ON THE FIRST MAIN THEOREM ON BLOCKS OF CHARACTERS OF FINITE GROUPS

BY
RICHARD BRAUER

1. Introduction[1]

Let G be a finite group and let p be a fixed prime number. The first main theorem on blocks establishes a one-to-one correspondence between the p-blocks B of G with the defect groups D and the p-blocks b of the normalizer $N_G(D)$ of D with the defect group D, cf. [2], [3]. It is the purpose of this note to show that this theorem can be derived easily from the results of [4]. We shall need only the results (2A), (2B), (3A), (3B) and (4A) of [4]. In particular, we shall not need any results dealing with fields of characteristic 0. A proof of the main theorem on blocks operating completely within a fixed field Ω of characteristic p has already been given by A. Rosenberg [5].

We use the same notation as in [4]. In particular, Ω will denote an algebraically closed field of characteristic p, $\Omega[G]$ will denote the group algebra of G over Ω, and $Z = Z(G)$ will be the class algebra of G over Ω (i.e., $Z(G)$ is the center of $\Omega[G]$). As remarked in [4], the results (3A), (3B), and (4A) of [4] remain valid, if instead of the decomposition of $Z(G)$ into block ideals we consider more generally any decomposition

$$Z = A_1 \oplus A_2 \oplus \cdots \oplus A_r \tag{1}$$

as a direct sum of ideals A_i. Each A_i is a direct sum of block ideals of Z. Let $\hat{Z}$ denote the dual space consisting of all linear functions defined on Z with values in Ω.

An ideal $A \neq (0)$ occurs as a summand in a decomposition (1), if and only if A has the form $\eta_A Z$ where η_A is an idempotent of Z. We may consider the dual space $\hat{A}$ of A as a subspace of $\hat{Z}$ by extending each $f \in \hat{A}$ linearly so that it vanishes on the complement $(1 - \eta_A)A$. (In [4], the notation F_A was used for this subspace of Z.) If Q is a p-subgroup of G, the multiplicity $m_A(Q)$ of Q as a lower defect group of A is defined as follows. Consider subspaces V of $\hat{A}$ with the following two properties.

(i) For each $f \neq 0$ in V, there exists a conjugate class K of G with the defect group Q such that $f(\mathcal{S}K) \neq 0$. (Here $\mathcal{S}K$ is the class sum of K.)

(ii) For $f \in V$, we have $f(\mathcal{S}K) = 0$ for all conjugate classes K of G whose defect group has lower order than Q.

Then $m_A(Q)$ is the maximal dimension of such Ω-spaces V.

Received November 24, 1969.

[1] The note was written while the author was a Visiting Professor at the University of Chicago. Conversations with John G. Thompson and Richard G. Swan have been helpful.

If $\mathcal{P} = \mathcal{P}(G)$ is a system of representatives for the classes of conjugate p-subgroups of G, the system $\mathfrak{D}(A)$ of lower defect groups of A consists of the members Q of $\mathcal{P}$, each Q taken with the multiplicity $m_A(Q) \geqq 0$. As already indicated, the results (3A), (3B), (4A) of [4] remain valid for arbitrary decompositions (1). This implies that the number of elements of $\mathfrak{D}(A)$ is equal to $\dim_\Omega A$. Another consequence is the following proposition.

(1A) *Let A be an ideal of Z which is a direct summand of Z. If A is a direct sum of two ideals $A_1 \oplus A_2$ then*

$$\mathfrak{D}(A) = \mathfrak{D}(A_1) \cup \mathfrak{D}(A_2).$$

Here, the union is meant as the union of *systems* of elements: The multiplicity of Q as member of $\mathfrak{D}(A_1) \cup \mathfrak{D}(A_2)$ is the sum of the multiplicities of Q in $\mathfrak{D}(A_1)$ and $\mathfrak{D}(A_2)$.

2. Proof of the first main theorem on blocks

Consider a block ideal B of Z. Then $\hat{B}$ contains a unique algebra homomorphism ω_B of Z onto Ω.

DEFINITION. A p-subgroup D of G is an (ordinary) *defect group* of the block B, if there exist conjugate classes K_0 of G with the defect group D for which $\omega_B(\mathcal{S}K_0) \neq 0$ while $\omega_B(\mathcal{S}K) = 0$ for all conjugate classes K whose defect group has smaller order than D.

It is clear that every defect group D of B is a lower defect group of B; $D \in \mathfrak{D}(B)$.

If $Q \in \mathfrak{D}(B)$, it follows from [4], (3A) and (4A) that there exist blocks b of $N_G(Q)$ such that $B = b^G$. Then for each conjugate class K of G, we have

$$\omega_B(\mathcal{S}K) = \textstyle\sum_L \omega_b(\mathcal{S}L) \tag{2}$$

where L ranges over all conjugacy classes L of $N_G(Q)$ with $L \subseteq K \cap C_G(Q)$. In particular, this holds when $Q = D$ is a defect group of B. If $\omega_B(\mathcal{S}K) \neq 0$, then (2) shows that $K \cap C_G(D) \neq \emptyset$. This implies that there exist defect groups of K which contain D. Thus, we have (cf. [3, (8A)]).

