RETRACTING THREE-MANIFOLDS ONTO FINITE GRAPHS

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1. Introduction and Definitions

A piecewise-linear 3-manifold-with-boundary M is called a *cube-with-handles of genus n* if M is orientable and is a regular neighborhood of a finite connected graph of Euler characteristic $1 - n$. If M is a polyhedral cube-with-handles of genus n in S^3, then $S^3 - \operatorname{Int} M$ is called a *cube-with-holes of genus n*. We call a cube-with-holes N *retractable* if N can be retracted onto a wedge of n simple closed curves, where n is the genus of N. If such a wedge can be chosen in $\operatorname{Bd} N$, then N is *boundary-retractable*.

In [2], Lambert showed that for each $n \geq 2$, there exists N_n, a cube-with-holes of genus n, such that no mapping of N_n onto a cube-with-handles of genus n, H_n, can take $\operatorname{Bd} N_n$ homeomorphically onto $\operatorname{Bd} H_n$. By our Theorem 2, N_n is retractable. It is our purpose here to note that the existence of such a "boundary-preserving" mapping for a cube-with-holes is equivalent to its being boundary-retractable, and to give examples of cubes-with-holes of arbitrary genus $n \geq 3$ which are not even retractable. Our examples also show that the fundamental group G of a cube-with-holes of genus $n \geq 3$ can be residually nilpotent (Theorem 6) and in fact can have $G/G_m \approx F/F_m$ for each $m \geq 1$, where F is free of rank n (see Corollary 5.14.1, page 353 of [3]), and yet G can fail to be a free group. See the definitions below.

We also give (Theorem 5) a necessary condition for a cube-with-holes to be boundary-retractable, and indicate how to apply it to Lambert's example. We are grateful to Joseph Martin for several helpful conversations about this matter. We have not been able to prove the existence or nonexistence of a non-retractable cube-with-holes of genus two, but Theorem 4 gives a criterion in terms of mappings into the torus which may prove useful.

If G is any group, and $a, b \in G$, we denote the *commutator* $a^{-1}b^{-1}ab$ *of a and b* by $[a, b]$. For non-empty subsets A and B of G, $[A, B]$ denotes the subgroup of G generated by the set S of all commutators $[a, b]$, where $a \in A$, $b \in B$. That is, $[A, B]$ is the smallest subgroup of G containing S. We let G_m denote the m^{th} term in the *lower central series of G*. Specifically, $G_1 = G$, $G_2 = [G_1, G]$, and, in general, $G_{m+1} = [G_m, G]$ for each $m \geq 1$. Each G_m is a normal subgroup of G. We call G_2 the *commutator subgroup of G*, and G/G_2 is *G abelianized.* It is also convenient to introduce the notation G_ω for the normal subgroup $\bigcap_{m \geq 1} G_m$. Finally, if $G_{m+1} = 1$, we say that G is *nilpotent of class m*, and if $G_\omega = 1$ we say that G is *residually nilpotent.*

We will be using some known results from the literature. For example, a theorem of Magnus (see pages 311 and 312 of [3]) asserts that any free group

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is residually nilpotent. Another useful fact is that a finitely-generated, residually nilpotent group is *Hopfian* (see Theorem 5.5, page 296 of [3]). That is, any homomorphism of such a group onto itself is necessarily an isomorphism. In particular, if F is a free group of finite rank, then F/F_m is Hopfian, for any $m \leq \omega$.

2. Constructing Retractions

The next theorem can also be proved by the more sophisticated methods of Stallings in [4].

THEOREM 1. *Let F be a free group of rank n. Suppose G is a group and h is a homomorphism of G into F which induces an isomorphism of G/G_2 onto F/F_2. Then the kernel of h is precisely G_ω.*

