THE ROYDEN BOUNDARY OF A RIEMANNIAN MANIFOLD[1]

BY

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An ever present theme in function-theory is the study of a given family of harmonic functions in terms of their boundary values. This theme persists for investigations on open Riemann surfaces which are not embedded in larger ones. In the absence of a natural boundary an ideal boundary can be tailored to suit the study of a particular type of harmonic function. Royden [9] showed that an open Riemann surface can always be compactified in such a fashion that the HBD-functions (harmonic, bounded, Dirichlet-finite) have continuous extensions and are determined by their behavior on the harmonic boundary, a small subset of the "new" points. Nakai [7] discovered that this same harmonic boundary serves as the basis for a representation theory for the HD-functions.

It is natural to ask how much of this theory can be extended to Riemannian manifolds. Not only do the known proofs rely on strictly Riemann surface techniques but the Royden algebra which determines the compactification is of an essentially different nature in higher dimensions. This difference lies in the following result of Nakai [6]: quasi-conformally equivalent Riemann surfaces have isomorphic Royden algebras but only quasi-isometrically equivalent manifolds have this property (also cf. [8]).

In this paper we show that the theory can be carried over in its entirety. In Sections 1 and 2 we introduce the fundamentals. We establish the maximum principle for HD-functions in terms of their values on the harmonic boundary in Sections 4 and 5. The Royden-Nakai decomposition theorem for Riemannian manifolds is given in Sections 6 and 7 and Section 8 contains some easy consequences. At this point the theory on Riemann surfaces easily generalizes and we conclude in Section 9 by simply stating two results. For a complete account we refer the reader to the forthcoming monograph of Sario and Nakai [10] on Riemann surfaces.

The authors have been informed that Loeb and Walsh [3] have obtained similar results for Banach sublattices of HB-functions in the axiomatic setting.

1. Let R be a noncompact orientable Riemannian manifold. We focus our attention on the Tonelli functions on R, the vector-lattice of continuous real-valued functions on R with locally square integrable first partial derivatives. The Dirichlet integral of Tonelli functions over relatively compact regions Ω

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is given by $D_\Omega(f, g) = \int_\Omega df \wedge {}^*dg$. A function f is said to have finite Dirichlet integral if

$$D(f) = \sup_\Omega D_\Omega(f) < \infty, \quad \text{where } D_\Omega(f) = D_\Omega(f, f).$$

The set of all bounded Tonelli functions with finite Dirichlet integral form an algebra, the *Royden algebra* M associated with R.

If $\{f_n\}$ is a sequence of functions which converge uniformly to f on compact subsets of R, then we write $f = B\text{-}\lim f_n$ or $f = C\text{-}\lim f_n$ depending on whether or not $\{f_n\}$ is bounded. If $\lim D(f_n - f) = 0$, then we write $f = D\text{-}\lim f_n$. The notations $f = BD\text{-}\lim f_n$, $f = CD\text{-}L \lim f_n$ are used to indicate both types of convergence.

The Royden algebra M is a lattice and is complete in the BD-topology. Another essential fact is that the C^1-functions are dense in M in the BD-topology. For the proofs of these two facts we refer to Nakai [5, p. 203–6].

A subregion Ω of R will be called *regular* if it is relatively compact and $\partial\Omega$ is smooth. If $u \in M$ is harmonic on a regular region Ω, then the denseness of the C^1-functions in M gives the following mild generalization of Green's formula, $D_\Omega(f, u) = \int_{\partial\Omega} f{*}\, du$ for every $f \in M$.

2. The *Royden compactification* R^* of R is the unique compact Hausdorff space such that R is dense in R^*, every $f \in M$ has a continuous extension to R^* (again denoted by f) and M separates the points of R^*. (For the existence of such compactifications see Loeb [2].) The set $\Gamma = R^* - R$ is called the *Royden* boundary.

The Royden algebra M has the *Urysohn property*: for disjoint compact sets $K_i \subset R^*$, $i = 0, 1$ there exists an $f \in M$ such that $f \mid K_i = i$ and $0 \leq f \leq 1$. This can be deduced from the denseness of M in the sup norm in the set of continuous functions on R^*.

The ideal in M consisting of the functions with compact support will be denoted by M_0 and its BD-closure by M_Δ. The set

$$\Delta = \{p \in R^* \mid f(p) = 0 \quad \text{for every} \quad f \in M_\Delta\}$$

is called the *harmonic boundary* of R. In passing we remark that it can be shown that Δ is nowhere dense in Γ. We reserve the symbol $\bar{A}$ for the closure of A in R^*.