(2A) *If D is a defect group of the block B of G and if $\omega_B(\mathcal{S}K) \neq 0$ for a conjugate class K of G, then D is contained in a defect group of K.*

The following proposition is an immediate consequence.

(2B) *The defect groups of a block are determined uniquely up to conjugacy.*

If Q_1 and Q_2 are subgroups of G, as in [4] we define $Q_1 \gtrsim Q_2$ to mean that Q_1 contains a conjugate of Q_2. In particular, this relation defines a partial order in $\mathcal{P}$.

(2C) *There exists a unique maximal element D in $\mathfrak{D}(B)$ and D is a defect group of B.*

Proof. We have already seen that $\mathfrak{D}(B)$ contains a defect group D of B. If $Q \in \mathfrak{D}(B)$, (2) applies for a block b of $N_G(Q)$ with $b^G = B$. Take for K the class K_0 in the definition of D. Then $\omega_B(\mathcal{S}K_0) \neq 0$ and, by (2), $K_0 \cap C_G(Q) \neq \emptyset$. This implies that Q is contained in a conjugate of the defect group D of K_0. Hence $D \gtrsim Q$ as we had to show.

(2D) (First main theorem). *Let Q be a p-subgroup of G and set $N = N_G(Q)$. The mapping*

$$b \rightarrow b^G = B$$

sets up a one-to-one correspondence between the blocks b of N with the defect group Q and the blocks B of G with the defect group Q.

Proof. We may assume that $Q \in \mathcal{P}(G)$. We have $Q \in \mathcal{P}(N)$. A simple group theoretical argument [3, (10A)] shows that if L is a conjugate class of N with the defect group Q, the conjugate class $K = L^G$ of G containing L has also the defect group Q. Conversely, if K is a conjugate class of G with the defect group Q, then $K \cap C_G(Q) = L$ is a conjugate class of N with the defect group Q and $K = L^G$.

If b is a block of N with the defect group Q, there exist conjugate classes L_0 of N with the defect group Q such that $\omega_b(\mathcal{S}L_0) \neq 0$. If $B = b^G$ and $K = L_0^G$, then (2) reads

$$\omega_B(\mathcal{S}K) = \omega_b(\mathcal{S}L_0) \neq 0. \tag{3}$$

On account of (2A), the defect group Q of K contains a conjugate of the defect group D of B. Thus $Q \gtrsim D$.

On the other hand, choose K_0 as in the definition of the defect group of a block. Then (2) with $K = K_0$ shows that there exist conjugate classes $L \subseteq K_0 \cap C_G(Q)$ of N with $\omega_b(\mathcal{S}L) \neq 0$. On account of (2A) applied to N and b, a defect group of L in N contains the defect group Q of b. Then a defect group of $K_0 \supseteq L$ in G contains Q. Hence $D \gtrsim Q$. Thus, D is conjugate to Q, and B has the defect group Q.

Suppose now that b_1 and b_2 are two distinct blocks of $N_G(Q)$ with the defect group Q. Then $A = b_1 \oplus b_2$ is a direct summand of the class algebra $Z(N)$. By (1A) and (2C) we see that $\mathfrak{D}(A)$ has the unique maximal element Q.

On the other hand, $f = \omega_{b_1} - \omega_{b_2}$ is an element of $\hat{A} \subseteq \hat{Z}(N)$. Let L be a conjugate class of N of minimal defect for which $f(\mathcal{S}L) \neq 0$. Then a defect group Q_1 of L in N belongs to $\mathfrak{D}(A)$. By (2C), Q_1 is contained in a conjugate of Q in N and hence $Q_1 \subseteq Q$. Since $f(\mathcal{S}L) \neq 0$ implies that $\omega_{b_1}(\mathcal{S}L) \neq 0$ or $\omega_{b_2}(\mathcal{S}L) \neq 0$, the defect group Q_1 of L in N contains a conjugate of the defect group Q of b_1 and b_2, cf. (2A). Thus $Q_1 = Q$; i.e., L has the defect group Q in N.

Suppose now that $b_1^G = b_2^G = B$. For $K = L^G$, as in (3) we have

$$\omega_B(\mathcal{S}K) = \omega_{b_i}(\mathcal{S}L)$$

for $i = 1$ and 2. However, this is impossible as

$$\omega_{b_1}(\mathcal{S}L) - \omega_{b_2}(\mathcal{S}L) = f(\mathcal{S}L) \neq 0.$$

Thus, $b_1^G \neq b_2^G$.

Our proof will be complete if we can show that if B is a block of G with the defect group Q, there exist blocks b of N with the defect group Q for which $b^G = B$. Since $Q \in \mathfrak{D}(B)$, there certainly exist blocks b of N with $b^G = B$. If K_0 is chosen as in the definition of the defect group of B, then K_0 has the defect group Q and (3) applies with $L_0 = K_0 \cap C_G(Q)$. Hence $\omega_b(\mathcal{S}L_0) \neq 0$. Now (2A) implies that the defect group Q of L_0 in N contains a conjugate of the defect group D^* of b. On the other hand, we have the Lemma [3, (9F)].