Proof. Let K be the kernel of h. Since $h(G_\omega) \subset F_\omega = 1$ (as remarked above), we have $G_\omega \subset K$. For the reverse implication, we consider the homomorphisms h_m, $m \geq 2$, induced by h:

$$h_m : G/G_m \rightarrow F/F_m .$$

Let $f_1, \cdots, f_n$ be a free basis for F, let α_m be the natural homomorphism of G onto G/G_m, and let β_m be the natural homomorphism of F onto F/F_m. By hypothesis, h_2 is an isomorphism of G/G_2 onto F/F_2. Choose $g_1, \cdots, g_n$, elements of G, so that $h_2 \alpha_2(g_i) = \beta_2(f_i)$ for each i, $1 \leq i \leq n$. Then

$$\alpha_2(g_1), \cdots, \alpha_2(g_n)$$

generate G/G_2. Using Lemma 5.9, page 350 of [3], $\alpha_m(g_1), \cdots, \alpha_m(g_n)$ generate G/G_m for each m. It follows that for each m, there is a homomorphism, k_m, of F/F_m onto G/G_m determined by $\beta_m(f_i) \rightarrow \alpha_m(g_i)$, $1 \leq i \leq n$.

Repeating the argument of the previous paragraph, the elements

$$\beta_m h(g_1), \cdots, \beta_m h(g_n)$$

generate F/F_m for each $m \geq 2$. Since $h_m \alpha_m = \beta_m h$, it follows that h_m is an epimorphism. By the Hopfian property of F/F_m, $h_m k_m$ is an isomorphism onto for each m. Hence, h_m is actually an isomorphism of G/G_m onto F/F_m for each m. We obtain $K \subset G_\omega$, as desired.

COROLLARY. *Let G be a group such that G abelianized is free abelian of rank n. Let h be a homomorphism of G onto F, the free group of rank n. Then the kernel of h is precisely G_ω.*

Proof. If h_2 denotes the induced homorphism of G/G_2 into F/F_2, then h_2 is a homomorphism of the free abelian group of rank n *onto* the free abelian group of rank n. Hence, h_2 is an isomorphism onto, and the theorem applies.

THEOREM 2. *Let M be a compact 3-manifold, possibly with boundary, and let G denote its fundamental group. Let F be a free group of rank n. If there is a* *homomorphism* φ *of* G *into* F *which induces an isomorphism of* G/G_2 *onto* F/F_2, *then there is a retraction of* M *onto a wedge of* n *simple closed curves.*

Proof. First we show that under the hypothesis of the theorem, there is a homomorphism of G *onto* F. By Theorem 1, the kernel of the homomorphism φ is G_ω. Hence, G/G_ω is a finitely generated, free group. Since G/G_ω abelianized is isomorphic to G/G_2, which is free abelian of rank n, we conclude that G/G_ω is free of rank n. Thus, there is a homomorphism ψ of G onto F.

Let B be a wedge of n simple closed curves, $S_1, \cdots, S_n$, and assume $x_1, \cdots, x_n$ are points of B, x_i is a point of S_i, and each x_i is distinct from the wedge point which we denote x_0. The homomorphism ψ of G onto F may be realized by a simplicial map f of M into B so that $f_* = \psi$. We choose x_i so that under the triangulation of B which makes f simplicial, x_i is not a vertex for any $i > 0$. Each component of $N_i = f^{-1}(x_i)$ is a regularly embedded, two-sided, compact, polyhedral surface in M.

Let $\bar{x}_i$ denote the generator of $\pi_1(S_i, x_0)$, $1 \leq i \leq n$. Since $f_* = \psi$ is an epimorphism, for each $i > 0$ there is a simple closed curve l_i in M based at some point in $f^{-1}(x_0)$ so that $f_*[l_i] = \bar{x}_i$, where $[l_i]$ denotes the homotopy class of l_i. Furthermore, l_i is in general position with respect to $\bigcup_{i=1}^n N_i$ and except for base point, the curve l_i is disjoint from the curve l_j, $i \neq j$.

By considering the orientation of $\bar{x}_i$ and looking at the inverse image under f of a small neighborhood of x_i, we can choose an orientation for l_i and find the word of $\pi_1(B, x_0)$ corresponding to $f_*[l_i]$ by following l_i in a positive direction and recording its intersections with $\bigcup_{i=1}^n N_i$; the sign of each entry is determined by the two-sidedness of each N_i. Using the fact that $\pi_1(B, x_0)$ is a free group and that the word corresponding to $f_*[l_i]$ is equal to $\bar{x}_i$, it must be true that either l_i meets only one component of N_i or that there is a cancellation of the form $\bar{x}_j\bar{x}_j^{-1}$ (or $\bar{x}_j^{-1}\bar{x}_j$) in $\pi_1(B, x_0)$.

A cancellation of the form $\bar{x}_j\bar{x}_j^{-1}$ (or $\bar{x}_j^{-1}\bar{x}_j$) in $\pi_1(B, x_0)$ has as its geometric counterpart in M a subarc α of l_i which meets $\bigcup_{i=1}^n N_i$ only in its endpoints which are both in N_j (possibly not the same component of N_j). We shall use this geometric interpretation of the reduction of $f_*[l_i]$ in $\pi_1(B, x_0)$ to $\bar{x}_i$ to obtain a collection of surfaces $L_1, \cdots, L_n$ in M so that $l_i \cap L_i$ is precisely one point and l_i pierces L_i at this point. Furthermore, $l_i \cap L_j$ is void for $i \neq j$. To be precise: if α is a subarc of l_i with its endpoints in N_j and corresponding to a cancellation $\bar{x}_j\bar{x}_j^{-1}$ (or $\bar{x}_j^{-1}\bar{x}_j$), we let Q denote a small, regular neighborhood of α which is disjoint from N_h, $h \neq j$, and which has been cut off at its intersection with N_j. Hence, Q meets N_j in two disks in Bd Q, D_1 and D_2. Each D_k, $k = 1, 2$, is a small regular neighborhood in N_j of an endpoint of α. Let N_j' be obtained from N_j and Bd Q by replacing the disks D_1 and D_2 in $N_j \cap$ Bd Q by the closed annulus in Bd Q complementary in Bd Q to Int $D_1 \cup$ Int D_2.

We now have a collection $N_1', \cdots, N_j', \cdots, N_n'$, the components of which are regularly embedded, two-sided, compact, polyhedral surfaces and $N_h' = N_h$, $h \neq j$ and N_j' is obtained from N_j as described. The word for the simple closed curve l_i has been reduced with respect to this collection since the cancellation $\bar{x}_j \bar{x}_j^{-1}$ (or $\bar{x}_j^{-1}\bar{x}_j$) has been eliminated as viewed geometrically. In other words, we have reduced the number of components of $l_i \cap \bigcup_{i=1}^n N_i$. In a finite number of steps, we obtain the collection of regularly embedded, two-sided, polyhedral surfaces $L_1, \cdots, L_n$ in M. The surface L_i will be the component meeting l_i, $1 \leq i \leq n$.

The collection of surfaces $L_1, \cdots, L_n$ does not separate M. Let $U(L_i)$, $1 \leq i \leq n$, denote the interior of a small regular neighborhood of L_i. For each i, let p_i be the point of L_i common to l_i. If W denotes the wedge determined by the n simple closed curves $l_1, \cdots, l_n$ and $U(L_i)$ is properly chosen, then $W - U(L_i)$ is a tree. Hence, the projection of L_i onto p_i, $1 \leq i \leq n$, can be extended to a retraction of M onto W. The construction of the retraction is similar to the construction of the map of Theorem 2 of [2].

COROLLARY. *Let M denote a cube-with-holes and let G denote the fundamental group of M. Let F be a free group of rank n. Then there is a homomorphism of G onto F if and only if there is a retraction of M onto a wedge of n simple closed curves.*

Proof. If there is a retraction r of M onto a wedge of n simple closed curves, B, then the homomorphism of G onto $\pi_1(B)$ induced by r is the desired epimorphism.

The converse follows by the techniques of the previous theorem.

THEOREM 3. *Let M be a cube-with-holes of genus n. Then M is boundary-retractable if and only if there is a map of M onto a cube with handles, H, which induces a homeomorphism of* Bd M *onto* Bd H.

Proof. Suppose there is a map f of M onto the cube with handles H so that $f \mid \text{Bd } M$ is a homeomorphism of Bd M onto Bd H. Let B denote a wedge of n simple closed curves $J_1, \cdots, J_n$ on Bd H so that H is a regular neighborhood of B. Choose disks $D_1, \cdots, D_n$ in H so that for each $i = 1, \cdots, n$, $D_i \cap \text{Bd } H = \text{Bd } D_i$, $D_i \cap J_j = \emptyset$, $D_i \cap D_j = \emptyset$, $i \neq j$, and $D_i \cap J_i = \{x_i\}$, a single point in Bd D_i.

