapping*

$$\phi \to I_\phi = \{f : \phi(|f|) = 0\}$$

is a bijection of the set Λ of extreme points of Φ onto the set of all maximal T-ideals. Moreover, every T-ideal I_ϕ ($\phi \in \Phi$) is the intersection of all maximal T-ideals containing it.

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§4 (present paper) introduces the radical of a positive operator T, that is, the intersection of all maximal T-ideals. For ergodic Markov operators, the radical is characterized by Theorem 4. §5 is concerned with the study of the peripheral spectrum (Def. 5). In addition to several new results obtained for the point spectrum, it is shown that the peripheral spectrum of every irreducible Markov operator is a subgroup of the circle group (Thm. 7). Finally, §6 analyzes the ideal structure of weakly compact positive operators without radical (Thm. 8).

4. The radical

In Theorem 2 we could assert that each T-ideal of the form I_ϕ is the intersection of all maximal T-ideals containing it, but for arbitrary T-ideals the assertion is, in general, false (Example 1 below). One reason for this situation is to be found in the fact that, in general, the intersection of all maximal T-ideals is not the zero ideal. This leads us to the following definition.

DEFINITION 4. For any positive operator T on $C(X)$, the T-**radical** R is defined to be the intersection of all maximal T-ideals. In case $R = (0)$, T is said to be **radical-free.**

Examples. 1. The Volterra operator T, where

$$Tf(s) = \int_0^s f(t)\, dt$$

for $s \in [0, 1] = X$, (§2, Example 1), has $R = \{f : f(0) = 0\}$. However, the intersection of all T-ideals is (0).

2. The operator $Tf(z) = f(\alpha z)$, where z runs through the circle group Γ, is radical-free, whatever $\alpha \in \Gamma$. Similarly, every permutation matrix P (Example 3, §2) defines a radical-free operator.

3. Obviously, every irreducible operator (Def. 2) is radical-free.

It is clear from this definition that the T-radical R is the ideal whose support is the closure of $\bigcup S_I$ where I runs through all maximal T-ideals. Thus a radical-free T is completely determined by the family (T_I) of its irreducible restrictions, so that irreducible operators appear as building blocks for radical-free operators. This is particularly evident when X is finite; in this case, each radical-free T is a (finite) direct sum of irreducible components (§6).

PROPOSITION 8. *If R is the radical of T, the operator T_R induced by T on $C(X)/R$ is radical-free.*

Proof. In fact, if S is the support of R and q the canonical homomorphism

$$C(X) \to C(X)/R \cong C(S),$$

then it follows from Proposition 2 that $q^{-1}(R_S) = R$ where R_S denotes the radical of T_R. Thus $R_S = q(R) = (0)$, since q is surjective, and hence T_R is radical-free.

This permits us to answer in part the question raised at the beginning of of this section.

PROPOSITION 9. *Suppose that the support of the T-radical R is the union of the supports of all maximal T-ideals.*[3] *If J is any T-ideal in $C(X)$, then $J + R$ is the intersection of all maximal T-ideals containing J.*

Proof. If I denotes this intersection, it is clear that $I \supset J + R$. Let us show that $I = J + R$. If again q denotes the canonical map

$$C(X) \to C(X)/R \cong C(S),$$

then $q(J + R)$ is a T_R-ideal (note that $J + R$ is closed in $C(X)$) with support $H \subset S$, say. Now each maximal T_R-ideal has its support either contained in H or disjoint from H so that, since T_R is radical-free (Proposition 8), $q(J + R)$ is the intersection of all maximal T_R-ideals containing it. Thus, by Proposition 2, $q(J + R) = \bigcap q(K)$, where K runs through all maximal T-ideals containing $J + R$. Application of q^{-1} shows that

$$\bigcap K = I = q^{-1}q(J + R) = J + R.$$

The following result, a direct consequence of Theorem 1, shows radical-free positive operators to have a property familiar from Hermitian operators in Hilbert space. (A topological nilpotent in a Banach algebra is an element with spectral radius 0.)

PROPOSITION 10. *If $T \geq 0$ is a radical-free topological nilpotent, then $T = 0$.*

Proof. Since $r(T) = 0$, by Theorem 1 each maximal T-ideal is actually a maximal ideal in $C(X)$, determined by a point $s \in X$ such that $Te(s) = 0$. But $R = (0)$ means that the set of these s is dense in X. Thus $Te = 0$ and, since T is positive, $T = 0$.

We now characterize the radical of Markov and of ergodic Markov operators. First, a sufficient condition for $f \in R$.

PROPOSITION 11. *If T is a Markov operator and if* $\lim_n T^n |f| (s) = 0$ *for each $s \in X$, then $f \in R$.*

Proof. Since $Te = e$, by Theorem 1 each maximal ideal is of the form I_ϕ, where $\phi \geq 0$ and $T'\phi = \phi$. Since $T^n |f|$ is bounded, the pointwise convergence to 0 implies that $\phi(|f|) = \lim_n \phi(T^n |f|) = 0$ for each ϕ determining a maximal T-ideal; hence $f \in R$.

