

ASYMPTOTIC DISTRIBUTION OF BEURLING'S GENERALIZED INTEGERS

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0. Introduction

This paper is a study of the asymptotic distribution of Beurling's generalized numbers which improves previous estimates of the error term. Our approach to the problem is first to show how measures of a certain type may be expressed as "exponentials." We then apply this formalism to the counting measure of generalized integers and Lebesgue measure and estimate the one measure by the other. With this representation we show how a hypothesis on the distribution of generalized primes leads to an estimate of the distribution of generalized numbers.

Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers subject to the following three conditions but otherwise arbitrary:

$$(i) \ p_1 > 1, \quad (ii) \ p_{n+1} \geq p_n, \quad (iii) \ p_n \rightarrow \infty.$$

Following Beurling [1] we call such a collection $\{p_i\}$ a set of generalized (henceforth g -) primes. The multiplicative semigroup generated by the $\{p_i\}$ is countable and may be arranged in a nondecreasing sequence $\{n_i\}_{i=1}^{\infty}$. Setting $n_0 = 1$, we call $\{n_i\}_{i=0}^{\infty}$ the set of g -integers associated with $\{p_i\}$.

The function $\pi(x)$ is defined to be the number of g -primes less than or equal to x , and $\Pi(x)$ is defined by

$$\Pi(x) = \sum_{n=1}^{\infty} (1/n)\pi(x^{1/n}).$$

Let $N(x)$ be the number of g -integers less than or equal to x .

With these definitions, our main results take the following form:

THEOREM. *Suppose there exist positive numbers c and α such that for large x the following relation holds:*

$$\int_1^x d\Pi(t)/t = \int_1^x (1 - t^{-1}) dt/(t \log t) + \log c + O(\log^{-\alpha} x).$$

Then $N(x) = cx + O(x \log^{2-\alpha} x)$.

THEOREM. *Suppose there exist numbers $c > 0$ and $a \in (0, 1)$ such that*

$$\int_1^x d\Pi(t)/t = \int_1^x (1 - t^{-1}) dt/(t \log t) + \log c + O\{\exp(-\log^a x)\}.$$

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Then

$$N(x) = cx + O\{x \exp(-[\log x \log \log x]^{a'})\}, \quad a' = a/(1+a).$$

These theorems sharpen results previously obtained by Nyman [5] and Malliavin [3] respectively. While Nyman and Malliavin use Fourier analysis and tauberian arguments, our method involves neither of these techniques. The proofs of the two theorems have a common beginning and are in principle similar, but they require separate estimates. We shall prove the second in detail and content ourselves with indicating the estimates necessary for the first.

We remark that one may show $N(x) \sim cx$ (with no error term) using weaker hypotheses than our first theorem, by methods of Wirsing [7].

1. Definitions and preliminaries

1.1. Measures. Let \mathfrak{M} denote the collection of complex valued set functions on the Borel subsets of $[1, \infty)$ with the property that on any finite interval $[1, x]$ each is a finite measure. The set functions that appear in what follows will all be members of \mathfrak{M} , and we shall call such functions measures. By "a set" we shall always mean a bounded Borel set contained in $[1, \infty)$.

We denote the elements of \mathfrak{M} by such symbols as $d\alpha$ or $d\mu$. We do this so that we may conveniently refer to the distribution function of $d\mu$, say, by $\mu(x)$. Precisely, for $d\mu \in \mathfrak{M}$ set $\mu(x) = 0$ for $x < 1$, and for $x \geq 1$ let $\mu(x) = \int_1^{x+} d\mu$. $\mu(x)$ as so defined is right continuous. To achieve additivity of the integral we adopt the convention that $\int_x^y d\mu$ means $\int_{x+}^{y+} d\mu$ except when $x = 1$, in which case we taken the lower limit to be $1-$.

Throughout this paper, the symbol δ will represent point mass 1 at 1.

1.2. Generalized numbers. If $N(x)$ is defined as in the introduction, dN is a pure point measure. We have made no assumption about unique factorization of the g -integers, and for l a g -integer, $dN\{l\}$ equals the number of distinct representations of l in prime factors.

The function Π was defined in the introduction by a formally infinite series. We show Π to be well defined and close to π in the following

LEMMA. *For each x the series for $\Pi(x)$ is terminating, and if we assume $\pi(x) = O(x)$, then $\Pi(x) - \pi(x) = O(\sqrt{x})$.*

Proof. $x^{1/n} < p_1$ for $n > \log x / \log p_1$. For such n , $\pi(x^{1/n}) = 0$, and the series is finite. Also,

$$0 \leq \Pi(x) - \pi(x) \leq \pi(x^{1/2}) + \frac{\log x}{\log p_1} \pi(x^{1/3}) \leq Kx^{1/2}.$$

As is usual in prime number theory, it is Π rather than π which is important in the sequel. The Borel-Stieltjes measure $d\Pi$ is a pure point measure whose

support is the set of all (positive integral) powers of g -primes. Precisely,

$$\begin{aligned} d\Pi\{l\} &= \alpha^{-1} + \beta^{-1} + \cdots + \gamma^{-1} && \text{if } l = p_i^\alpha = p_j^\beta = \cdots = p_k^\gamma \\ &= 0 && \text{if } l \text{ is not a prime power.} \end{aligned}$$

We define the generalized Chebychev ψ function by setting $d\psi(t) = \log t d\Pi(t)$. One verifies that $\psi(x) = \sum \log p$, where the summation extends over all g -integers $n_i \leq x$ with $n_i = p^\alpha$ for some g -prime p .

1.3. Convolution. We define a map $*$: $\mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$ called multiplicative convolution or, briefly, convolution. For $d\alpha, d\beta \in \mathfrak{M}$ and a set E let

$$\int_E d\alpha * d\beta = \iint_{st \in E} d\alpha(s) d\beta(t).$$

It is well known [2; pp. 643–644] that $d\alpha * d\beta$ as so defined is a Borel measure. Since $d\alpha$ and $d\beta$ have support contained in $[1, \infty)$, the same holds for $d\alpha * d\beta$, and $d\alpha * d\beta \in \mathfrak{M}$. Convolution is associative and commutative because \mathbf{R} and \mathbf{C} have these properties. $(\mathfrak{M}, +, *, \delta)$ is an algebra over \mathbf{R} or \mathbf{C} with δ as unit.

Define $d\alpha^0 = \delta$ for all $d\alpha \in \mathfrak{M}$, and define $d\alpha^n$ by $d\alpha^{n-1} * d\alpha$, $n = 1, 2, \dots$.