We choose the map f to be simplicial and choose the disks $D_1, \cdots, D_n$ to be in general position with respect to the subdivision of H for which f is simplicial. Hence, the inverse image under f of B gives the desired wedge on Bd M. The construction of the retraction is like that of Theorem 2 using the component of $f^{-1}(D_i)$ containing $f^{-1}(\text{Bd } D_i)$ for the surface L_i, $i = 1, \cdots, n$. This establishes the conclusion in this direction.

Conversely, suppose there is a retraction r of M onto the wedge $X = \bigcup_{i=1}^n J_i$ of n simple closed curves $J_1, \cdots, J_n$ in Bd M. We choose r simplicial. Note that this may be done preserving the property that r is a retraction, see for example [6]. Let x_0 denote the wedge point of X and for each $i = 1, \cdots, n$, let $x_i \in J_i - \{x_0\}$ be a nonvertex point of the subdivision for which r is simplicial.

Let F_i denote the component of $r^{-1}(x_i)$ containing x_i, and let K_i denote the boundary component of Bd F_i containing x_i, $i = 1, \cdots, n$. The simple closed curve K_i crosses J_i at the point $x_i \in K_i \cap J_i$.

We have on Bd M simple closed curves $J_1, \cdots, J_n$ and $K_1, \cdots, K_n$ so that $J_i \cap J_j = \{x_0\}$, $J_i \cap K_j = \emptyset$, $K_i \cap K_j = \emptyset$, $i \neq j$, and $J_i \cap K_i = \{x_i\}$ for each $i = 1, \cdots, n$. Let $X' = \bigcup_{i=1}^n K_i$.

Then $X \cup X'$ does not separate Bd M. This follows from the fact that K_i crosses J_i at x_i and the manner in which the mutually exclusive collection of curves $K_1, \cdots, K_n$ meets X. Furthermore, if N is a regular neighborhood in Bd M of $X \cup X'$, then Bd N has only one component and thus Bd N is a simple closed curve. Using an Euler characteristic argument, we have that Bd M $-$ Int N is a disk.

It now follows that each boundary component of F_i distinct from K_i bounds a disk on Bd M missing $X \cup X'$. Thus using standard techniques, we cap off any such boundary components and push the newly obtained surfaces into the interior of M except for the one boundary component K_i. Let $L_1, \cdots, L_n$ denote the collection of surfaces obtained in this manner. The collection $L_1, \cdots, L_n$ does not separate M, $L_i \cap L_j = \emptyset$, $i \neq j$ and L_i has precisely the one boundary component K_i which remains in Bd M, $i = 1, \cdots, n$. Also, the collection $K_1, \cdots, K_n$ does not separate Bd M.

The argument establishing the desired map is like that of Theorem 2 of [2] and that of Theorem 2 of this article.

Suppose f is a mapping of a space X onto a space Y. Then f is called *unstable* if it is homotopic to a mapping into a proper subset of Y. Otherwise, f is *stable*.

THEOREM 4. *Let M be a compact 3-manifold, possibly with boundary, with $H_1(M; Z)$ a free abelian group of rank two. Then, there is a retraction of M onto a wedge of two simple closed curves if and only if every mapping of M into the torus $T = S^1 \times S^1$ is unstable.*

Proof. Suppose each mapping of M into T is unstable. Since $H_1(M; Z)$ is free abelian of rank two, there is a homomorphism φ of $G = \pi_1(M)$ onto $\pi_1(T)$ with kernel $\varphi = G_2$.

Let f be a mapping of M into T so that $f_* = \varphi$. By hypothesis, we may choose f so that f maps M into $B = (S^1 \times q) \cup (p \times S^1)$, a wedge of two circles at $(p, q) \in T$. Since $\pi_1(B)$ is a free group of rank two and the inclusion of B into T induces an isomorphism of $H_1(B; Z)$ onto $H_1(T; Z)$, the homomorphism f_* satisfies the hypothesis of Theorem 2, and hence the required retraction exists.

Conversely, suppose there is a retraction r of M onto B, a wedge of two simple closed curves. Then r_* is a homomorphism of $G = \pi_1(M)$ onto $F = \pi_1(B)$.