THEOREM 4. *Let T be a Markov operator with radical R. If T is ergodic, then $f \in R$ if and only if* $\lim_n M_n |f| = 0$.[4] *If T is uniformly ergodic, then $f \in R$ if and only if* $\lim_n T^n |f| = 0$.

We note the following corollary.

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[3] This is true, e.g., for weakly compact T.

[4] $M_n = n^{-1}(I + T + \cdots + T^{n-1})$.

COROLLARY. *Let T be an ergodic Markov operator, and denote by P the projection which is the strong limit of the averages M_n. Then the radical of T is identical with the radical of P, and the largest ideal in $C(X)$ contained in the null space of P.*

Proof of Theorem 4. Suppose T is ergodic. The adjoint P' of P maps $M_R(X)$ onto the space of real T-invariant measures on X; thus if $\psi \geq 0$ is any normalized measure we have, by Theorem 2, $P'\psi$ contained in the weak convex closure of the set Λ each of whose elements determines a maximal T-ideal. Now if $f \in R$ then $|f| \in R$ and hence $P'\psi(|f|) = \psi(P|f|) = 0$, which shows that $P|f| = 0$. Conversely, if $P|f| = 0$ then $\lambda(|f|) = 0$ for each $\lambda \in \Lambda$, since $P'\lambda = \lambda$; it follows that $|f| \in R$, hence $f \in R$.

Suppose now that T is uniformly ergodic. We shall show that in these circumstances, the restriction of T to R has spectral radius $r' < 1$. Assuming this to be true, we conclude that for each $f \in R$ the C. Neumann's series $\sum_0^\infty \lambda^{-(n+1)} T^n |f|$ converges for all λ, $r' < \lambda < 1$. This implies, clearly, that $\lim_n T^n |f| = 0$. On the other hand, if $\lim_n T^n |f| = 0$ then $f \in R$ by Proposition 11. It remains to prove this lemma.

LEMMA. *If T is a uniformly ergodic Markov operator, then the restriction of T to the radical R has spectral radius $r' < 1$.*

In fact, suppose that $r' = 1$. There exists a normalized sequence $(f_n) \subset R$, $f_n \geq 0$, for which $\lim_n \| f_n - Tf_n \| = 0$. This sequence defines an element $\hat{f} \geq 0$, $\| \hat{f} \| = 1$, of the space $C(Y)$ constructed in the imbedding theorem (§1), and from the preceding it follows that $Pf_n = 0$ for all n. On the other hand, the uniform ergodicity of T is equivalent with the uniform ergodicity of T_1 (which is clearly a Markov operator on $C(Y)$), and it follows now that $P_1 \hat{f} = 0$ where P_1 is the uniform limit of the averages of T_1. Hence from the first part of Theorem 4 (applied to the operator T_1) we conclude that $\hat{f} \in R_1$ (the radical of T_1) while, evidently, $\hat{f} = T_1 \hat{f}$. Thus we have $\hat{f} = P_1 \hat{f} = 0$, which is contradictory, Q.E.D.

Theorem 4 permits us to establish a simple condition for an ergodic Markov operator on $C(X)$ to be radical-free.

PROPOSITION 12. *Let T be an ergodic Markov operator. For T to be radical-free, it is sufficient (and if X is metrizable, necessary) that there exist a strictly positive T-invariant measure on X.*

Proof. If again $Pf = \lim M_n f (f \in C(X))$, we have $P'\phi = \phi$ for each T-invariant measure ϕ. Thus if ϕ is strictly positive, P is strictly positive, whence $R = (0)$ by the corollary of Theorem 4. Conversely, if X is metrizable, then X (being compact) has a countable base of open sets, and if $R = (0)$ then the union of the supports of all maximal T-ideals is dense in X. So by Theorem 1 there exists a sequence (ϕ_n) of normalized positive T-invariant measures whose supports have a union dense in X. Obviously, $\phi = \sum_n 2^{-n} \phi_n$ is a T-invariant measure which is strictly positive.

For ergodic Markov operators, Theorem 4 reduces the characterization of the radical of T to the corresponding problem for a positive projection P satisfying $Pe = e$. Let us indicate a result on somewhat more general projections which is not difficult to prove. (For such projections, see also [13].)

PROPOSITION 13. *For a positive projection P of norm 1 on $C(X)$, the following are equivalent assertions:*

(a) *P is radical-free;*

(b) *X is the union of two disjoint, open and closed subsets X_0, X_1 such that $Pe_0 = 0$ and $Pe_1 = e_1$ for the respective characteristic functions, and such that P is strictly positive on $C(X_1)$.*

5. The peripheral spectrum

Much of the motivation and stimulation for the present study has its origin in known spectral properties of positive linear operators, cf. [6], [8], [9]. The two first propositions of this section are contained in [9] (see also [10, Appendix]).