We shall frequently use the following convolution formulas. Let $d\alpha, d\beta \in \mathfrak{M}$, let E be a set, and let f be a Borel function which is bounded on every bounded set $[1, X]$. Then

$$(1.3a) \quad \int_1^x d\alpha * d\beta = \int_1^x \alpha(x/t) d\beta(t) = \int_1^x \beta(x/t) d\alpha(t),$$

$$(1.3b) \quad \int_E f(d\alpha * d\beta) = \iint_{st \in E} f(st) d\alpha(s) d\beta(t).$$

The first formula follows from the definition of convolution and Fubini's Theorem. In the case $f = \chi_F$ for some set F , equation (1.3b) is just the definition of $\int_{E \cap F} d\alpha * d\beta$. By linearity, this formula is valid for simple functions, and finally (1.3b) holds for all bounded Borel functions by uniform approximation by simple functions.

1.4. The derivation L . We define a map $L : \mathfrak{M} \rightarrow \mathfrak{M}$ by

$$\int_E Ld\alpha = \int_{t \in E} \log t d\alpha(t).$$

L is a derivation, i.e. a linear operator satisfying

$$(1.4a) \quad L(d\alpha * d\beta) = d\alpha * Ld\beta + Ld\alpha * d\beta \quad \text{for } d\alpha, d\beta \in \mathfrak{M}.$$

To show (1.4a) note that

$$\int_{t \in E} \log t (d\alpha * d\beta)(t) = \iint_{xy \in E} \log xy d\alpha(x) d\beta(y).$$

Define L^n by $L^0 d\alpha = d\alpha$, $L^n d\alpha = L(L^{n-1} d\alpha)$, $n = 1, 2, \dots$. The following formulas involving L may be verified by induction:

$$(1.4b) \quad L(d\alpha^n) = n d\alpha^{n-1} * L d\alpha, \quad n = 1, 2, \dots$$

$$(1.4c) \quad L^n(d\alpha * d\beta) = \sum_{j=0}^n \binom{n}{j} L^j d\alpha * L^{n-j} d\beta, \quad n = 0, 1, 2, \dots$$

1.5. The Mellin transform. Although we shall not make use of Fourier techniques in proofs, we mention here the Mellin transform (Fourier transform on a multiplicative group) to illustrate analogues with classical methods and results.

Let

$$\mathfrak{M}' = \left\{ d\alpha \in \mathfrak{M} : \text{there exists } n \text{ such that } \limsup_{x \rightarrow \infty} x^{-n} \int_1^x |d\alpha| = 0 \right\}.$$

For $d\alpha \in \mathfrak{M}'$ there exist $s \in \mathbb{C}$ for which the integral $\int_1^\infty x^{-s} d\alpha(x)$ converges, indeed absolutely. This integral is called the Mellin transform of $d\alpha$ and is an analytic function of s (at least) in the interior of the region of absolute convergence.

This transform is important in analytic arguments involving multiplicative convolution because it preserves algebraic structure. Precisely, if $A(s) = \int x^{-s} d\alpha(x)$ and $B(s) = \int x^{-s} d\beta(x)$ are absolutely convergent for some s , then for such s we have $\int x^{-s} (d\alpha * d\beta)(x) = A(s)B(s)$.

To see the effect of the L mapping on the Mellin transform, note that if $A(s) = \int x^{-s} d\alpha(x)$, then $A'(s) = -\int x^{-s} \log x d\alpha(x)$ (provided the latter integral converges absolutely). Thus the L mapping in \mathfrak{M}' corresponds to $-d/ds$ in the transform space, and formulas (1.4a, b, and c) correspond to well-known rules in calculus.

1.6. Topology. We define a set of functionals on \mathfrak{M} by $\|d\nu\|_x = \int_1^x |d\nu|$ for all $x \in [1, \infty)$. One verifies that $\{\|\cdot\|_x\}$ is a family of semi norms. Let $(\mathfrak{M}, +, *, \|\cdot\|)$ be the algebra \mathfrak{M} with the set of semi norms $\{\|\cdot\|_x\}$. The topological space $(\mathfrak{M}, \|\cdot\|)$ satisfies the first axiom of countability. By convergence in \mathfrak{M} we mean convergence with respect to each semi norm $\|\cdot\|_x$. Let $(\mathfrak{M}_{[1,x]}, +, *, \|\cdot\|_x)$ be the algebra of measures on $[1, x]$ with norm $\|\cdot\|_x$.

PROPOSITION 1.6. $(\mathfrak{M}, +, *, \|\cdot\|)$ is a semi normed algebra complete under each semi norm.

Proof. It is well known that $(\mathfrak{M}_{[1,x]}, \|\cdot\|_x)$ is complete (cf. [2; pp. 160–162]). The norm inequality

$$(1.6a) \quad \|d\alpha * d\beta\|_x \leq \|d\alpha\|_x \|d\beta\|_x$$

follows from the fact that $|d\alpha(s)| |d\beta(t)|$ is integrated over the set

$$\{(s, t) : 1 \leq s \leq x, 1 \leq t \leq x\}$$

on the right hand side of (1.6a), while on the left hand side, it is integrated

over the subset

$$\{(s, t) : s \geq 1, t \geq 1, st \leq x\}.$$

In the sequel we will use the well-known fact that in a Banach space, an absolutely convergent series is convergent. Thus, to show $\sum d\lambda_n$ convergent in \mathfrak{M} it suffices to show $\sum \|d\lambda_n\|_x < \infty$, for each $x \geq 1$.

The semi norms induce a partial ordering on \mathfrak{M} . We say $d\alpha$ is smaller than $d\beta$ on $[1, x]$ and write $d\alpha <_x d\beta$ if $\int_1^x d\beta = \int_1^x |d\beta|$ and $\|d\alpha\|_y \leq \|d\beta\|_y$ for all $y \leq x$. If these two conditions hold for all x , we say $d\alpha$ is smaller than $d\beta$ and write $d\alpha < d\beta$. Examples of $<_x$ and $<$ are the following: $d\alpha <_x \|d\alpha\|_x \delta$ and $d\beta < |d\beta|$, for all $d\alpha, d\beta \in \mathfrak{M}$.

This partial ordering is preserved under addition and convolution. Precisely, $d\alpha <_x dA$ and $d\beta <_x dB$ imply $d\alpha + d\beta <_x dA + dB$ and $d\alpha * d\beta <_x dA * dB$. Verification of these statements is easy.

2. The exponential representation theorem

2.1. Power series. Power series of a measure will play a central role in this chapter, and we begin with a discussion of convergence. A treatment of functions of operators by means of power series may be found in [4; pp. 369–372].

PROPOSITION 2.1. *Let $f(z)$ be an arbitrary function holomorphic at zero whose Maclaurin series $\sum a_n z^n$ has radius of convergence ρ . Let $d\nu \in \mathfrak{M}$ and $|d\nu\{1\}| < \rho$. Then $f(d\nu) = \sum a_n d\nu^n$ converges in \mathfrak{M} .*

Proof. It suffices to establish the following “spectral radius formula”: For each $x \in [1, \infty)$,

$$(2.1a) \quad \lim_{n \rightarrow \infty} (\|d\nu^n\|_x)^{1/n} = |d\nu\{1\}|.$$

On the one hand, we have for all n and $x \geq 1$,

$$|d\nu\{1\}| = (\|d\nu^n\|_1)^{1/n} \leq (\|d\nu^n\|_x)^{1/n}.$$

On the other hand, we shall show that $|d\nu\{1\}| < \gamma$ implies that

$$\limsup (\|d\nu^n\|_x)^{1/n} < \gamma.$$

Since $\|d\nu\|_x$ is right continuous as a function of x and $\lim_{x \rightarrow 1+} \|d\nu\|_x < \gamma$, there is a number $c > 1$ such that $\|d\nu\|_c < \gamma$. Set

$$r = \|d\nu\|_c \quad \text{and} \quad d\beta(t) = \chi_{[c, \infty)}(t) |d\nu|(t).$$

For fixed x , note that $d\beta^j <_x 0$ for $j \geq J = [(\log x)/(\log c) + 1]$. This is so since $d\beta(t)$ has no support for $t < c$ and for all $y \leq x$ we have

$$\int_1^y d\beta^j = \int_{s_1 \cdots s_j \leq y} \cdots \int d\beta(s_1) \cdots d\beta(s_j).$$

For $j \geq J$, there is at least one s_i in the product $s_1 \cdots s_j$ satisfying $s_i \leq y^{1/j} \leq x^{1/j} < c$, and the multiple integral is zero.

Now

$$d\nu^n < (r\delta + d\beta)^n = \sum_{j=0}^n \binom{n}{j} r^{n-j} d\beta^j,$$

and for $n \geq J$,

$$\|d\nu^n\|_x \leq r^n \sum_{j=0}^J r^{-j} \binom{n}{j} \|d\beta^j\|_x \leq cn^J r^n,$$

where $c = c(x)$. Thus

$$\limsup (\|d\nu^n\|_x)^{1/n} \leq r < \gamma,$$

and (2.1a) is established. The proposition follows at once from this formula, since $\sum a_n d\nu^n$ converges absolutely on each interval $[1, x]$.

Remarks. If $d\nu\{1\} > \rho$, $\sum a_n d\nu^n$ does not converge in \mathfrak{M} . It is clear from the proof that if $d\nu$ has no support in some neighborhood of 1, then for each x there exists a J such that $d\nu^j <_x 0$ for all $j \geq J$, and the formally infinite series reduces to a polynomial in $d\nu$.

Within its circle of convergence, an ordinary power series is a continuous function. An analogous result holds for power series in \mathfrak{M} .

COROLLARY 2.2 (Continuity). *Let f and $d\nu$ be as in Proposition 2.1. If $d\nu_n \rightarrow d\nu$ in \mathfrak{M} , then $f(d\nu_n) \rightarrow f(d\nu)$ in \mathfrak{M} .*

Proof. Let r, c , and $d\beta$ be as in the proof of Proposition 2.1, and let r' satisfy $r < r' < \rho$. Let x be arbitrary but fixed. Since $d\nu_n \rightarrow d\nu$ in \mathfrak{M} , $\|d\nu - d\nu_n\|_x < r' - r$ for all sufficiently large n . For such n ,

$$d\nu_n <_x |d\nu| + (r' - r)\delta <_x r'\delta + d\beta.$$

Also $d\nu <_x r'\delta + d\beta$. We set $g(z) = \sum |a_n| z^n$ and note that g is holomorphic for $|z| < \rho$. By the proposition, $f(d\nu_n), f(d\nu)$, and $g'(r'\delta + d\beta)$ are all convergent, and indeed absolutely convergent. Now

$$\begin{aligned} f(d\nu) - f(d\nu_n) &= \sum_{j=1}^{\infty} a_j (d\nu^j - d\nu_n^j) \\ &= \sum_{j=1}^{\infty} a_j \left\{ \sum_{k=0}^{j-1} d\nu^{j-k-1} * d\nu_n^k \right\} * (d\nu - d\nu_n) \\ &<_x \sum_{j=1}^{\infty} |a_j| j (r'\delta + d\beta)^{j-1} * |d\nu - d\nu_n| \\ &= g'(r'\delta + d\beta) * |d\nu - d\nu_n| \rightarrow 0 \quad \text{in } \mathfrak{M}_{[1,x]}, \end{aligned}$$

proving the assertion.

COROLLARY 2.3. *Let f and $d\nu$ be as in Proposition 2.1, and let L be the log mapping defined in Section 1.4. Then*

$$(2.3a) \quad Lf(d\nu) = f'(d\nu) * Ld\nu.$$

Proof. By Proposition 2.1, $f(d\nu)$ converges. The L map is linear and continuous, as is the map defined by convolution with a fixed element of \mathfrak{M} . Thus

$$\begin{aligned} L(\sum a_n d\nu^n) &= \sum a_n Ld\nu^n = \sum (a_n n d\nu^{n-1} * L(d\nu)) \\ &= (\sum a_n n d\nu^{n-1}) * Ld\nu. \end{aligned}$$

2.4. Inverses. Let $d\alpha \in \mathfrak{M}$. We say $d\alpha$ has an inverse if there exists $d\beta \in \mathfrak{M}$ such that $d\alpha * d\beta = \delta$. Obviously, $d\beta$ then has an inverse as well. The usual algebraic proof shows that an element of \mathfrak{M} has at most one inverse.

The invertible elements of \mathfrak{M} are characterized in the following

PROPOSITION 2.4. $d\alpha \in \mathfrak{M}$ is invertible iff $d\alpha\{1\} \neq 0$.

Proof. If $d\alpha$ is invertible with inverse $d\beta$, then $d\alpha\{1\}d\beta\{1\} = 1$ and $d\alpha\{1\} \neq 0$. Conversely, if $d\alpha\{1\} = a \neq 0$, we will exhibit an inverse for $d\alpha$. By an easy normalization, we may assume $a = 1$ or, equivalently, $d\alpha = \delta + d\gamma$ with $d\gamma \in \mathfrak{M}$ and $d\gamma\{1\} = 0$.

We proceed with the same proof that shows an element of a Banach algebra that is close to the identity is invertible. $f(z) = (1 + z)^{-1} = \sum (-z)^n$ converges for $|z| < 1$. Since $d\gamma\{1\} = 0$, by Proposition 2.1 $f(d\gamma)$ converges in \mathfrak{M} . Now $f(d\gamma) * d\alpha = \delta$. This may be seen by estimating f by a partial sum and passing to the limit.

Remark. The algebra \mathfrak{M} has precisely one maximal ideal, namely

$$\{d\alpha \in \mathfrak{M} : d\alpha\{1\} = 0\}.$$

2.5. Quotient space. Anticipating the multivalued property of the logarithm of a measure, we introduce here a quotient space modulo which the logarithm is single valued. Let \mathfrak{M}_1 denote the elements of \mathfrak{M} which have non zero point mass at 1. Let $(\overline{\mathfrak{M}}, +)$ be the quotient of the groups $(\mathfrak{M}, +)$ and $(2\pi i\mathbb{Z}\delta, +)$, where the latter group consists of all integral multiples of $2\pi i\delta$. Let \mathfrak{R}_{1p} be the subset of \mathfrak{M} consisting of real-valued measures with *positive* point mass at 1. And let $(\mathfrak{R}, +)$ be the subgroup of real valued measures in $(\mathfrak{M}, +)$.

By Proposition 2.4, $(\mathfrak{M}_1, *)$ and $(\mathfrak{R}_{1p}, *)$ are abelian groups. Since \mathfrak{M} is a semi normed algebra, each of

$$(\overline{\mathfrak{M}}, +, \|\cdot\|), (\mathfrak{M}_1, *, \|\cdot\|), (\mathfrak{R}, +, \|\cdot\|) \text{ and } (\mathfrak{R}_{1p}, *, \|\cdot\|)$$

is a topological group.

To help achieve clarity, the following convention will be in force for the remainder of this chapter. A Roman letter will denote a member of $(\mathfrak{M}_1, *, \|\cdot\|)$, a Greek letter in parentheses represents a coset in $(\overline{\mathfrak{M}}, +, \|\cdot\|)$, and the same Greek letter without parentheses is a representative of the coset.

A neighborhood of a point $d\alpha$ in $(\mathfrak{M}_1, *, \|\cdot\|)$ is given by

$$\{d\beta \in \mathfrak{M}_1 : \|d\beta * d\alpha^{-1} - \delta\|_x < \varepsilon\},$$

while a neighborhood of $(d\nu)$ in $(\overline{\mathfrak{M}}, +, \|\cdot\|)$ is given by

$$\{(d\mu) \in \overline{\mathfrak{M}} : \min_{d\nu \in (d\nu)} \|d\mu - d\nu\|_x < \varepsilon\}.$$

With proper identification, convergence in $(\overline{\mathfrak{M}}, +, \|\cdot\|)$ is equivalent to con-

vergence in \mathfrak{M} . It is easy to see that the convergence of a sequence of elements in $(\mathfrak{M}_1, *, \|\cdot\|)$ implies its convergence in \mathfrak{M} . However, the example δ/n shows the converse to be false. The same remarks hold for \mathfrak{R} except that cosets are unnecessary.

The following result, the representation theorem, shows the pair of \mathfrak{M} (respectively, \mathfrak{R}) groups to be algebraic and topological copies of one another.

2.6. The exponential representation.

THEOREM 2.6. *Let $E : (\overline{\mathfrak{M}}, +, \|\cdot\|) \rightarrow (\mathfrak{M}_1, *, \|\cdot\|)$ be defined for $(d\lambda) \in \overline{\mathfrak{M}}$ by $E((d\lambda)) = e^{d\lambda}$ where $d\lambda$ is a representative of $(d\lambda)$ in \mathfrak{M} . E is a group isomorphism and a homeomorphism.*

Let $da \in \mathfrak{M}_1$, $da\{1\} = k \neq 0$. The measure $d\lambda$ satisfies $e^{d\lambda} = da$ iff $d\lambda$ is given by either of the following equivalent conditions:

$$(2.6a) \quad L d\lambda = L da * da^{-1}, \quad d\lambda\{1\} = \log k$$

$$(2.6b) \quad d\lambda = \log da.$$

The symbolic expression (2.6b) has the following meaning: For any c satisfying $|1 - k/c| < 1$,

$$(2.6b') \quad d\lambda = (\log c)\delta - \sum_{n=1}^{\infty} (1/n)(\delta - da/c)^n = (\log c)\delta + \log (da/c).$$

*Moreover, $E_{\mathfrak{R}}$, the restriction of E to \mathfrak{R} , maps $(\mathfrak{R}, +, \|\cdot\|)$ isomorphically and homeomorphically onto $(\mathfrak{R}_{1p}, *, \|\cdot\|)$.*

Proof. First, E is a well-defined map from $\overline{\mathfrak{M}}$ into \mathfrak{M}_1 . For $(d\lambda) \in \overline{\mathfrak{M}}$, the expression $e^{d\lambda}$ is convergent in \mathfrak{M} by Proposition 2.1. The range of E is contained in \mathfrak{M}_1 since $e^{d\lambda}\{1\} = e^{d\lambda\{1\}} \neq 0$. If $d\lambda'$ is another representative of $(d\lambda)$, then $d\lambda' = d\lambda + 2\pi n i \delta$ and (by the next paragraph)

$$e^{d\lambda + 2\pi i n \delta} = e^{d\lambda} * e^{2\pi i n \delta} = e^{d\lambda}.$$

We now show $E((d\lambda + d\mu)) = E((d\lambda)) * E((d\mu))$. The series for $e^{d\lambda}$, $e^{d\mu}$, $e^{d\lambda + d\mu}$, and $e^{d\lambda} * e^{d\mu}$ converge absolutely in \mathfrak{M} . Thus we may sum the doubly infinite matrix

$$\left\{ \frac{d\lambda^m}{m!} * \frac{d\mu^n}{n!} \right\}_{m,n=1}^{\infty}$$

by rectangles to yield $e^{d\lambda} * e^{d\mu}$ or by triangles (Cauchy product) to yield $e^{d\lambda + d\mu}$.

By Corollary 2.2, the exponential is a continuous map from \mathfrak{M} to \mathfrak{M}_1 and remains so in the topology of $(\overline{\mathfrak{M}}, +, \|\cdot\|)$. Thus E is continuous.