Now let f be any mapping of M into T. Since F is a free group and since

$$\text{kernel } r_* = G_\omega \subset G_2 \subset \text{kernel } f_*,$$

there is a homomorphism ψ of F into $\pi_1(T)$ so that $\psi r_* = f_*$. Let h be a piecewise-linear mapping of B into T inducing ψ, so that $(hr)_* = \psi r_* = f_*$. By a standard argument using the asphericity of T, f is homotopic to hr. But since B is a one-dimensional complex, hr is not onto. Hence f is unstable.

3. Cubes-with-holes

If X is an arcwise-connected space, and f is any mapping of S^1 into X, we denote by $\{f\}$ the conjugate class of elements in $\pi_1(X)$ determined by f. Thus, if N is a normal subgroup of $\pi_1(X)$, the statement "$\{f\} \in N$" is well defined. Using the notation introduced earlier, we define the *second derived group* $G^{(2)}$ of the group G to be $[G_2, G_2]$ (see page 293 of [3]). $G^{(2)}$ is a normal subgroup of G.

THEOREM 5. *Let M be a cube-with-holes and let $G = \pi_1(M)$. If M is boundary-retractable and J is a simple closed curve in* Bd M *such that $\{J\} \in G_\omega$, then $\{J\} \in G^{(2)}$.*

Proof. Let H be a cube-with-handles of the same genus as M. By Theorem 3 there is a piecewise-linear mapping of M onto H so that $f \mid \text{Bd } M$ is a homeomorphism onto Bd H. Assuming, as we may, that J is polyhedral, $f(J)$ is a polyhedral simple closed curve in Bd H. Since $\{J\} \in G_\omega$, $f(J)$ is contractible in H and hence bounds a polyhedral disk D in H such that $D \cap \text{Bd } H = \text{Bd } D$.

By choosing the disk D in general position with respect to f (possibly adjusting J slightly to achieve this), we ensure that each component of $f^{-1}(D)$ is a properly embedded, 2-sided polyhedral surface in M. Let F be the component of $f^{-1}(D)$ containing J. Since

$$\text{kernel } f_* = G_\omega \subset G_2,$$

each loop in F is in G_2, and since $J = \text{Bd } F$ it follows that $\{J\} \in G^{(2)}$.

Example. Let M be the cube-with-holes of genus two used in Theorem 1 of [2] (see pages 151 and 152 of [5] for the details of computing $G = \pi_1(M)$). G has the presentation

$$G = \{c, g, x; [c, [g, x]] = x\},$$

and the class $\{x\}$ can be represented by a simple closed curve in Bd M. As Zeeman shows in [5], $\{x\} \in G_\omega$. Hence, if M were boundary-retractable, it would follow from Theorem 5 that $\{x\} \in G^{(2)}$. But the homomorphism φ of G onto the permutation group on three symbols defined by

$$\varphi(x) = (1\ 2\ 3), \qquad \varphi(c) = \varphi(g) = (1\ 2),$$

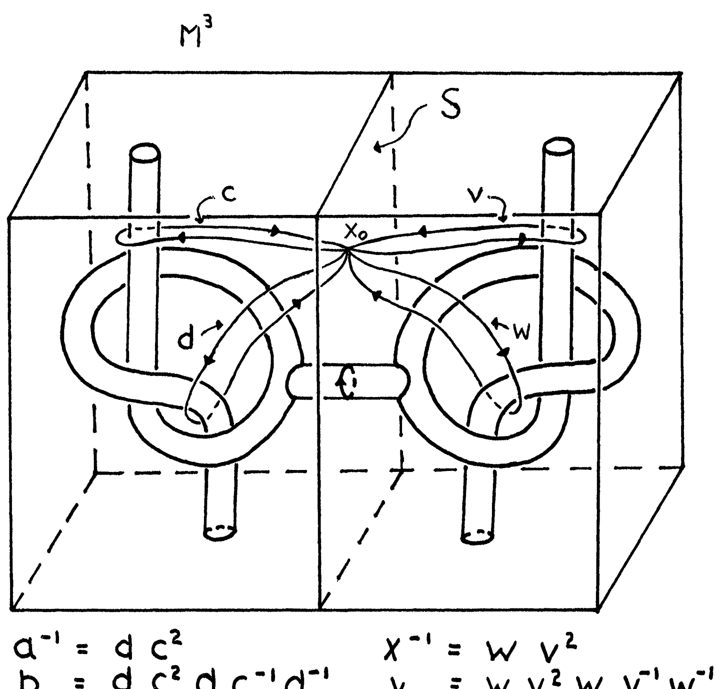
does not send x to (1), while the second derived group of this permutation group is (1). Hence, Theorem 5 gives an alternate proof that M is *not* boundary-retractable. It is clear that there is a homomorphism of G onto the free group of rank two, so that M *is* retractable.