3. We shall denote by $H(A)$, $H^c(A)$, $B(A)$, $P(A)$ and $D(A)$ the set of functions on A which are harmonic, harmonic with continuous extension to $\bar{A} \cap R$, bounded, nonnegative and Dirichlet-finite, respectively. The intersection of these sets will be denoted by juxtaposition. The class of subregions G with ∂G smooth for which the set of functions $u \in H^cX(G)$ with $u \mid \partial G = 0$ consists of only 0 will be denoted by SO_{HX}.

LEMMA. *If a region G belongs to SO_{HBD} then it belongs to SO_{HPD}.*

Choose an exhaustion R_n of R such that $G_n = R_n \cap G$ is a regular region. Let

$u \in H^c PD(G)$ such that $u \mid \partial G = 0$ and set $u_m = \min (u, m)$. Consider the continuous function v_{mn} on G with $v_{mn} \in H^c(G_n)$ and

$$v_{mn} \mid (G - G_n) \cup \partial G_n = u_m \mid (G - G_n) \cup \partial G_n .$$

The superharmonicity of v_{mn} implies that $\{v_{mn}\}_{n=0}^{\infty}$ is decreasing and consequently $v_n = B\text{-}\lim_n v_{mn} \in H^c(G)$. By Green's formula

$$D_G(u_m - v_{mn}) = D_G(u_m) - D_G(v_{mn}) \geq 0$$

and

$$D_G(v_{mn} - v_{mp}) = D_G(v_{mn}) - D_G(v_{mp}) \geq 0 \quad \text{for } p \geq n.$$

Therefore $\infty > d = \lim_n D_G(v_{mn})$ exists. Since $|\operatorname{grad} v_{mn}|$ converges pointwise in G to $|\operatorname{grad} v_m|$, we conclude by Fatou's lemma that

$$D_G(v_{mn} - v_n) \leq D_G(v_{mn}) - d.$$

This implies that $v_m = D\text{-}\lim_n v_{mn}$ and that $v_m \in H^c BD(G)$ with $v_m \mid \partial G = 0$.

We shall complete the proof by showing that if $v_m \equiv 0$ for all m, then $u \equiv 0$. We know that $u = D\text{-}\lim_m u_m$ and $u_m = D\text{-}\lim_n g_{mn}$, where $g_{mn} = u_m - v_{mn}$. Since $\operatorname{supp} g_{mn} \subset \bar{G}_n$, we have

$$D_G(g_{mn}, u) = \int_{\partial G_n} g_{mn} * du = 0$$

and therefore $D_G(u) = 0$.

4. The role played by Δ in determining HD-functions is already implicit in the following theorem and corollary established by A. Mori [4] and Kusunoki-Mori [1] for Riemann surfaces.

THEOREM. *Every region G with ∂G smooth and $\bar{G} \cap \Delta = \emptyset$ belongs to SO_{HBD}.*

Since $\bar{G} \cap \Delta = \emptyset$, for every $p \in \bar{G}$ there exists an $f_p \in M_\Delta$ such that $f_p(p) > 1$. The compactness of $\bar{G}$ allows us to find points $p_1, \cdots, p_n$ such that

$$\bar{G} \subset \cup_1^n \{x \in R^* \mid f_{p_i}(x) > 1\}.$$

Since M_Δ is also a lattice, we can construct using the f_{p_j} a function $f \in M_\Delta$ which is identically 1 on $\bar{G}$. We choose a sequence $\{f_n\} \subset M_0$ with $f = BD\text{-}\lim_n f_n$.

Suppose that $u \in H^c BD(G)$ and $u \mid \partial G = 0$. Then the functions uf_n converge to u in the Dirichlet norm on G. In fact for any compact subset K of G we have

$$\tfrac{1}{2} D_G(uf_n - u)$$
$$\leq \sup_G |u|^2 D_G(f_n) + \sup_K |f_n - 1|^2 D_K(u) + \sup_G |f_n - 1|^2 D_{G-K}(u).$$

Thus $\lim \sup_n D_G(uf_n - u) \leq \sup_G |f_n - 1|^2 D_{G-K}(u)$. Since we can make the right-hand side arbitrarily small by choosing K appropriately, we have $u = D\text{-}\lim_n uf_n$.

We choose an exhaustion $\{R_n\}$ of R such that supp $f_n \subset R_n$ and $G_n = G \cap R_n$ is a regular region. We have that $D_G(uf_n, u) = \int_{\partial G_n} uf_n {*du} = 0$, since $uf_n \mid \partial G_n = 0$. Our assertion follows from the fact that $D_G(u) = \lim_n D_G(uf_n, u) = 0$.

We can now make a stronger statement.

COROLLARY. *Every region G with ∂G smooth and $\bar{G} \cap \Delta = \emptyset$ belongs to SO_{HD}.*

Assume that there exists a nonconstant $u \in H^cD(G)$ with $u \mid \partial G = 0$ and we may assume in addition that $\sup_G u > 0$. By Sard's theorem the critical values of u are nowhere dense and therefore we can find an $a \in (0, \sup_G u)$ with $u^{-1}(a)$ smooth. Let G' be a component of $\{x \in G \mid u(x) > a\}$. Since $\partial G'$ smooth, $\bar{G}' \cap \Delta = \emptyset$, $u - a \in H^cPD(G')$ and $u - a \mid \partial G' = 0$, it follows that $u - a \equiv 0$ in G'. This contradicts the definition of G' and establishes the corollary.

5. We are now able to state the following counterpart of Nakai's result [7].

THEOREM. *Let F be an arbitrary region in R and $u \in HD(F)$. If $\limsup_{x \in F, x \to p} u(x) \le m$ for every $p \in (\bar{F} \cap \Delta) \cup \partial F$, then $u \le m$.*

Assume that there exists a point $x_0 \in F$ with $u(x_0) > m$. Choose an $a \in (m, u(x_0))$ with $u^{-1}(a)$ smooth and let G be a component of $\{x \in F \mid u(x) > a\}$. We immediately arrive at a contradiction by noting that G satisfies the hypotheses of the corollary to Theorem 4.

6. Let $\{R_n\}$ be an exhaustion of R by regular regions. Let ω_n be the continuous function on R with $\omega_n \mid \bar{R}_0 = 1$, $\omega_n \mid R - R_n = 0$ and $\omega_n \in H(R_n - \bar{R}_0)$. The sequence $\{\omega_n\} \subset M_0$ is BD-Cauchy and consequently $\omega = BD\text{-}\lim \omega_n \in M_\Delta$. If $\omega \equiv 1$, then we shall write $R \in O_G$ and if $\dim HD(R) = 1$, then we shall write $R \in O_{HD}$.

LEMMA. *If $R \in O_G$, then $R \in O_{HD}$. Moreover, $R \in O_G$ if and only if $\Delta = \emptyset$.*

Let $u \in HD(R)$ and set $u_m = \min(m, \max(u, -m))$. As in the proof of Theorem 4 we conclude that $u_m = D\text{-}\lim_n \omega_n u_m$. But $D(\omega_n u_m, u) = \int_{\partial R_n} \omega_n u_m {*du} = 0$ and $u = D\text{-}\lim u_m$ which implies that $D(u) = 0$.

If $R \in O_G$ then it is immediate from the definitions that $\Delta = \emptyset$. On the other hand if $\Delta = \emptyset$, then $(R - R_0)^- \cap \Delta = \emptyset$ and by Theorem 4 we conclude that $R - \bar{R}_0 \in SO_{HBD}$. Note that $1 - \omega \in H^cBD(R - \bar{R}_0)$ and $1 - \omega \mid \partial R_0 = 0$; therefore, $\omega \equiv 1$.

7. Denote by $\tilde{M}$ the set of Tonelli functions on R which have finite Dirichlet integral. It is easily seen that every function in $\tilde{M}$ has an extension to a continuous extended real-valued function on R^*.

A compact subset K of R^* will be called *distinguished* if $(K \cap R)^- = K$ and $\partial(K \cap R)$ is smooth. Note that the empty set is distinguished. The set of functions f in $\tilde{M}$ such that $f = 0$ on $\Delta \cup K$ will be denoted by $\tilde{M}_{\Delta \cup K}$. The following decomposition theorem holds Riemannian manifolds (cf. [7], [9]).