Suppose $e^{d\lambda} = da$, $da\{1\} = k \neq 0$. Then $k = e^{d\lambda}\{1\} = e^{d\lambda\{1\}}$ yields $d\lambda\{1\} = \log k \pmod{2\pi i}$. By (2.3a), $Lda = Le^{d\lambda} = Ld\lambda * da$. Convolving both sides of the last expression by da^{-1} , we see that (2.6a) holds.

To show E is 1-1, it suffices to show $e^{d\lambda} = \delta \Rightarrow (d\lambda) = 0$. If $e^{d\lambda} = \delta$, (2.6a)

implies $L d\lambda = L\delta = 0$. Thus support $d\lambda = \text{support } \delta$, and $d\lambda = c\delta$, where c is a constant which may be found by evaluation at 1. That is,

$$1 = \delta\{1\} = e^{d\lambda(1)} \Rightarrow d\lambda\{1\} = \log 1 = 2\pi ni,$$

n some integer. Thus $d\lambda = 2\pi ni\delta$ or $(d\lambda) = 0$.

We must prove that the conditions (2.6a) and (2.6b') are equivalent. First we note that, for $da \in \mathfrak{M}_1$ with $da\{1\} = k$ and c satisfying $|1 - k/c| < 1$, the following equation holds:

$$(2.6c) \quad L \left\{ (\log c)\delta - \sum_{n=1}^{\infty} (1/n) \left(\delta - \frac{da}{c} \right)^n \right\} \\ = \sum_{n=1}^{\infty} (\delta - da/c)^{n-1} * L da/c = L da * da^{-1}.$$

The first equality is obtained by applying L under the summation sign, the second by summing $\sum (\delta - da/c)^{n-1}$ to $c da^{-1}$. If (2.6a) is valid, then by (2.6c) we have

$$L\{(\log c)\delta - \sum (1/n)(\delta - da/c)^n\} = L d\lambda.$$

This equality proves (2.6b') for $(1, \infty)$ and direct evaluation at 1 establishes (2.6b') at that point. Conversely, if (2.6b') holds, then $d\lambda\{1\} = \log k$, and we read from (2.6c) that $L d\lambda = L da * da^{-1}$, proving the validity of (2.6a).

Because of the $\log^{-1} t$ factor in its representation, it is not evident *a priori* that a set function $d\lambda$ defined by (2.6a) lies in \mathfrak{M} . However, it is not hard to see that $d\lambda$ as given by (2.6b') has finite total variation on any compact set and thus is an element of \mathfrak{M} .

Now suppose $d\lambda$ satisfies (2.6a). We show that $e^{d\lambda} = da$ from which we deduce that E maps onto \mathfrak{M}_1 . Our method is to prove that $L(e^{d\lambda} * da^{-1}) = 0$. This implies that $e^{d\lambda} * da^{-1} = c\delta$ and by evaluation at 1 with the second condition of (2.6a), one sees that $c = 1$. We need the following equation, obtained by expanding $0 = L\delta = L(da * da^{-1})$:

$$(2.6d) \quad da * L da^{-1} = -da^{-1} * L da.$$

Now

$$L(e^{d\lambda} * da^{-1}) = e^{d\lambda} * (L d\lambda * da^{-1} + L da^{-1}) \\ = e^{d\lambda} * da^{-1} * (L d\lambda + L da^{-1} * da) = 0,$$

where we have successively used (2.3a) to evaluate $Le^{d\lambda}$, (2.6d) to transform $L da^{-1} * da$, and the hypotheses (2.6a).

Given $da \in \mathfrak{M}_1$, define a measure $d\lambda$ by (2.6a). By the above paragraph $e^{d\lambda} = da$, and thus there exists $(d\lambda) \in \mathfrak{M}$ such that $E((d\lambda)) = da$, which shows that E maps onto \mathfrak{M}_1 .

We now show E^{-1} is continuous. Let $da_n \rightarrow da$ in $(\mathfrak{M}_1, *, \|\cdot\|)$. This means $da_n * da^{-1} \rightarrow \delta$ in \mathfrak{M} . In particular, for sufficiently large n we have

$| (da_n\{1\})/c - 1 | < 1$ where $c = da\{1\}$. For such n ,

$$d\lambda_n = (\log c)\delta - \sum_{j=1}^{\infty} (1/j)(\delta - da_n/c)^j$$

converges to $d\lambda = (\log c)\delta - \sum_1^{\infty} (1/j)(\delta - da/c)^j$ by Corollary 2.2. Thus E^{-1} is continuous.

Now $(\mathfrak{R}, +, \|\cdot\|)$ can be regarded as a closed subgroup of $(\overline{\mathfrak{M}}, +, \|\cdot\|)$ and thus $E_{\mathfrak{R}}(\mathfrak{R})$ is a closed subgroup of $(\mathfrak{M}_1, *, \|\cdot\|)$. We claim that $E_{\mathfrak{R}}(\mathfrak{R}) = (\mathfrak{R}_{1p}, *, \|\cdot\|)$. Indeed,

$$d\lambda \in \mathfrak{R} \Rightarrow e^{d\lambda} \in \mathfrak{R}_{1p} \quad \text{and} \quad da \in \mathfrak{R}_{1p} \Rightarrow \log (da) \in \mathfrak{R}$$

(with proper choice of the branch of log). It is clear that $E_{\mathfrak{R}}$ is an isomorphism and a homeomorphism. This completes the proof of the representation theorem.

Remarks. The representation theorem yields certain canonical decompositions for measures in \mathfrak{M}_1 .

If $da \in \mathfrak{M}_1$ with $da = e^{d\lambda}$, we decompose $d\lambda$ into real and imaginary parts $d\lambda = d\alpha + i d\beta$. Then $da = dA * dB$, where $dA = e^{d\alpha} \in \mathfrak{R}_{1p}$ and $dB = e^{id\beta} = \cos (d\beta) + i \sin (d\beta)$, giving a representation for \mathfrak{M}_1 by measures in "polar form." If $\lim_{x \rightarrow \infty} \|d\beta\|_x < \infty$, $|\int_1^{\infty} dB| = 1$.

If $da \in \mathfrak{R}_{1p}$ with $da = e^{d\lambda}$, we make a Jordan decomposition of $d\lambda$ into a difference of two non-negative mutually singular measures, $d\lambda = d\alpha - d\beta$. Let $e^{d\alpha} = dA$ and $e^{d\beta} = dB$. Then $da = dA * dB^{-1}$, and every member of \mathfrak{R}_{1p} is expressible as a quotient of positive exponentials.