THEOREM 6. *For each $n \geq 3$, there is a cube-with-holes M^3 of genus n whose fundamental group is residually nilpotent yet is not a free group. Hence, this group admits no homomorphism onto the free group of rank n.*

Proof. The last part of the conclusion follows from the first part and from Theorem 1. Consider first the case $n = 3$, and let M^3 be the cube-with-holes of genus three shown in the figure. A routine computation using van Kampen's Theorem gives the following presentation for $G = \pi_1(M^3, x_0)$:

$$G = \{a, b, x, y : a^2b^3 = x^2y^3\}.$$

Perhaps it will be helpful to the reader in verifying the presentation given, to remark that G is obtained by amalgamating two free groups of rank two along an infinite cyclic subgroup and that the geometric counterpart of this fact is that M can be expressed as $M_1 \cup M_2$, where M_i is a cube-with-two-handles and $M_1 \cap M_2$ is an annulus S.



To see that G is not a free group, we note first that G abelianized is free abelian of rank three, and hence it suffices to show that $\rho(G) = 4$, where $\rho(G)$ is the minimum number of generators of G. We remark also that if G_1 and G_2 are finitely-generated groups, then it follows from the Grushko-Neumann Theorem that $\rho(G_1 * G_2)$ is defined and is equal to $\rho(G_1) + \rho(G_2)$ (see [3], page 192). In particular, if

$$G_i = \{\alpha, \beta : \alpha^2\beta^3 = 1\}$$

for $i = 1, 2$, then $\rho(G_i) = 2$ (G_i is the group of the trefoil knot) and there is a homomorphism of G onto $G_1 * G_2$. Hence, $\rho(G) \geq \rho(G_1 * G_2) = 4$. Thus G is not a free group.

In [1], Baumslag proves that the (unrestricted) direct product J of countably many copies of the free group of rank two contains an isomorphic copy of G as a subgroup. Since the product of residually nilpotent groups is residually nilpotent, J and each of its subgroups is residually nilpotent. This completes the proof for $n = 3$.

We digress for a moment to give a brief description of an explicit embedding of G in J for the reader not familiar with [1]. Let F be the free group with free basis a, b and let $u = a^2b^3 \in F$. Consider the epimorphisms

$$\delta_i : G \rightarrow F \qquad (i = 1, 2, 3, \cdots),$$

defined by $\delta_i(a) = a$, $\delta_i(b) = b$, $\delta_i(x) = u^{-i}au^i$, and $\delta_i(y) = u^{-i}bu^i$. Then Proposition 1 of [1] and the fact that an element $f \in F$ commutes with u if and only if it is a power of u, imply that the homomorphism δ of G given by

$$\delta(g) = (\delta_1(g), \delta_2(g), \cdots) \in J$$

is one-to-one. It is perhaps of interest to note that G cannot be embedded in the direct product of a *finite* number of copies of F (this follows from Theorem 3 of [1].

The required examples of genus $n > 3$ are obtained by adding $(n - 3)$ orientable handles of index one to Bd M^3. If N^3 is such a cube-with-holes of genus $n > 3$, then

$$H^n = \pi_1(N^3) = G * (\text{free group of rank } n - 3),$$

and $\rho(H^n) = 4 + (n - 3) = n + 1$, while $\rho(H^n/H_2^n) = n$. Hence H^n is not a free group.

To see that H^n is residually nilpotent when n is odd and greater than three, we can argue just as before to obtain an embedding of H^n in the direct product of countably many copies of the free group of rank $(n + 1)/2$. If n is even and greater than three, we have only to note that $H^{n+1} = H^n * Z$, so that H^n is a subgroup of the residually nilpotent group H^{n+1}.

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