ON AN INEQUALITY OF BANACH ALGEBRA GEOMETRY AND SEMI-INNER PRODUCT SPACE THEORY

BY

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Introduction

A fundamental result, due to Lumer [4], in the theory of complex semi-inner product spaces, is that if T is an operator on a s.i.p.s., then

$$(1) \quad \| T \| \leqq 4 \,| W(T)|, \qquad \text{where} \quad | W(T)| = \sup \{ |[Tx, x]| : \| x \| = 1 \}.$$

Bohnenblust and Karlin, in an earlier study [1] of the geometry of the unit sphere of a Banach algebra, showed that if A is a Banach algebra with identity, then

$$(2) \quad \| a \| \leqq e\psi(a), \qquad \text{all} \quad a \in A,$$

$$\text{where} \quad \psi(a) = \text{Max}_{|z|=1} \lim_{r \to 0+} (\| 1 + rza \| - 1)/r.$$

If A is an algebra of operators on a s.i.p.s., it follows immediately from Lemma 12 of [4] that $| W(a)| = \psi(a)$, all $a \in A$; thus (1) may be replaced by

$$(1') \qquad \| T \| \leqq e \,| W(T)|.$$

Here we first sharpen the estimates used in [4] to obtain a direct "semi-inner product space" proof of (1′). To do this we introduce the integral formula

$$T = \frac{1}{2\pi i N} \int_C \frac{\zeta^N}{(\zeta - T)^N} \, d\zeta, \quad \text{all positive integers } N,$$

in place of the more usual

$$T = \frac{1}{2\pi i} \int_C \frac{\zeta d\zeta}{\zeta - T}.$$

Then we give an example of a two-dimensional s.i.p.s. X so that the shift operator T on X satisfies $\| T \| = 1$ and $| W(T)| = 1/e$. This proves that e is the best possible positive constant in the inequalities (1′) and (2). If Y is the closed interval $[1/e, 1]$ and Z the unit circle, then X is the subspace of $C(Y \times Z)$ generated by $f^*(y, z) = -ezy(\log y)$ and $g^*(y, z) = y$.

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The notation and terminology used here for semi-inner product space notions is that of [4]. All normal spaces considered here have complex scalars and are complete. If h is a continuous function on a compact Hausdorff space, $\| h \|$ will denote the "sup" norm of h. r and s will always denote non-

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negative real numbers, while λ, μ, ζ, and z will denote complex numbers. If T is an operator, $r(T)$ will denote its spectral radius.

1. The proof of $\|T\| \leqq e\,|W(T)|$

We take the following lemma virtually intact from [4, p. 33].

LEMMA 1.1. *Let T be an operator on the semi-inner product space X so that $|W(T)| < 1$. Then*

$$(3) \qquad \|(I - T)^{-1}\| \leqq (1 - |W(T)|)^{-1}.$$

Proof. Since $r(T) \leqq |W(T)| < 1$, $I - T$ is invertible. Now if $\|x\| = 1$, we have via the Schwarz inequality that

$$\|(I - T)x\| \geqq |[(I - T)x, x]| \geqq [x, x] - |[Tx, x]| \geqq 1 - |W(T)|.$$

Therefore, for all x,

$$\|(I - T)x\| \geqq (1 - |W(T)|)\|x\|.$$

Set $x = (I - T)^{-1}y$, where $\|y\| = 1$, to obtain

$$1 \geqq (1 - |W(T)|)\|(I - T)^{-1}y\|,$$

from which (3) immediately follows.

Next we present the integral formula referred to in the introduction.

LEMMA 1.2. *If T is an operator on the Banach space X so that $r(T) < 1$, then*

$$(4) \qquad T = \frac{1}{2\pi i N}\int_C \frac{\zeta^N}{(\zeta - T)^N}\,d\varphi, \quad \textit{for each positive integer } N,$$

where C is the unit circle in the complex plane.

Proof. Observe that (4) can be obtained by differentiating

$$T^N = \frac{1}{2\pi i}\int_C \frac{\zeta^N}{\zeta - T}\,d\zeta$$

formally $N - 1$ times under the integral sign with respect to T. This differentiation is both natural and easily justifiable within the framework of the Lorch theory of analytic functions in commutative Banach algebras [3]. A proof via the standard operational calculus [2, p. 566], goes as follows: First notice that (4) is valid if T is replaced by a complex number z of absolute value <1. But if r is a positive number between $r(T)$ and 1, C' is the circle of radius r about 0 in the complex plane, and φ is a complex number of magnitude 1, the operational calculus shows that

$$\frac{1}{2\pi i}\int_{C'} \frac{dz}{(\zeta - z)^N(z - T)} = \frac{1}{(\zeta - T)^N}.$$