LEMMA 1. *Suppose that a finite group H has a normal p-subgroup Q. Then the defect group of each block of H contains* Q.

If this lemma is applied to $H = N$, we find $D^* \supseteq Q$. Hence $D^* = Q$ and the proof of (2D) is complete.

As to the proof of Lemma 1, we can follow Rosenberg [5]. Rosenberg has given a very simple proof of the following lemma.

LEMMA 2. *Let H be as in Lemma 1. If L is a conjugate class of H which does not meet $C_H(Q)$, then $\mathcal{S}L$ is a nilpotent element of the class algebra $Z(H)$ of H.*

This implies Lemma 1 since if b is a block of H, $\omega_b(z) = 0$ for every nilpotent element z of $Z(H)$. Hence if L is a conjugate class of H and if $\omega_b(\mathcal{S}L) \neq 0$, by Lemma 2, $L \cap C_H(Q) \neq \emptyset$, i.e., the defect group of L contains Q. In particular, the defect group of b contains Q.

An an immediate corollary of (2D), we mention

(2E) *The theorem* (2D) *remains valid if we replace N by a subgroup H of G with*

$$G \supseteq H \supseteq N_G(Q).$$

Indeed, if the notation is as in (2D), we have to associate the block $B^* = b^H$ of H with the block $(B^*)^G = (b^H)^G = b^G$ of G.

3. Ground fields which are not algebraically closed

Rosenberg in [5] also considers the case of ground fields K of characteristic p which are not algebraically closed. We shall show that this case can be reduced to that of an algebraically closed field Ω.

Let $Z_K(G)$ denote the class algebra of G with regard to an arbitrary field K of characteristic p. Suppose that A is a block ideal of $Z_K(G)$. As is well known, Wedderburn's theorem on finite division algebras implies that the field Λ of $|G|$-th roots of unity over K is a splitting field of the group algebra K[G], cf. [1]. It follows that Λ is a splitting field of $Z_K(G)$.

If we now work in the class algebra $Z_\Lambda(G) \supseteq Z_K(G)$, the ideal $AZ_\Lambda(G)$ is a direct summand of $Z_\Lambda(G)$ and hence $AZ_\Lambda(G)$ is a direct sum of block ideals B_i $(i = 1, 2, \cdots, r)$.

Let τ be an element of the Galois group $G(\Lambda/K)$ of the separable field extension field Λ of K. If $f \in \hat{B}_i$, $(1 \leqq i \leqq r)$, let f^τ denote the linear function on $Z_\Lambda(G)$ whose value for a class sum $\mathcal{S}K$ is equal to $f(\mathcal{S}K)^\tau$. It is easy to see that there exists a block ideal B_j with $1 \leqq j \leqq r$ such that $f^\tau \in \hat{B}_j$. We set $B_j = B_i^\tau$. Moreover, we see that $G(\Lambda/K)$ acts transitively on the set $\{B_1, B_2, \cdots, B_r\}$.

Let Ω now denote an algebraic closure of Λ. Then

$$AZ_\Omega(G) = \oplus \sum_{i=1}^{r} B_i Z_\Omega(G).$$

Since Λ is a splitting field of $Z_\Lambda(G)$, each $B_i Z_\Omega(G)$ remains indecomposable, i.e., it is a block ideal of the class algebra $Z_\Omega(G)$ of G. It is clear that the r block ideals $B_i Z_\Omega(G)$ have the same system of lower defect groups. Hence, by (1A),

$$\mathfrak{D}(AZ_\Omega(G)) = r \cdot \mathfrak{D}(B_1 Z_\Omega(G)). \tag{4}$$

We may now define the system of lower defect groups $\mathfrak{D}(A)$ of the block ideal A of $Z_K(G)$ as the system (4) and the (ordinary) defect group of A as the ordinary defect group of the $B_i Z_\Omega(G)$. It is immediate that (2B), (2C), (2D), (2E) remain valid, if the underlying field is not algebraically closed.

REFERENCES

1. R. BRAUER, *Über Systeme hyperkomplexer Zahlen*, Math. Zeitschrift, vol. 30 (1929), pp. 79–107.
2. ———, *On the arithmetic in a group ring*, Proc. Nat. Acad. Sci. U.S.A., vol. 30 (1944), pp. 109–114.
3. ———, *Zur Darstellungstheorie der Gruppen endlicher Ordnung I*, Math. Zeitschrift, vol. 63 (1956), pp. 406–444.
4. ———, *Defect groups in the theory of representations of finite groups*, Illinois J. Math., vol. 13 (1969), pp. 53–73.
5. A. ROSENBERG, *Blocks and centres of group algebras*, Math. Zeitschrift, vol. 76 (1961), pp. 209–216.

HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS