MEAN GROWTH AND COEFFICIENTS OF H^p FUNCTIONS[1]

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Let $f(z)$ be analytic in the unit disk $|z| < 1$, and let

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}, \qquad 0 < p < \infty;$$

$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

The function f is said to belong to the class H^p $(0 < p \leq \infty)$ if $M_p(r,f)$ is bounded for $0 \leq r < 1$. Hardy and Littlewood [4], [5] proved that $f \in H^p$ implies

$$M_q(r,f) = o((1-r)^{1/q-1/p}), \quad 0 < p < q \leq \infty,$$

and they pointed out that the exponent $(1/q - 1/p)$ is best possible. In the present paper, we show that the Hardy-Littlewood estimate is best possible in a stronger sense, and we apply this result to prove that several known theorems on the Taylor coefficients of H^p functions are also best possible.

THEOREM 1. *Let $0 < p < q \leq \infty$, and let $\phi(r)$ be positive and non-increasing on $0 \leq r < 1$, with $\phi(r) \to 0$ as $r \to 1$. Then there exists a function $f \in H^p$ such that*

$$M_q(r,f) \neq O(\phi(r)(1-r)^{1/q-1/p}).$$

For $q = \infty$, this theorem was obtained in [6]. The more general result is now deduced from this special case. We shall need the following elementary lemma (see [2, Kap. IX, §5]).

LEMMA. *Let $1 < p < \infty$, and let $\rho = (1+r)/2$, where $0 < r < 1$. Then as $r \to 1$,*

$$\int_0^{2\pi} |\rho e^{it} - r|^{-p} \, dt = O((1-r)^{1-p}).$$

Proof of Theorem 1. Let $f \in H^p$, $p < q$, and suppose first that $1 < q < \infty$. If $\rho = (1+r)/2$, we have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z = re^{i\theta}.$$

Thus, by Hölder's inequality and the lemma,

$$M_\infty(r,f) \leq C(1-r)^{-1/q} M_q(\rho, f).$$

From this it is clear that the theorem for $1 < q < \infty$ follows from the case

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$q = \infty$, which was proved in [6]. If $q = 1$, essentially the same argument can be used to obtain the desired conclusion. Finally, suppose $0 < q < 1$, and observe that for $f \in H^p$,

$$M_1(r, f) \leq \{M_\infty(r, f)\}^{1-q}\{M_q(r, f)\}^q$$
$$\leq C(1 - r)^{-(1/p)(1-q)}\{M_q(r, f)\}^q.$$

Thus if

$$M_q(r, f) = O(\phi(r)(1 - r)^{1/q-1/p}),$$

for some $p < q$ and all $f \in H^p$, it follows that

$$M_1(r, f) = O([\phi(r)]^q(1 - r)^{1-1/p}),$$

which contradicts what we have already proved.

We now turn to coefficient theorems for H^p functions. Hardy and Littlewood [5] proved that if

$$f(z) = \sum_{n=0}^\infty a_n z^n \in H^p, \qquad 0 < p \leq 1,$$

then $a_n = o(n^{1/p-1})$, and the exponent $(1/p - 1)$ is best possible. The following theorem shows that the estimate cannot be improved at all. This result is due to Evgrafov [1], but we believe our proof is simpler and more natural.

THEOREM 2. *Let $\{\delta_n\}$ be an arbitrary sequence of positive numbers tending monotonically to zero. Then for each p $(0 < p \leq 1)$, there exists*

$$f(z) = \sum a_n z^n \in H^p$$

such that

$$a_n \neq O(\delta_n n^{1/p-1}).$$

Proof. If the theorem were false, then for each $f \in H^p$ there would exist a constant C such that

$$\{M_2(r, f)\}^2 = \sum_{n=0}^\infty |a_n|^2 r^{2n}$$
$$\leq C \sum_{n=1}^\infty \delta_n^2 n^{2/p-2} r^{2n}$$
$$\leq C\delta_1^2 \sum_{n=1}^{\nu(r)} n^{2/p-2} + C\delta_{\nu(r)}^2 \sum_{n=\nu(r)}^\infty n^{2/p-2} r^{2n},$$