Thus

$$\frac{1}{2\pi i}\int_{C'}\frac{\zeta^N}{(\zeta - T)^N}\,d\zeta = \frac{1}{(2\pi i)^2 N}\int_C\int_{C'}\frac{\zeta^N\,dz\,d\zeta}{(\zeta - z)^N(z - T)}$$

$$= \frac{1}{(2\pi i)^2 N}\int_{C'}\int_C\frac{\zeta^N\,d\zeta\,dz}{(\zeta - z)^N(z - T)}$$

$$= \frac{1}{2\pi i}\int_{C'}\frac{dz}{z - T} = T.$$

Lemmas 1.1 and 1.2 together yield

LEMMA 1.3. *Under the hypotheses of* 1.1,

$$(5)\qquad \| T \| N(1 - | W(T)|)^N \leqq 1, \qquad \textit{all positive } N.$$

Proof. The usual absolute estimates applied to (4) yield

$$\| T \| \leqq N^{-1} \max_{|\zeta|=1}\left\| \frac{1}{\zeta - T}\right\|^N, \quad \text{all } N.$$

(5) now follows from a simple application of (3).

Remark. When $N = 1$ in (5) we have $\| T \| (1 - | W(T)|) \leqq 1$. If $\| T \|$ is taken to be 2, then $1 - | W(T)| \leqq \frac{1}{2}$, so $4 | W(T)| \geqq 2 = \| T \|$. This is the proof of (1) that appears in [4].

THEOREM 1.4. *If T is an operator on a* s.i.p.s. *X, then* $\| T \| \leqq e | W(T)|$.

Proof. If $| W(T)| = 0$, it follows directly from (5) that $T = 0$. If $| W(T)| \neq 0$, we may assume without loss of generality that $| W(T)| = 1$. For each positive integer N, apply (5) to the operator $T/(N + 1)$ to obtain

$$\| T \| (N/(N + 1))^N \leqq 1.$$

Let $N \to \infty$ to obtain $\| T \| \leqq e$.

2. An example in which $\| T \| = e \, | W(T) |$

We begin by constructing a two-dimensional positive cone P of continuous non-negative functions on the closed interval $Y = [1/e, 1]$. Let f and g be defined on Y by $f(y) = ey \log y$ and $g(y) = y$, and set

$$P = \{rf + sg : r, s \geqq 0\}.$$

The following relevant properties of f are verified via the calculus:

(a) $f(1/e) = 1, f(1) = 0, f \geqq 0$, f and f' are strictly decreasing on Y, $f'(1/e) = 0$, and $f'(1) = 0$.

(b) $f(y)/y - f'(y) = e$, all $y \in Y$.

From (a) and elementary calculus we have the following:

(c) If $h \in P$ and $h \neq 0$, h assumes its maximum at exactly one point, which we denote by y_h. For notational convenience, choose some point of Y and denote it by y_0.

We now define a mapping $[\ ,\ ]$ of $P \times P$ into the non-negative reals by

$$[u, v] = u(y_v)v(y_v) = u(y_v)\| v \|, \qquad u, v \in P.$$

$[\ ,\ ]$ will be called a semi-inner product for P (although, of course, $(P, [\ ,\ ])$ is not a semi-inner product space) because $[\ ,\ ]$ satisfies the conditions

(i) $[u_1 + u_2, v] = [u_1, v] + [u_2, v]$,
(ii) $[ru, v] = r[u, v]$, $r \geqq 0$,
(iii) $[u, u] = \| u \|^2$ and
(iv) $[u, v] \leqq \| u \| \| v \|$.

Let S be the "shift" mapping of P into itself given by $S(rf + sg) = rg$. We can now define $\| S \|$ and $| W(S)|$ by

$$\| S \| = \sup \{Sh : h \in P, \| h \| = 1\} \quad \text{and}$$

$$| W(S)| = \sup \{[Sh, h] : h \in P, \| h \| = 1\}.$$

We will show that $\| S \| = 1$ and $| W(S)| = 1/e$. Then we will construct a complex s.i.p.s. X (described in the introduction) modeled closely enough on P so that the results $\| S \| = 1$ and $| W(S)| = 1/e$ can be carried over to $\| T \| = 1$ and $| W(T)| = 1/e$, where T is the shift operator on X which is analogous to S.

LEMMA 2.1. $\| S \| = 1$.

Proof. Clearly $\| S(rf + sg)\| = r \| g \| \leqq \| rf + sg \|$, so $\| S \| \leqq 1$. Since $S(f) = g$, and $\| f \| = \| g \| = 1$, $\| S \| = 1$.

LEMMA 2.2. $| W(S)| = 1/e$.