THEOREM. *Let $f \in \tilde{M}$ and a distinguished compact set K be given. There exists a unique pair of functions u, g with $u \in \tilde{M} \cap H(R - K)$ and $g \in \tilde{M}_{\Delta \cup K}$ such that $f = u + g$. Moreover, $D(\varphi, u) = 0$ for every $\varphi \in \tilde{M}_{\Delta \cup K}$ and*

$$|u| \leq \sup_{\partial(K \cap R) \cup (\Delta \cap (R^* - K))} |f|$$

on $R - K$.

We shall also use the symbol $\pi_K f$ for u. If $R \in O_G$ and $K = \emptyset$, then in view of Lemma 6 there is nothing to prove and we exclude this case from consideration. The uniqueness of the decomposition and the maximum principle are direct consequences of Theorem 5.

Let $\{R_n\}$ be an exhaustion of R with $\bar{R}_0 \subset R - K$ and $R_n - K$ regular. Let u_n, u_n', u_n'' be the continuous functions on R which are harmonic in $R_n - K$ and $u_n = f$, $u_n' = f^+$, $u_n'' = f^-$ on $R - (R_n - K)$. Green's formula gives

$$D(f^+ - u_n') = D(f^+) - D(u_n') \geq 0 \text{ and } D(u_n' - u_p') = D(u_n') - D(u_p') \geq 0$$

for $p \geq n$. Thus we see that $\{u_n'\}$ is D-Cauchy.

Denote by w_n the continuous function with

$$w_n \mid \bar{R}_0 = 1, \; w_n \mid R - (R_n - K) = 0 \quad \text{and} \quad w_n \in H((R_n - K) - \bar{R}_0)$$

and set $g_n' = f^+ - u_n'$. It follows that

$$D(g_n', w_n) = \int_{\partial R_0} f^+ {*dw_n} - \int_{\partial R_0} u_n' {*dw_n}.$$

Let $a_n = \inf_{\partial R_0} u_n'$ and $b = \sup_{\partial R_0} f^+$. Using the above identity we obtain

$$\begin{aligned} a_n D(w_n) &= a_n \int_{\partial R_0} *dw_n \leq \int_{\partial R_0} u_n' {*dw_n} \\ &\leq b \int_{\partial R_0} *dw_n + |D(g_n', w_n)| \leq bD(w_n) + D^{1/2}(f)\, D^{1/2}(w_n). \end{aligned}$$

Note that we have eliminated from consideration the case where $\lim D(w_n) = 0$ and therefore $\limsup a_n < \infty$.

The Harnack inequality now gives a uniform bound for the sequence $\{u_n'\}$ on every compact subset of $R - K$. Consequently we can invoke the Harnack principle to obtain a subsequence, denoted by $\{u_n'\}$, and a function $u' \in H(R - K)$ such that $u' = CD\text{-}\lim u_n'$. Since the sequence $\{u_n''\}$ lends itself to the same argument, we conclude that there is a function $u \in \tilde{M} \cap H(R - K)$ with $u = f$ on K.

Set $g_n = f - u_n$, $g = f - u$ and observe that $g_n \in M_0$, $g_n = 0$ on K and $g = CD\text{-}\lim g_n$. It is easily seen that

$$BD\text{-}\lim \frac{g_n}{1 + |g_n|} = \frac{g}{1 + |g|} \in M_\Delta.$$

Therefore $g = 0$ on $\Delta \cup K$ and the existence of the decomposition is established.

To prove the orthogonality of $\tilde{M}_{\Delta \cup K}$ and $\tilde{M} \cap H(R - K)$, choose a $\varphi \in \tilde{M}_{\Delta \cup K}$ and a sequence $\{\varphi_n\} \subset M_0$ with $\varphi_n = 0$ on K and $\varphi = D\text{-}\lim \varphi_n$. Since $\pi_K \varphi \equiv 0$, we may choose for φ_n the g_n constructed above for $\varphi = f$. For any $u \in \tilde{M} \cap H(R - K)$ Green's formula gives $D(\varphi_n, u) = 0$ and consequently $D(\varphi, u) = 0$.

8. We now give several applications of the decomposition theorem. We begin by reestablishing (cf. [11]) the Virtanen relation, $O_{HD} = O_{HBD}$, for Riemannian manifolds.

COROLLARY 1. *If u is a nonconstant function in $HD(R)$, then there is a nonconstant $v \in HBD(R)$.*

By Theorem 5 we know that $u \mid \Delta$ is nonconstant and therefore for sufficiently large m, $u_m \mid \Delta$ is nonconstant, where $u_m = \min(m, \max(u, -m))$. The function πu_m is a possible choice for v.