where $\nu(r) = [(1 - r)^{-1/2}]$, the greatest integer not exceeding $(1 - r)^{-1/2}$. Hence

$$\{M_2(r, f)\}^2 = O([\nu(r)]^{2/p-1}) + O(\delta_{\nu(r)}^2(1 - r)^{1-2/p})$$
$$= O(\delta_{\nu(r)}^2(1 - r)^{1-2/p})$$

for each $f \in H^p$ $(0 < p \leq 1)$, which contradicts Theorem 1. Implicit here is the assumption, which can be made without loss of generality, that $\{\delta_n\}$ tends to zero so slowly that the second term dominates. Thus the proof of Theorem 2 is complete.

Hardy and Littlewood [3] also proved that if

$$f(z) = \sum a_n z^n \in H^p, \qquad 0 < p \le 2,$$

then

$$\sum_{n=1}^{\infty} n^{p-2} \mid a_n \mid^p < \infty.$$

For $p = 1$ this is a theorem of Hardy; for $p = 2$ it is Parseval's relation. In the case $0 < p \le 1$ it may be viewed as a slight sharpening of the fact that $n^{p-1} \mid a_n \mid^p \to 0$. The Hardy-Littlewood theorem is best possible in the following sense.

THEOREM 3. *Let $\{\lambda_n\}$ be an arbitrary sequence of positive numbers tending monotonically to infinity. Then for each p $(0 < p \le 2)$, there exists $f(z) = \sum a_n z^n \in H^p$ such that*

$$\sum_{n=1}^{\infty} \lambda_n n^{p-2} \mid a_n \mid^p = \infty.$$

Proof. First consider the case $0 < p \le 1$. If Theorem 3 were false, we would have for each $f \in H^p$ $(0 < p \le 1)$,

$$\begin{aligned} \{M_2(r, f)\}^2 &= \sum_{n=0}^{\infty} \mid a_n \mid^2 r^{2n} \\ &\le C \sum_{n=1}^{\infty} \mid a_n \mid^{p/2} n^{(1/p-1)(2-p/2)} r^{2n} \\ &\le C \{\sum_{n=1}^{\infty} \lambda_n n^{p-2} \mid a_n \mid^p\}^{1/2} \{\sum_{n=1}^{\infty} \lambda_n^{-1} n^{4/p-3} r^{4n}\}^{1/2}, \end{aligned}$$

by the Cauchy-Schwarz inequality. But, as in the proof of Theorem 2, this would imply

$$M_2(r, f) = O(\lambda_{\nu(r)}^{-1/4} (1 - r)^{1/2-1/p})$$

for every $f \in H^p$, contradicting Theorem 1.

Now suppose $1 < p < 2$. If there is a sequence $\{\lambda_n\}$ such that

$$\sum \lambda_n n^{p-2} \mid a_n \mid^p < \infty \quad \text{for each } f \in H^p,$$

then by Hölder's inequality

$$\begin{aligned} \{M_2(r, f)\}^2 &= \sum_{n=0}^{\infty} \mid a_n \mid^2 r^{2n} \\ &\le \mid a_0 \mid^2 + \{\sum_{n=1}^{\infty} \lambda_n n^{p-2} \mid a_n \mid^p\}^{1/p} \{\sum_{n=1}^{\infty} \mu_n n^{\alpha} \mid a_n \mid^q r^n\}^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$, $\mu_n = \lambda_n^{-q/p}$, and $\alpha = (2 - p)q/p$. Thus

$$\begin{aligned} \{M_2(r, f)\}^{2q} &\le C \sum_{n=1}^{\infty} \mu_n n^{\alpha} \mid a_n \mid^q r^n \\ &\le C_1 \sum_{n=1}^{\nu(r)} n^{\alpha} + C\mu_{\nu(r)} \sum_{n=\nu(r)}^{\infty} n^{\alpha} \mid a_n \mid^q r^n, \end{aligned}$$

where now $\nu(r) = [(1 - r)^{(p-2)/2}]$. But a calculation gives $(2 - p)(\alpha + 1) = \alpha$, so we have

$$\sum_{n=1}^{\nu(r)} n^{\alpha} = O([\nu(r)]^{\alpha+1}) = O((1 - r)^{-\alpha/2}).$$