By analogy with common fractions and meromorphic functions, we call a measure in \mathfrak{R}_{1p} integral if its quotient representation has a unit denominator. $c\delta = e^{(\log c)\delta}$ is an integral measure iff $c \geq 1$. More significant examples of integral measures, as we shall see, are dn , the counting measure of positive integers, and $\delta + dt$, where dt is Lebesgue measure on $[1, \infty)$. Integral measures are clearly non negative and form a semi group under convolution.

An important property of an integral measure is that it majorizes its inverse: If $da = e^{d\lambda}$, $d\lambda \geq 0$; $da^{-1} = e^{-d\lambda}$ and

$$(2.6e) \quad da^{-1} = \sum (-d\lambda)^n/n! < \sum d\lambda^n/n! = da.$$

From the exponential form we see that any measure $da \in \mathfrak{M}_1$ possesses precisely n distinct n^{th} roots in \mathfrak{M}_1 , where an n^{th} root of da is a measure db satisfying $db^n = da$. For a fixed determination of $\log(da)$ they are given by $db_k = \omega_k e^{(1/n)\log da}$, ω_k an n^{th} root of unity, $k = 0, 1, 2, \dots, n - 1$. It is routine to check that these are n^{th} roots, are distinct, and include all n^{th} roots. Note that an integral measure has an integral n^{th} root.

3. Distribution of generalized integers

3.1. Representation of dN as an exponential. By Theorem 2.6 we know that dN may be written as the exponential of some measure: We show this

measure to be $d\Pi$ in

PROPOSITION 3.1. $dN = e^{d\Pi}$

Proof. It suffices to show

$$(3.1a) \quad d\psi * dN = L dN.$$

If this expression is valid, convolution by dN^{-1} will yield $L d\Pi = d\psi = L dN * dN^{-1}$ which by (2.6a) implies $dN = e^{d\Pi}$.

It is clear that the support of the measures in both sides of (3.1a) is contained in $\{n_i\}$, the set of g -integers. Thus it suffices to prove (3.1a) is valid at each $l \in \{n_i\}$.

In case $dN\{l\} = 1$, l is uniquely expressible as $l = \prod_i p_i^{\alpha_i}$ and we may use the following classical argument to prove (3.1a).

$$(d\psi * dN)\{l\} = \sum_{jk=l} d\psi\{j\} dN\{k\}$$

where the sum need only extend over such pairs (j, k) for which $j = p_i^\beta$, $1 \leq \beta \leq \alpha_i$ and $k = l/j$. For all other (j, k) with $jk = l$ at least one of $d\psi\{j\}$, $dN\{k\}$ vanishes. Since unique factorization holds for l , it holds for all divisors of l . Thus we have $d\psi\{p_i^\beta\} = \log p_i$, $1 \leq \beta \leq \alpha_i$ and $dN\{l/p_i^\beta\} = 1$, $\beta \leq \alpha_i$. Thus

$$(d\psi * dN)\{l\} = \sum_i \sum_{j=1}^{\alpha_i} \log p_i = \sum \alpha_i \log p_i = \log l.$$

This proves (3.1a) in case $dN\{l\} = 1$.

Where we do not have unique factorization, it is possible for a number to divide a product without dividing any of the factors. For example, consider the set of g -primes $2, \sqrt{6}, 3, \dots$ for which the g -prime $\sqrt{6}$ divides $2 \cdot 3$ but divides neither factor. To facilitate discussion in the case of nonunique factorization, we make the following definition. We say q^β belongs to the factorization $\prod p_i^{\alpha_i}$ and write $q^\beta \in \prod p_i^{\alpha_i}$ if $q = p_k$, some k , and $1 \leq \beta \leq \alpha_k$. The above example shows that a prime power may belong to one factorization of a number and not belong to another.

We now prove (3.1a) holds at l in the case $dN\{l\} = \nu > 1$. We could make a slight "perturbation" of the g -primes to achieve unique factorization, use the result we have just proved in this case, and pass to the limit. Such a proof would require a weaker topology than that of $(\mathfrak{N}, \|\cdot\|)$, and it is more convenient to give instead a combinatorial proof.

Let $\prod_i p_i^{\alpha_{ij}}$, $j = 1, \dots, \nu$, be the ν representations of l . Let (s, t) be a pair for which $st = l$ and $d\psi\{s\} dN\{t\} \neq 0$, i.e. $s = q_1^{\beta_1} = \dots = q_\mu^{\beta_\mu}$, $\mu \geq 1$ and $t = \prod_i q_i^{\gamma_i} = \dots = \prod_i q_i^{\gamma_i}$, $\lambda \geq 1$. The contribution to $d\psi * dN$ at (s, t) is $\lambda \sum_{k=1}^\mu \log q_k$. By assigning the value $\log q_k$ to the representation $(q_k^{\beta_k}, \prod_i q_i^{\gamma_{ij}})$ and summing over all representations of (s, t) , we also get the number $\lambda \sum_{k=1}^\mu \log q_k$. Thus $\sum_{st=l} d\psi\{s\} dN\{t\} = \sum \sum \log q$ where the inner sum is over $\{q^\beta : q^\beta \in \prod_i p_i^{\alpha_{ij}}\}$ and the outer sum is over $j = 1, \dots, \nu$. By the proof in the case of unique factorization, the inner sum is $\log l$. This establishes (3.1a) and completes the proof of Proposition 3.1.

3.2. Lebesgue measure as an exponential. For c a fixed real number, define a map $T_c : \mathfrak{M} \rightarrow \mathfrak{M}$ by $\int_{\mathfrak{E}} T_c d\alpha = \int_{\mathfrak{E}} t^c d\alpha(t)$. From the definition of convolution and the distributivity of multiplication over addition, T_c is an algebra isomorphism of $(\mathfrak{M}, +, *)$ into itself.

We denote by dt Lebesgue measure on the Borel subsets of $[1, \infty)$. The measure $T_{-1} dt$ is translation invariant on the multiplicative semigroup of real numbers greater than or equal to 1, and therefore has the following property:

LEMMA. For any $d\alpha \in \mathfrak{M}$,

$$(3.2a) \quad d\alpha * T_{-1} dt = \alpha(t) dt/t.$$

Proof. $\int_1^x d\alpha * dt/t = \int_1^x \alpha(x/t) dt/t = \int_1^x \alpha(u) du/u$. Since the distribution functions coincide, so must the measures.