Proof. For each $y \in Y$ let

$$\Gamma_y = \{h : h \in P, \| h \| = 1, \ \text{and} \ \ y_h = y\},$$

and set

$$W_y = \sup \{[Sh, h] : h \in \Gamma_y\}.$$

It is sufficient to show that each $W_y = 1/e$. Observe that when $h = rf + sg \in P$, and $\| h \| = 1$,

$$[Sh, h] = [rg, h] = ry_h,$$

so

$$W_y = \sup \{ry : rf + sg \in \Gamma_y\}.$$

Now when $y = 1/e$, $\Gamma_{1/e} = \{f\}$, so $W_{1/e} = 1/e$.

When $1/e < y < 1$, it is not hard to see that $rf + sg \in \Gamma_y$ iff the two linear equations in r and s,

(1) $$rf(y) + sy = 1 \quad \text{and} \quad rf'(y) + s = 0,$$

are satisfied. But (1) has the unique solution

$$r = (f(y) - f'(y)y)^{-1} = 1/ey$$

and

$$s = -f'(y)(f(y) - f'(y)y)^{-1} = -f'(y)/ey.$$

Thus $W_y = (1/ey)y = 1/e$.

Finally we consider the case $y = 1$. $rf + sg$ lies in Γ_1 iff $r \leqq -1/f'(1)$ and $s = 1$. Therefore $A_1 = -1/f'(1) = 1/e$; 2.2 is proved.

Now let Z denote the unit circle in the complex plane and define f^* and g^* on $Y \times Z$ by $f^*(y, z) = zf(y)$ and $g^*(y, z) = g(y)$. Set

$$X = \{\lambda f^* + \mu g^* : \lambda, \mu \text{ complex}\}.$$

We provide X with a semi-inner product as follows: Select, for each $\psi \in X$, some point (y_ψ, z_ψ) of $Y \times Z$ at which $|\psi|$ attains its maximum. For φ, ψ in X define

$$[\varphi, \psi] = \varphi(y_\psi)\overline{\psi(y_\psi)}.$$

Clearly the norm induced on X by $[\ ,\ ]$ is the sup norm.

We now establish the strong relation between the norms in P and in X. Define $U : X \to P$ by

$$U(\lambda f^* + \mu g^*) = |\lambda| f + |\mu| g.$$

Clearly U maps X onto P.

LEMMA 2.3. *$\|U(\psi)\| = \|\psi\|$, all $\psi \in X$.*

Proof. Write $\psi = \lambda f^* + \mu g^*$.

If $(y, z) \in Y \times Z$, then since

$$|\psi(y, z)| \leqq |\lambda| f^*(y, z)| + |\mu| |g^*(y, z)| = |\lambda| f(y) + |\mu| g(y),$$

$\|\psi\| \leqq \|U(\psi)\|$.

If $y \in Y$, there are real numbers σ and τ so that

$$|\lambda| f(y) + |\mu| g(y) = \lambda e^{i\sigma} f(y) + \mu e^{i\tau} g(y)$$
$$= |\lambda f^*(y, e^{i(\sigma-\tau)}) + \mu g^*(y, e^{i(\sigma-\tau)})|.$$

Therefore $\|U(\psi)\| \leqq \|\psi\|$.

We shall also need the following lemma, which establishes a link between the semi-inner products of P and of X.

LEMMA 2.4. *If $\psi \in X$ and $\psi \neq 0$, then $y_\psi = y_{U(\psi)}$.*

Proof. Since

$$\| \psi \| = | \psi(y_\psi, z_\psi)| \leqq U(\psi)(y_\psi) \leqq \| U(\psi)\| = \| \psi \|,$$

all the terms in the preceding inequality are equal. Therefore $U(\psi)$ assumes its maximum at y_ψ; so $y_\psi = y_{U(\psi)}$.

Now let T be the shift operator on X defined by

$$T(\lambda f + \mu g) = \lambda g.$$

Note that $UT = SU$.

THEOREM 2.5. *$\| T \| = 1$, and $| W(T)| = 1/e$.*

Proof.

$$\begin{aligned} \| T \| &= \sup \{\| T\psi \| : \| \psi \| = 1\} \\ &= \sup \{\| UT\psi \| : \| \psi \| = 1\} \qquad \text{(by 2.3)} \\ &= \sup \{\| SU\psi \| : \| \psi \| = 1\} \\ &= \| S \| = 1. \end{aligned}$$

Now consider $\psi = \lambda f^* + \mu g^* \in P$, where $\| \psi \| = 1$.

$$\begin{aligned} |[T\psi, \psi] | &= | \lambda | | g^*(y_\psi, z_\psi)| | \psi(y_\psi, z_\psi)| \\ &= | \lambda | y_\psi = | \lambda | y_{U(\psi)} \qquad \text{(by 2.4)} \\ &= [SU\psi, U\psi]. \end{aligned}$$

Therefore $| W(T)| = \sup \{[SU\psi, U\psi] : \psi \in P, \| \psi \| = 1\} = W(S) = 1/e$.

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