On the other hand, because

$$A_n = \sum_{k=1}^{n} |a_k|^q$$

is bounded (by the Hausdorff-Young theorem), we find after summation by parts that

$$\begin{aligned}\sum_{n=1}^{\infty} n^\alpha |a_n|^q r^n &= \sum_{n=1}^{\infty} \{n^\alpha - (n+1)^\alpha r\} A_n r^n \\ &= \sum_{n=1}^{\infty} \{n^\alpha - (n+1)^\alpha\} A_n r^n \\ &\quad + (1-r) \sum_{n=1}^{\infty} (n+1)^\alpha A_n r^n \\ &= O((1-r)^{-\alpha}) = O((1-r)^{(1-2/p)q}).\end{aligned}$$

Therefore, for each $f \in H^p$ $(1 < p < 2)$ we have

$$M_2(r, f) = O(\mu_{\nu(r)}^{1/2q}(1-r)^{1/2-1/p}),$$

which again contradicts Theorem 1. This concludes the proof of Theorem 3, since the case $p = 2$ is trivial.

As a final application of Theorem 1, we point out that the following result of Hardy and Littlewood [3] is also best possible. If $1 \leq p \leq 2$ and $q = p/(p-1)$ is the conjugate index, then $f \in H^p$ implies

$$\sum_{n=1}^{\infty} n^{-k} |a_n|^s < \infty, \qquad k = 1 - s/q,\ p \leq s \leq q.$$

This result may be viewed as an interpolation between the Hardy-Littlewood theorem considered in Theorem 3 and the Hausdorff-Young theorem.

THEOREM 4. *Let $\{\lambda_n\}$ be a positive sequence tending monotonically to infinity. Then for each p $(1 \leq p \leq 2)$ and for each s $(p \leq s \leq q)$, there exists $f(z) = \sum a_n z^n \in H^p$ such that*

$$\sum_{n=1}^{\infty} \lambda_n n^{-k} |a_n|^s = \infty.$$

Proof. Since the argument is similar to the ones already given, we shall only sketch it. If $1 < p \leq s < 2$, Hölder's inequality gives

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq \{\sum_{n=1}^{\infty} \lambda_n n^{-k} |a_n|^s\}^{1/\alpha} \{\sum_{n=1}^{\infty} \mu_n^{\beta/\alpha} n^\gamma |a_n|^q r^{2\beta n}\}^{1/\beta},$$

where $\alpha = (q-s)/(q-2)$, $\beta = \alpha/(\alpha - 1)$, $\gamma = k\beta/\alpha$, and $\mu_n = \lambda_n^{-1}$. Summation by parts and the Hausdorff-Young theorem now gives a contradiction, as in the last part of the proof of Theorem 3. The case $1 = p \leq s < 2$ is handled similarly. If $2 < s \leq q$, we have

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq \{\sum_{n=1}^{\infty} \lambda_n n^{-k} |a_n|^s\}^{1/\alpha} \{\sum_{n=1}^{\infty} \mu_n^{\beta/\alpha} n^\gamma r^{2\beta n}\}^{1/\beta},$$

where now $\alpha = s/2$ and β, γ, and μ_n are as above; we then obtain a contradiction as in the proof of Theorem 2. In the case $p < s = 2$, we write

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n|^2 r^{2n} &\leq \sum_{n=1}^{\nu(r)} \lambda_n n^{-k} |a_n|^2 \mu_n n^k r^{2n} + \sum_{n=\nu(r)}^{\infty} \\ &\leq C[\nu(r)]^k + \mu_{\nu(r)} \sum_{n=1}^{\infty} \lambda_n n^{-k} |a_n|^2 n^k r^{2n},\end{aligned}$$

where $\nu(r) = [(1-r)^{-1/2}]$; a summation by parts then leads to a contradiction.

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