PROPOSITION 14. *If T is irreducible and $r(T) = 1$,[5] the space F of fixed vectors of T has dimension at most 1.*

Proof. It is sufficient to consider $C_R(X)$. We note first that F is a vector sublattice of $C_R(X)$. For, by Theorem 1, there exists a $\phi = T'\phi$ which is strictly positive; if $f \in F$ then $|f| \leq T|f|$ but $\phi(T|f| - |f|) = 0$ implies $|f| \in F$. Now if $f \in F$ then $f^+ \in F$ and $f^- \in F$, and since the closed ideals generated by f^+ and f^- are T-ideals and thus either (0) or $C_R(X)$, it follows that at least one of f^+, f^- is zero. Consequently, F is a totally ordered Archimedean vector sublattice of $C_R(X)$, and hence of dimension at most 1.

COROLLARY. *If $f \in C_R(X)$ is fixed under T, then f is either strictly positive, or strictly negative, or identically zero.*

The **peripheral point spectrum** of T is the set of all eigenvalues (if any) contained in $|\lambda| = r(T)$.

PROPOSITION 15. *Let T be an irreducible operator satisfying $r(T) = 1$,[5] and suppose that the peripheral point spectrum Γ_0 of T is $\neq \emptyset$. Then Γ_0 is a subgroup of the circle group, and T is similar to an irreducible Markov operator.*

Proof. Let $\alpha f = Tf, f \neq 0, \alpha \in \Gamma_0$. Then $|f| \leq T|f|$ hence $|f| = T|f|$ (proof of Proposition 14), and $|f|$ is strictly positive. Clearly, if g is an eigenfunction of T for another unimodular eigenvalue, then from Proposition 14 we conclude that $|f| = |g|$ whenever $\|f\| = \|g\| = 1$. The operator U on $C(X)$ defined by $Uh = |f|^{-1}T(|f|h)$ is an irreducible Markov operator similar to T. Of course, U and T have identical peripheral point spectra, and the group property of Γ_0 follows now from the lemma on unimodular eigenfunctions (§1).

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[5] The assumption $r(T) = 1$ is essentially a normalization (Prop. 10). Moreover, if $T = 0$ is irreducible then $C(X)$ is one-dimensional.

Remark. Clearly, the unimodular eigenfunctions of U form a group (under ordinary multiplication); it can be shown that each unimodular eigenvalue is simple [10, p. 272 Theorem 3.3]. Moreover, for each $\alpha \in \Gamma_0$ a corresponding eigenfunction can be selected so that $\alpha \rightarrow g_\alpha$ is an isomorphism of Γ_0 into the group of unimodular functions on X.

Definition 5 ([9], [10]). Let T be positive on $C(X)$ with $r(T) = 1$. The peripheral spectrum of T is said to be **cyclic** if it is a union of cyclic subgroups of the circle group. The peripheral point spectrum is called **fully cyclic** if, whenever $\alpha(|\alpha| = 1)$ is an eigenvalue with an eigenfunction $f = |f| g$, then $\alpha^n |f| g^n = T(|f| g^n)$ for all integers n.

The notion of cyclic peripheral spectrum is obviously meaningful for operators on arbitrary (complex) Banach spaces. We point out that H. Lotz [6] has shown that (the complexification of) each positive operator on a Banach lattice, such that $(\lambda - 1)(\lambda - T)^{-1}$ is uniformly bounded for $\lambda > 1 = r(T)$, has cyclic peripheral spectrum.

Theorem 5. *If T is radical-free and $r(T) = 1$, the peripheral point spectrum of T is fully cyclic. If, in addition, T is a Markov operator, then the fixed space of T is a closed subalgebra (hence a sublattice) of $C(X)$.*

Proof. Let $\alpha f = Tf$, where $|\alpha| = 1$ and $f = |f| g$ (g unimodular). If I is a maximal T-ideal and subscripts 1 denote the restrictions of $f, g, \cdots$ to the support S_I of I, then $\alpha f_1 = T_I f_1$. Since T_I is irreducible (corollary of Proposition 2), then from Proposition 14 and the lemma on unimodular eigenfunctions (§1) we conclude that $\alpha^n |f_1| g_1^n = T_I(|f_1| g_1^n)$ for all $n \in Z$. The first assertion is now an immediate consequence of the fact that T is radical-free (i.e., the union of the supports of maximal T-ideals is dense in X).

Suppose now, in addition, that $Te = e$. If $f \in C(X)$ and $Tf = f$, then $T_I f_1 = f_1$ and it follows from Proposition 14 that f_1 is constant. Thus if f, h are fixed vectors of T, we have $f_1 h_1 = T_I(f_1 h_1)$ for each maximal ideal I and hence $fh = T(fh)$, Q.E.D.

The reader will notice from the proof that (for the second assertion) the assumption $Te = e$ can be replaced by the weaker condition that Te assumes only the values 0 and 1. From this observation and Proposition 13, we obtain this corollary.

Corollary. *If P is a positive, radical-free projection of norm 1, then the range of P is a closed subalgebra of $C(X)$.*

Examples. 1. A typical example for Theorem 5 is furnished by the operator T of §2, Example 2. Here the eigenfunctions for α^m are the functions $f_m(z) = z^m$.