LEMMA.

$$(3.2b) \quad (dt)^n = \log^{n-1} t dt / (n-1)!, \quad n = 1, 2, \dots$$

Proof. We will show by induction on n that $dt^n/t = \log^{n-1} t dt/t(n-1)!$ which implies the lemma. The equality for dt^n/t is true for $n = 1$, and we assume its truth for the n^{th} case. We apply successively the isomorphic property of the T_{-1} map, the inductive hypothesis, and the formula for convolution by dt/t from the previous lemma, giving

$$\frac{(dt)^{n+1}}{t} = \frac{(dt)^n}{t} * \frac{dt}{t} = \frac{\log^{n-1} t dt}{(n-1)! t} * \frac{dt}{t} = \frac{\log^n t dt}{n! t}.$$

This proves the truth of the $(n+1)^{\text{st}}$ case, whence the lemma.

LEMMA.

$$(3.2c) \quad (\delta + dt) * (\delta - dt/t) = \delta.$$

Proof. This formula for the inverse of $\delta + dt$ can be verified by noting with the help of Lemma 3.2a that $dt/t * (\delta + dt) = dt$. Without prior knowledge of the result, we could obtain the formula from the proof of existence of inverses (Section 2.4) by summing the series for $\sum_0^\infty (-dt)^n$ with the aid of the previous lemma.

Define a measure $d\tau$ by $\tau(x) = \int_1^x (1 - t^{-1}) dt/\log t$.

LEMMA.

$$(3.2d) \quad \delta + dt = e^{d\tau}$$

Proof. By (2.6a) it suffices to show that

$$(3.2e) \quad \{L(\delta + dt)\} * (\delta - dt/t) = L d\tau.$$

The left hand side of (3.2e) = $L dt * (\delta - dt/t) = dt * dt * (\delta - dt/t) = dt * \{(\delta + dt) - \delta\} * (\delta - dt/t) = dt - dt * (\delta - dt/t) = dt * dt/t = (t-1) dt/t =$ the right hand side of (3.2c).

This establishes the lemma. Another method of proof would be to evaluate $\log(\delta + dt)$ with the aid of Lemma 3.2b.

An example of $\sim R$. H. Malliavin [3] showed that the Riemannian hypothesis could not hold for g -numbers, even if $N(x) - x = O(1)$. We show how to simplify his example by use of the above representation. For $c \in (0, \frac{1}{2})$, we consider the "zeta function"

$$\frac{s-1+c}{s-1} = \int x^{-s}(\delta + c dx), \quad \text{Re } s > 1,$$

whose continuation has a zero at $s = 1 - c > \frac{1}{2}$. Now

$$\int_1^x (\delta + c dt) = cx + O(1)$$

and

$$\delta + c dx = e^{d\tau_c},$$

where the measure $d\tau_c$ is defined by

$$\tau_c(x) = \int_1^x \frac{1-t^{-c}}{\log t} dt.$$

$\tau_c(x)$ is an increasing function and

$$\tau_c(x) = li(x) - \frac{x^{1-c}}{(1-c)\log x} + O\left(\frac{x^{1-c}}{\log^2 x}\right).$$

Thus we have represented the "integer counting measure" by an exponential of a positive measure, and the Riemannian hypothesis is false, at least for such general types of "number systems."

3.3. The distribution of generalized integers. We now have the machinery with which to prove our main theorem, which asserts that if the g -primes are well distributed, then so too are the g -integers. Although formulated with the measures $d\Pi$ and dN associated with generalized numbers, the theorem is valid for an arbitrary integral measure dN . After proving the theorem we shall consider the effect of a hypothesis upon the behavior of $\Pi(x)$ or $\psi(x)$ instead of $\int_1^x d\Pi/t$.

THEOREM 3.3a. *Suppose there exist positive numbers c and α such that for large x the following inequalities hold:*

$$\int_1^x d\Pi(t)/t = \int_1^x (1-t^{-1}) dt/(t \log t) + \log c + O(\log^{-\alpha} x).$$

Then $N(x) = cx + O(x \log^{2-\alpha} x)$.

THEOREM 3.3b. *Suppose there exist numbers $c > 0$ and $a \in (0, 1)$ such that*

$$\int_1^x d\Pi(t)/t = \int_1^x (1-t^{-1}) dt/(t \log t) + \log c + O\{\exp(-\log^a x)\}.$$

Then

$$N(x) = cx + O\{x \exp(-[\log x \log \log x]^{a'})\}, \quad a' = a/(1+a).$$

Proof. Let $\nu_j(x) = \int_1^x t^{-1} \{d\Pi - d\tau - (\log c)\delta\}^j$ and let $\nu_1(x) = \nu(x)$. We have

$$dN = e^{d\Pi} = c\delta * (\delta + dt) * e^{d\Pi - d\tau - (\log c)\delta} = c(\delta + dt) * e^{td\nu(t)}$$

and

$$\begin{aligned} N(x) &= c \int_1^x \left\{ \int_1^{x/t} (\delta + du) \right\} e^{td\nu(t)} = c \int_1^x \frac{x}{t} e^{td\nu} \\ &= cx \int_1^x t^{-1} \sum_{j=0}^{\infty} \{td\nu(t)\}^j / j! = cx \sum_{j=0}^{\infty} \nu_j(x) / j!. \end{aligned}$$

Thus we have

$$(3.3c) \quad N(x) - cx = cx \sum_{j=1}^{\infty} \nu_j(x) / j!,$$

and we will show the sum to be suitably small.

The essential step in the proof of (3.3a) is to show that there exist constants A_0 and A_1 , independent of n , such that

$$|\nu_n(x)| \leq nA_0(2 \log \log x + A_1)^{n-1} \log^{-\alpha} x.$$

The theorem follows by inserting this estimate into (3.3c). We shall not pursue further the proof of (3.3a) but turn our attention to estimating $\nu_n(x)$ for Theorem 3.3b.

Define $\lambda(x) = \int_1^x d\tau(t)/t = \int_1^x (1 - t^{-1}) dt/(t \log t)$. Note that for $x \geq e$,

$$\lambda(x) \leq \int_1^e dt/t + \int_e^x dt/(t \log t) = 1 + \log \log x.$$

LEMMA. Suppose that for some $b \geq 0$ and $a \in (0, 1)$,

$$|\nu(x)| \leq A_0 \log^b ex \exp\{- (\log x)^a\}, \quad 1 \leq x < \infty.$$

Then there exists a constant A_1 such that for $1 \leq x < \infty$,

$$(3.3d) \quad |\nu_n(x)| \leq nA_0(2\lambda(x) + A_1)^{n-1} \log^b ex \exp\{-(n^{-1} \log x)^a\}.$$

Remark. The *raison d'être* of the factor $\log^b ex$ is Corollary 3.3.