COROLLARY 2. *If $G \in SO_{HPD}$, then $\bar{G} \cap \Delta - (\partial G)^- = \emptyset$.*

Suppose that there is a point $p \in \bar{G} \cap \Delta - (\partial G)^-$. Then $K = (R - G)^-$ is a distinguished compact set and is disjoint from p. The Urysohn property enables us to find an $f \in M$ such that $0 \leq f \leq 1, f \mid K = 0$ and $f(p) = 1$. The function $\pi_K f \in H^c PD(G)$ vanishes on ∂G but is nonzero on G and thus $G \notin SO_{HPD}$.

Lemma 3, Theorem 5 and its corollary may be summarized as follows: $SO_{HD} \subset SO_{HBD} \subset SO_{HPD}$ and $\bar{G} \cap \Delta = \emptyset$ is a sufficient condition for G to belong to SO_{HD}. By virtue of Theorem 5 the latter statement has a stronger formulation: $\bar{G} \cap \Delta - (\partial G)^- = \emptyset$ *implies that* $G \in SO_{HD}$. This combined with Corollary 2 gives the following complete result.

COROLLARY 3. $SO_{HPD} = SO_{HBD} = SO_{HD}$.

Indeed if $G \in SO_{HPD}$, then $\bar{G} \cap \Delta - (\partial G)^- = \emptyset$ and consequently $G \in SO_{HD}$.

COROLLARY 4. *The vector space $HD(R)$ is a lattice under the operations greatest harmonic minorant and least harmonic majorant.*

This lattice structure is derived from the structure of $\tilde{M} \mid \Delta$ by means of the decomposition theorem. It is easy to see that the greatest harmonic minorant of u and v is $\pi(\min(u, v))$.

COROLLARY 5. *The dimension of $HD(R)$ is n if and only if Δ consists of n points, $n \geq 2$ and it is 1 if and only if Δ is empty or consists of 1 point.*

If Δ is empty, then by Lemma 6, $\dim HD(R) = 1$. Now suppose that Δ consists of n points $p_1, \cdots, p_n$, $n \geq 1$. The decomposition theorem allows us to find n linearly independent functions $u_i \in HBD(R)$ such that $u_i(p_j) = \delta_{ij}$. For any $v \in HD(R)$ we must have $v(p_i) \neq \pm\infty$, that is $v \in HBD(R)$. Since $v = v_1 - v_2$, $v_1 = \pi v^+$, $v_2 = \pi v^-$, it is enough to show that $v_1(p_i) < \infty$. If this were not the case, then we would have by Theorem 5 that $nu_i \leq v_1$ for all

n, contradicting the fact that u_i is nonzero. Therefore every $v \in HD(R)$ can be represented by a linear combination of the u_i, $v = \sum v(p_i)u_i$. This proves the corollary in one direction.

Now suppose that dim $HD(R) = n$. If Δ consisted of more than n points, then we could find $n + 1$ linearly independent functions in $HD(R)$. Thus Δ is either empty or consists of m points, $m \leq n$. Then by the first part of the argument we must have $m = n$, if $n \geq 2$ and if $n = 1$, then either $m = 1$ or Δ is empty.

9. As in the two-dimensional case it turns out that the harmonic boundary supports lower bounded superharmonic functions (cf. Sario-Nakai [10]).

THEOREM. *Let F be an arbitrary region in R and u a superharmonic function on F with* $\inf u > -\infty$. *If* $\liminf_{x \in F, x \to p} u(x) \geq m$ *for every* $p \in (\bar{F} \cap \Delta) \cup \partial F$, *then* $u \geq m$.

A Poisson-Schwarz type representation is also a consequence of this theory (cf. [10]).

THEOREM. *There exists a kernel $P(x, p)$ on $R \times \Gamma$ and a positive regular Borel measure μ on Γ with the following properties*:

(i) *for fixed x, $P(x, p)$ is a nonnegative Borel function on Γ;*

(ii) *for fixed p, $P(x, p) \in HP(R)$;*

(iii) *if $f(p)$ is μ-integrable, then $u(x) = \int P(x, p)f(p)\, d\mu$ is the C-limit of functions in $HBD(R)$, in particular, $u \in H(R)$;*

(iv) *if in addition $f(p)$ is bounded and continuous at $q \in \Delta$, then* $\lim_{x \to q} u(x) = f(q)$.

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