2. Even if X is finite and the resolvent of T has at $\lambda = r(T) = 1$ a pole of order 1, the peripheral point spectrum of T is not necessarily fully cyclic.

An example is furnished by the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

The radical R of T is the ideal $\{x : x_1 = x_2 = x_3 = 0\}$.

3. If $X = \{1, 2\}$, consider the positive projections P given by

$$\begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$$

where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. If $\alpha = 0$ and $\beta = 1$, P is not radical-free. Hence the radical-freeness of P is not a necessary condition for the range of P to be a subalgebra. (Cf. [4], [13].)

THEOREM 6. *If $T \geq 0$ is a uniformly ergodic operator satisfying $\| T \| = r(T) = 1$, then each element of the peripheral spectrum is an eigenvalue of the second adjoint of T.*

Proof. If $\alpha \in \sigma(T)$, $|\alpha| = 1$, there exists a normalized sequence (f_n) such that $\lim_n \| \alpha f_n - T f_n \| = 0$. Unless (f_n) converges weakly to 0, the sequence has a weak adherent point $h \neq 0$ in the bidual $C(X)''$, for which $\alpha h = T'' h$. We shall show that the weak convergence of (f_n) to 0 is barred by our assumptions; for this it will be sufficient, by a well known result of measure theory, to prove that $\limsup \phi(|f_n|) > 0$ for a suitable measure $\phi \geq 0$ on X.

The proof uses the imbedding theorem (§1). The uniform ergodicity of T is equivalent with the uniform ergodicity of T_1, and if P and P_1 denote (as before) the respective limits of averages, then P_1 is indeed the operator induced on $C(Y)$ by P. The above sequence (f_n) defines an $\hat{f} \in C(Y)$ for which $\alpha \hat{f} = T_1 \hat{f}$, hence $|\hat{f}| \leq T_1 |\hat{f}|$. By [9, p. 305 Satz 1] there exists a positive measure $\psi_1 = T_1' \psi_1$ such that $\psi_1(|\hat{f}|) > 0$. From Corollary 2 of Theorem 2 it follows now that there exists a maximal T_1-ideal $I_1 \subset C(Y)$ not containing $\hat{f}$. Also, by Proposition 5 the ideal $I = C(X) \cap I_1$ is a maximal T-ideal.

Now denote by J_1 the T_1-ideal in $C(Y)$ generated by I; then still $I = C(X) \cap J_1$, since I is maximal. If, for each $f \in C(X)$, $\tilde{f}$ denotes its restriction to the support S_I of I, the elements of $C(Y)/J_1$ can be viewed as sequences $(\tilde{f}_n)$ modulo null sequences, and $C(X)/I$ can be canonically imbedded in $C(Y)/J_1$. Now from $|\hat{f}| \leq T_1 |\hat{f}|$ it follows that $|\hat{f}| \leq P_1 |\hat{f}|$. Taking canonical images in $C(Y)/J_1$, we conclude from $\hat{f} \notin J_1$ that

$$0 < \| P_1 |\hat{f}| \|_{J_1} = \limsup \| P_I |\tilde{f}_n| \|.$$

But T_I is irreducible in $C(X)/I$ which implies, by Proposition 14, that P_I

is of rank 1, hence of the form $\phi \otimes k$, where $k \in C(X)/I$ and ϕ is a (T-invariant) positive measure. Therefore, we have $\| P_I | \tilde{f}_n | \| = \phi(|f_n|) \| k \|$ and, consequently, $\limsup \phi(|f_n|) > 0$, Q.E.D.

Some remarks are in order on the implications of uniform ergodicity.[6] Via the imbedding theorem (§1) it is not very difficult to prove: If T is uniformly ergodic, then so is T_1 (and conversely); thus $r(T) \leq 1$. If $r(T) = 1$ (and, as always, $T \geq 0$), 1 is a simple pole of the resolvent with residue $P = \lim_n M_n$ (cf. introduction). It follows (see remark after Def. 5) that the peripheral spectrum of T is finite; however, its elements need not be eigenvalues and, in fact, not isolated elements of $\sigma(T)$; thus the assertion of Theorem 6 is not true in $C(X)$. However, if T is uniformly ergodic and irreducible with $r(T) = 1$, then by a deep result of Niiro and Sawashima [14], the peripheral spectrum of T consists of all n-th roots of unity for some $n \geq 1$, which are all simple poles of the resolvent. In particular, the peripheral spectrum of such T is a group.

We are now going to show that for any irreducible Markov operator T on $C(X)$, the peripheral spectrum is a group. It is clear (§2, Example 2b) that, in general, the peripheral spectrum is not pure point spectrum.