Proof (by induction). The case $n = 1$ is true. Assume the truth of the n^{th} case, and set $y = x^{1/(n+1)}$ and $z = x/y$.

$$\begin{aligned} \nu_{n+1}(x) &= \iint_{st \leq x} d\nu^n(s) d\nu(t) \\ &= \int_1^y \nu_n(x/t) d\nu(t) + \int_1^x \nu(x/t) d\nu^n(t) - \nu(y)\nu_n(z). \end{aligned}$$

$$\begin{aligned}
& \left| \int_1^y \nu_n(x/t) \, d\nu(t) \right| \\
& \leq \int_1^y nA_0(2\lambda(x/t) + A_1)^{n-1} \log^b(ex/t) \exp \{-(n^{-1} \log(x/t))^a\} |d\nu|(t) \\
& \leq nA_0(2\lambda(x) + A_1)^{n-1} \log^b ex \exp \{-(n^{-1} \log z)^a\} \int_1^y |d\nu| \\
& \leq nA_0(2\lambda(x) + A_1)^{n-1} \log^b ex \exp \{-(n+1)^{-1} \log x)^a\} \\
& \quad \cdot (2\lambda(x) + B + 2 |\log c|),
\end{aligned}$$

where $B = \sup |\nu(x)| < \infty$.

$$\begin{aligned}
& \left| \int_1^z \nu(x/t) \, d\nu^n(t) \right| \\
& \leq \int_1^z A_0 \log^b(ex/t) \exp \{-(\log x/t)^a\} |d\nu|^n(t) \\
& \leq A_0 \log^b ex \exp \{-(\log x/z)^a\} \left(\int_1^z |d\nu| \right)^n \\
& \leq A_0 \log^b ex \exp \{-(n+1)^{-1} \log x)^a\} (2\lambda(x) + B + 2 |\log c|)^n. \\
& |\nu(y)\nu_n(z)| \leq BnA_0(2\lambda(x) + A_1)^{n-1} \log^b ex \exp \{-(n+1)^{-1} \log x)^a\}.
\end{aligned}$$

Thus

$$|\nu_{n+1}(x)| \leq (n+1)A_0(2\lambda(x) + A_1)^n \log^b ex \exp \{-(n+1)^{-1} \log x)^a\}$$

with $A_1 = 2B + 2 |\log c|$.

Proof of the theorem.

$$\begin{aligned}
& |N(x) - cx|/(cx) \\
& \leq \sum_1^\infty |\nu_n(x)|/n! \\
& \leq A_0 \sum_1^\infty n(2\lambda(x) + A_1)^{n-1} \exp \{-(n^{-1} \log x)^a\}/n! = \sum_1^K + \sum_{K+1}^\infty.
\end{aligned}$$

To estimate the first sum we maximize

$$\exp \{-(n^{-1} \log x)^a\}/(n-1)!$$

on n . For ε a fixed arbitrarily small positive number, there exists x_0 such that for all $x \geq x_0$ and all n ,

$$\begin{aligned}
(3.3e) \quad & \exp \{-(n^{-1} \log x)^a\}/(n-1)! \\
& < \exp \{(\varepsilon - (1+a)^{1/(1+a)})(\log x \log \log x)^{a'}\}.
\end{aligned}$$

Let $A(\varepsilon, x, a) = A(\varepsilon)$ denote the right hand side of (3.3e).

$$\sum_1^K < A_0 A(\varepsilon) \cdot 2(2\lambda(x) + A_1)^{K-1},$$

provided $2\lambda(x) + A_1 \geq 2$.

$$\sum_{K+1}^\infty < A_0 \sum_K^\infty (2\lambda(x) + A_1)^n/n! \leq 2A_0(2\lambda(x) + A_1)^K/K!$$

provided $K \geq 2(2\lambda(x) + A_1)$.

Take $K = (\log x)^{a'}$, and recall that $\lambda(x) = O(\log \log x)$. For ε' any number greater than ε , there exists an x_1 such that for $x \geq x_1$, we have $\sum_1^K + \sum_{K+1}^\infty < A(\varepsilon')$. Choose ε and ε' such that $(1 + a)^{1/(1+a)} - \varepsilon' > 1$. This proves (3.3b) with "a little to spare."

COROLLARY 3.3. *Let $d\Pi \geq 0$ and*

$$(3.3f) \quad \Pi(x) = \tau(x) + O(x \exp \{- (\log x)^a \}).$$

Then if $a' = a/(1 + a)$, there exists a constant c such that

$$N(x) = cx + O(x \exp \{- (\log x \log \log x)^{a'} \}).$$

Remark. (3.3f) is the hypothesis that appears in Malliavin's paper [3].

Proof of Corollary 3.3.

$$\begin{aligned} \int_1^x t^{-1}(d\Pi - d\tau) &= (\Pi(x) - \tau(x))/x + \int_1^x (\Pi(t) - \tau(t)) dt/t^2 \\ &= O\{e^{-(\log x)^a}\} + \left(\int_1^\infty - \int_x^\infty \right) (\Pi(t) - \tau(t)) dt/t^2 \\ &= O\{e^{-(\log x)^a}\} + \log c + O\left\{ \int_x^\infty e^{-(\log t)^a} dt/t \right\}. \end{aligned}$$

The last integral is asymptotic to $(1/a)(\log x)^{1-a}e^{-(\log x)^a}$ as may be seen by use of l'Hospital's rule. Thus (3.3f) implies that there is a number $\log c$ such that

$$(3.3g) \quad \int_1^x t^{-1}\{d\Pi - d\tau - (\log c)\delta\} = O\{(\log x)^{1-a}e^{-(\log x)^a}\}.$$

The left hand side of (3.3g) is the function $\nu(x)$. We estimate $\nu_n(x)$ by (3.3d), taking $b = 1 - a$. Proceeding with the proof of the theorem as before, we obtain

$$|N(x) - cx|/x < (\log x)^{1-a}A(\varepsilon', x, a), \quad x \geq x_1.$$

We may absorb the $(\log x)^{1-a}$ factor into A by choosing an $\varepsilon'' > \varepsilon'$ but still so small that $(1 + a)^{1/(1+a)} - \varepsilon'' \geq 1$. Thus, there exists x_2 such that for $x \geq x_2$,

$$|N(x) - cx|/x < A(\varepsilon'', x, a) \leq \exp \{- (\log x \log \log x)^{a'} \}, \quad a' = a/(1 + a).$$

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