THEOREM 7. *If T is an irreducible Markov operator,*[7] *then the peripheral spectrum of T is a subgroup of the circle group.*

Proof. The proof uses the imbedding theorem (§1). Indeed, if α, β are unimodular numbers in $\sigma(T)$, then α, β are eigenvalues of T_1; we shall show that $\alpha\beta^* \in \sigma(T_1)$, assuming the existence of normalized eigenfunctions f_0, $g_0 : \alpha f_0 = T_1 f_0$, $\beta g_0 = T_1 g_0$, and a point $y \in Y$ such that $|f_0(y)| = |g_0(y)| = 1$. Taking this for granted,[8] we observe that

$$S_1 = \{y \in Y : |f_0(y)| = 1\} \quad \text{and} \quad S_2 = \{y \in Y : |g_0(y)| = 1\}$$

are T_1-invariant subsets of Y (that is, support T_1-invariant ideals); in fact, since $T_1 e_0 = e_0$ and $\| f_0 \| = 1$, hence $|f_0| \leq e_0$, an elementary computation shows that

$$T_1(e_0 - |f_0|) \leq e_0 - |f_0|,$$

which shows the closed ideal generated by $e_0 - |f_0|$ (≥ 0) to be T_1-invariant. Thus $S_1 \cap S_2$ supports a T_1-ideal, and the restrictions of f_0 and g_0 to $S = S_1 \cap S_2$ are unimodular eigenfunctions of the Markov operator $\hat{T}$ induced by T_1 on

[6] For the related notions of (weak and strong) almost periodic operators T, see Rosenblatt [11] and Glicksberg [12], and Wolff [16].

[7] Note that by Proposition 15, the following assertions are equivalent for irreducible T with $r(T) = 1$: (a) $\Gamma_0 \neq \emptyset$, (b) T has a fixed vector $\neq 0$, (c) T is similar to an irreducible Markov operator.

[8] Since T_1 is a Markov operator on $C(Y)$, the assertion of the lemma would be satisfied for all $y \in Y$ if T_1 were irreducible; the difficulty of the proof is thus due to the fact that irreducibility is lost under the transition $T \to T_1$.

$C(S)$; hence by the lemma on unimodular eigenfunctions (§1), $\alpha\beta^* \in \sigma(\hat{T})$ and, therefore, $\alpha\beta^* \in \sigma(T_1) = \sigma(T)$.

The proof rests on the following lemma.

LEMMA. *If T is an irreducible Markov operator and α and β are unimodular eigenvalues of T_1 (§1, imbedding theorem), there exist respective eigenfunctions f_0, g_0, $\| f_0 \| = \| g_0 \| = 1$ of T_1, such that $| f_0(y) | = | g_0(y) | = 1$ for some $y \in Y$.*

Proof. (i) For $f \in C(X)$ and $0 < \eta < 1$, let

$$M(f; \eta) = \{t \in X : | f(t) | \geq 1 - \eta\}.$$

Further let $\mu_s = T'\varepsilon_s$ (ε_s = Dirac measure at s), so that $Tf(s) = \int f(t)\, d\mu_s(t)$, and denote by $\Sigma_s \subset X$ the support of μ_s. We claim:

If $\varepsilon > 0$, $\| f \| = 1$ and $| f | \leq T | f | + (\varepsilon/2)e$, then $s \in M(f; \varepsilon/2)$ implies

$$\mu_s(\Sigma_s \cap M(f; k\varepsilon)) \geq 1 - 1/k$$

for all integers $k \geq 1$.

Proof. Let

$$A = \{t : | f(t) | \geq 1 - k\varepsilon\}, \qquad B = X - A,$$

and

$$x = \mu_s(\Sigma_s \cap M(f; k\varepsilon)).$$

Suppose $s \in M(f; \varepsilon/2)$. By hypothesis, $1 - \varepsilon \leq \int | f | \, d\mu_s$. On the other hand, since $\mu_s(X) = 1$,

$$\int | f | \, d\mu_s = \int_A | f | \, d\mu_s + \int_B | f | \, d\mu_s \leq x + (1 - k\varepsilon)(1 - x).$$

Hence it follows that $x \geq 1 - k^{-1}$.

(ii) We consider arbitrary sequences $F = (f_n)$ on $C(X)$ satisfying (a) $\| f_n \| = 1$ for all n, (b) $\| \alpha f_n - T f_n \| = \delta_n(F) \rightarrow 0$. (Such sequences exist, since $\alpha \in \sigma(T)$.) We say the sequence $(s_n) \subset X$ has property P with respect to F (briefly, has $P(F)$), if there exists a mapping (subsequence) $n \rightarrow k(n)$ such that $\lim_n | f_{k(n)}(s_n) | = 1$.

DEFINITION. Let $M = M(F)$ be the set of all $s \in X$ having the property: For each neighborhood $U(s)$ of s, there exists a sequence (s_n) having $P(F)$ and such that $s_n \in U(s)$ is true for infinitely many integers n. (Note that $P(F)$ is inherited by arbitrary infinite subsequences.)

We claim:

If T is irreducible and Markov, then $M(F) = X$ for each sequence F satisfying (a) and (b) above.

Proof. Let F be fixed. It is clear that M is nonempty and closed. We show that M is T-invariant, that is, $s \in M$ implies $\Sigma_s \subset M$ (notation of (i), above); then $M = X$ since T is irreducible.

If not, there exists $s_0 \in M$ and $t_0 \in \Sigma_{s_0}$ such that $t_0 \notin M$. $t_0 \notin M$ means there exists a closed neighborhood $V(t_0)$ such that for each sequence (s_n) having $P(F)$, the relation $s_n \in V(t_0)$ takes place only finitely often. By Urysohn's theorem, we can determine $h \in C(X)$, $h \geq 0$, having its support in $V(t_0)$ and yielding $Th(s_0) = 1$. Now let

$$U(s_0) = \{s \in X : Th(s) > 2^{-1}\}.$$

By the definition above, there exists a sequence (s_n) having $P(F)$ and such that $s_n \in U(s_0)$ for all n; let us fix such a sequence.

With $F = (f_n)$ and (s_n) fixed, we can now determine a null sequence of positive numbers ε_n and a subsequence $n \to k(n)$ of the positive integers, such that for all n

$$| f_{k(n)}(s_n) | \geq 1 - 2^{-1}\varepsilon_n , \qquad \delta_{k(n)}(F) \leq 2^{-1}\varepsilon_n .$$

$\delta_{k(n)} = \| \alpha f_{k(n)} - Tf_{k(n)} \| \leq 2^{-1}\varepsilon_n$ implies

$$| f_{k(n)} | \leq T \, | f_{k(n)} | + 2^{-1}\varepsilon_n \, e,$$

hence from (i) we conclude

$$(*) \qquad \mu_{s_n}(\Sigma_{s_n} \cap M(f_{k(n)} \, ; l\varepsilon_n)) \geq 1 - l^{-1} \qquad (l = 1, 2, \cdots).$$

Since $s_n \in U(s_0)$, hence $Th(s_n) = \int h \, d\mu_{s_n} > 2^{-1}$, it follows that $\mu_{s_n}(V(t_0)) \geq \gamma$ for all n and some $\gamma > 0$. On the other hand, $\mu_{s_n}(X) = 1$ since T is Markov; thus in view of $(*)$, $V(t_0)$ must intersect $M(f_{k(n)} \, ; l\varepsilon_n)$ for every n provided that $l^{-1} < \gamma$. Thus, fixing some $l_0 > \gamma^{-1}$, we obtain a sequence

$$t_n \in V(t_0) \cap M(f_{k(n)} \, ; l_0 \, \varepsilon_n)$$

that evidently has $P(F)$, contradicting the choice of $V(t_0)$.

(iii) If W_0 denotes the topological direct sum of a countably infinite number of copies X_n of X, $W_0 = \oplus_1^\infty X_n$, and W denotes the Stone-Čech compactification of W_0, then $Y = W \backslash W_0$ (see the imbedding theorem, §1). Let (g_n) be a normalized sequence for which $\| \beta g_n - Tg_n \| \to 0$; $(g_n) = G$ defines a bounded continuous function on W_0 which has a unique continuous extension to W, $\bar{G}$ say. The restriction g_0 of $\bar{G}$ to $Y = W \backslash W_0$ satisfies $\beta g_0 = T_1 \, g_0$ and $\| g_0 \| = 1$. Let $S_2 = \{y \in Y : | g_0(y) | = 1\}$. The final step in the proof is now to show the existence of a normalized function f_0 on Y, $\alpha f_0 = T_1 f_0$, such that $| f_0(y) | = 1$ for some $y \in S_2$.

Let $U_n = \{x \in W : | \bar{G}(x) | > 1 - n^{-1}\}$ $(n \in N)$; then (U_n) is a decreasing sequence of open neighborhoods of S_2. There exists a sequence $n \to \bar{k}(n)$ such that, if $Y_m = X_{\bar{k}(m)}$ and $m_0(n)$ is suitably chosen, we have $U_n \cap Y_m \neq \emptyset$ for all $m \geq m_0(n)$. (Since W_0 is dense in W, each U_n must intersect infinitely many X_l, $X_{k(m,n)}$ say $(m = 1, 2, \cdots)$. Because of $U_{n+1} \subset U_n$, $(k(m, n + p))$ $(m = 1, 2, \cdots)$ is a subsequence of $k(m, n)$ for each $p \in N$. Now let $\bar{k}(m) = k(m, m)$.)

Inductively define $j(m)$ to be the smallest integer $l \geq 1$ for which $l > j(m-1)$ whenever $m \geq 2$, and for which $U_m \cap Y_{l+p} \neq \emptyset$ for all $p = 0, 1, 2, \cdots$. Setting $Z_m = Y_{j(m)}$, $U_m \cap Z_m$ is an open subset $\neq \emptyset$ of Z_m. Hence by (ii) above, for each m there exists $f_m \in C(Z_m)$ of norm 1 satisfying

$$\| \alpha f_m - Tf_m \| < m^{-1} \quad \text{and} \quad | f_m(s_m) | > 1 - m^{-1},$$

for some suitably chosen $s_m \in U_m \cap Z_m$.

The sequence $F = (f_m)$ defines a unique continuous function $\bar{F}$ on W. We claim that its restriction f_0 to $Y = W \backslash W_0$ satisfies the requirement of the lemma. Indeed, the sequence $(s_m) \subset W$ has a cluster point $y \in Y$, and $| f_0(y) | = 1$. On the other hand, $y \in \bar{U}_n$ for each n, hence $y \in \bigcap_1^\infty \bar{U}_n$. It follows that $y \in S_2$, Q.E.D.

6. Weakly compact operators

This final section leaves the domain of Markov operators and attempts an analysis of the ideal structure of radical-free positive operators T on $C(X)$. To make the task less complex we impose, generally, the condition that T be weakly compact. It is known that this implies the compactness of T^2; what is actually used in the following is the compactness of some power of T.

DEFINITION 6. We say that a number ρ (≥ 0) is a **distinguished eigenvalue** of the adjoint T' of T if there exists a positive eigenvector of T' for ρ.

PROPOSITION 16. *If T (≥ 0) is weakly compact and Te strictly positive, the number of maximal T-ideals is finite.*

Proof. Let $Te(s) \geq \delta > 0$ for all $s \in X$; this implies $\rho \geq \delta$ for each distinguished eigenvalue of T'. It follows now from Theorem 1 that each maximal T-ideal is of the form I_ϕ where $\rho\phi = T'\phi$ and $\rho \geq \delta$. By Proposition 3 any number of these measures ϕ are mutually orthogonal, hence linearly independent. Thus the weak compactness of T implies that their number is finite.

PROPOSITION 17. *Suppose T is weakly compact and $r(T) > 0$, and that the pole of $(\lambda - T)^{-1}$ at $\lambda = r(T)$ is simple. For $\rho = r(T)$ to be the only distinguished eigenvalue of T', it is necessary and sufficient that there exist a strictly positive function f satisfying $\rho f = Tf$.*

Proof. The condition is necessary. It is evident from the weak compactness of T and Theorem 1 that T possesses but a finite number of maximal T-ideals $I_{\phi_i} = I_i$ $(i = 1, \cdots, n)$, since all these belong to ρ. Now let $S = \bigcup S_i$ be the union of their respective supports; then $X = S \cup Y$ where Y (the complement of S) is an open set on which live the functions f in the T-radical R. If P is the projection

$$\lim_{\lambda \downarrow \rho} (\lambda - \rho)(\lambda - T)^{-1},$$

$P'\phi_i = \phi_i$ implies that there exists an eigenfunction $f_i \geqq 0$, $\rho f_i = Tf_i$, which does not vanish on S_i and hence is strictly positive on S_i. If $f = \sum_i f_i$, f is strictly positive on S and satisfies $\rho f = Tf$. It follows that $Y_0 = \{s : f(s) = 0\}$ is a compact set contained in Y and, in fact, unless $Y_0 = \emptyset$, the support of a T-ideal. Thus $Y_0 \neq \emptyset$ implies the existence of a maximal T-ideal not containing R, which is clearly impossible. Therefore, $Y_0 = \emptyset$ and f is strictly positive.

The condition is sufficient. Clearly, if f is strictly positive and $\rho f = Tf$, then $r_1 \phi = T'\phi$ $(\phi \geq 0)$ is possible only when $\rho = r_1$, since otherwise we would have $\phi(f) = 0$.

It is a natural question to ask whether to each distinguished eigenvalue ρ of T', there belongs some eigenvector $\phi \geq 0$ of T' such that I_ϕ is T-maximal. This is true for $\rho = 0$ by Theorem 1; for $\rho > 0$ it is false as the following example shows.

For $X = \{1, 2, 3\}$ the matrix

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

defines a positive operator with $\sigma(T) = \{0, 2\}$. The radical (and only maximal T-ideal) is the ideal $\{x : x_3 = 0\}$ which belongs to $\rho = 0$. However, we can assert the following

PROPOSITION 18. *If T is weakly compact and radical-free, then to each distinguished eigenvalue ρ of T' there exists an eigenvector $\phi \geq 0$ such that I_ϕ is maximal.*

In fact, this is a consequence of the following stronger

PROPOSITION 19. *If T is weakly compact, radical-free, and if $\rho\psi = T'\psi$ then each maximal T-ideal containing I_ψ belongs*[9] *to ρ (hence I_ψ is the intersection of these maximal T-ideals).*

Proof. Immediate from Theorem 1 when $\rho = 0$. For the case $\rho > 0$, we shall need the following lemma.

LEMMA. *Each distinguished eigenvalue $\rho > 0$ such that $\rho\psi = T'\psi$, is a simple pole of the resolvent of the operator T_ψ induced by T on $C(X)/I_\psi$.*

Since ψ is strictly positive on S_ψ, ρ is the spectral radius of T_ψ on $C(S_\psi) \cong C(X)/I_\psi$ (cf. proof of Proposition 4). Since T_ψ is weakly compact, ρ is a pole of the resolvent. Suppose this pole is of order $k \geq 2$, then

$$Q = \lim_{\lambda \downarrow \rho} (\lambda - \rho)^k (\lambda - T_\psi)^{-1}$$

is a positive operator that leaves each maximal T_ψ-ideal I invariant, hence induces an operator Q_I on $C(S_\psi)/I$. Evidently Q_I is the coefficient of

[9] A maximal T-ideal I belongs to ρ if $r(T_I) = \rho$.

$(\lambda - \rho)^{-k}$ of the Laurent expansion of the resolvent of $T_{\psi,I}$. On the other hand, $T_{\psi,I}$ is irreducible, and thus either the spectral radius of $T_{\psi,I}$ is $< \rho$ or else ρ is a pole of order 1 of the corresponding resolvent ([10], p. 270, Thm. 3.2). In either case, $Q_I = 0$. Since T_ψ is radical-free (cf. Prop. 9 and footnote 3), it follows that $Q = 0$ and the lemma is proved.

Thus, since ρ is a simple pole of the resolvent of T_ψ, the corresponding residuum is a positive projection P of $C(S_\psi)$ onto the eigenspace of T_ψ (for ρ). Now if I_ϕ is any maximal T-ideal containing I_ψ, where $\lambda\phi = T'\phi$, then $S_\phi \subset S_\psi$, and $\lambda < \rho$ implies that each function f satisfying $\rho f = T_\psi f$ vanishes on S_ϕ. In other words, if S denotes the support of the intersection of all T-maximal ideals containing I_ψ and which belong to ρ, then for each $f \in C(S_\psi)$, Pf vanishes on $S_\psi \backslash S$. On the other hand, we have $P'\psi = \psi$, and this now implies that ψ vanishes on the open set $S_\psi \backslash S$. Thus (since ψ is strictly positive on S_ψ) the latter set is empty, and Proposition 19 is proved.

From the lemma used in the proof of Proposition 19, and from Theorem 1 we obtain this corollary.

COROLLARY. *If T is weakly compact and radical-free, then each distinguished eigenvalue >0 of T' is a simple pole of the resolvent.*

We are now ready to prove the main result of this section.

THEOREM 8. *Suppose $T \geq 0$ is weakly compact and radical-free. Then for each distinguished eigenvalue of T', the corresponding real eigenspace is a vector sublattice of $M_R(X)$, and these sublattices are mutually orthogonal. Moreover, each positive normalized eigenvector of T' is a unique convex combination (barycenter) of those (normalized, positive) eigenvectors that belong to the same eigenvalue and determine maximal T-ideals.*

Proof. Denote by (ρ_n) the (finite or infinite) sequence of distinguished eigenvalues of T', letting $\rho_0 = 0$ if $Te(s) = 0$ for some $s \in X$ (Theorem 1). Denote by J_n the intersection of all maximal T-ideals originating from ρ_n (Proposition 18), and let S_n be the support of J_n. The assertion concerning ρ_0 is clear by Theorem 1 and the theorem of Choquet–Bishop–de Leeuw [1]. When $n > 0$, Proposition 19 implies that each measure $\psi \geq 0$ such that $\rho_n \psi = T'\psi$, has its support contained in S_n and that ρ_n is the only distinguished eigenvalue of T_n', where T_n denotes the operator on $C(X)/J_n$ induced by T. Thus, by the lemma contained in the proof of Proposition 19, and by Proposition 17 it follows that there exists a strictly positive eigenfunction of $T_n : \rho_n f_n = T_n f_n$. Obviously, $\rho_n^{-1} T_n$ is similar to a Markov operator, which is ergodic (Proposition 6). Thus, we can apply Proposition 7 and Theorem 2 and, finally, Proposition 3, Q.E.D.

COROLLARY. *Let T (≥ 0) be any weakly compact operator on $C(X)$. If $\psi \geq 0$ is a normalized eigenvector of T' vanishing on the radical R of T, then ψ is a (unique) convex combination of those normalized, positive eigenvectors of T' that belong to the same eigenvalue and that determine maximal T-ideals.*

For the proof, it suffices to apply Theorem 8 to T_R (Proposition 9).

We shall say that T is the (finite) direct sum of operators T_i if there exists a finite partition of X into open and closed subsets X_i ($i = 1, \cdots, n$), such that the direct sum $C(X) = \oplus C(X_i)$ reduces T (in the usual operator theoretic sense).

PROPOSITION 20. *Suppose T is weakly compact and radical-free. If either Te is strictly positive or X is finite, T is a (finite) direct sum of irreducible operators.*

Proof. In fact, from Proposition 16 (or by direct verification if X is finite) we conclude that there exists but a finite number of maximal T-ideals I_i, with respective supports X_i. Clearly, the X_i are disjoint since I_i are maximal T-ideals, and $X = \bigcup X_i$ since T is radical-free. The remainder is clear.

In particular, if $C(X)$ is C^n (i.e., if X contains exactly n points), each radical-free positive operator can be represented by a positive matrix

$$\begin{pmatrix} A_1 & & & & & 0 \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & A_r \end{pmatrix}$$

where each A_k is a square matrix, and such that possibly $A_1 = 0$ and all A_k ($k = 2, \cdots, r$) are indecomposable (in the sense of Frobenius), or else such that all A_k are indecomposable.

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