ON CONJUGACY OF HOMOMORPHISMS OF TOPOLOGICAL GROUPS II

BY

D. H. LEE AND T. S. WU[1]

Let F be a compact topological group and G a locally compact topological group. Let Hom(G, F) denote the space of all continuous homomorphisms from F into G with uniform convergence topology. Assume that θ and φ are elements in Hom(F, G) and that G is either compact or a Lie group. Then we have shown in [3] that if θ and φ are in the same connected component of the space Hom(F, G), then θ and φ are conjugate, that is, there exists an element $g \in G$ such that $\theta(x) = g\varphi(x)g^{-1}$ for all $x \in G$.

In this note, we continue the investigation of the space Hom(F, G) and extend the previous result to the case where G is any locally compact topological group.

Let I_g, for $g \in G$, denote the inner automorphism of G induced by g. The following are the main results:

THEOREM I. *Let F be a compact group and G a locally compact group. Let $\mathcal{C} \subseteq$ Hom(F, G) be a connected component of the space* Hom(F, G). *Then if θ and φ are in $\mathcal{C}$, then θ and φ are conjugate.*

THEOREM II. *Let G be a locally compact group such that G is compact modulo its identity component, and F a compact subgroup of G. If $\{g_\lambda : \lambda \in \Lambda\}$ is a net in G such that the restrictions $I_{g_\lambda} \mid F$ of I_{g_λ} to F converge to an element $\theta \in$* Hom(F, G), *then there exists an element $h \in G$ such that θ is identical with I_h on F.*

1. The proof of Theorem I

Throughout this section, we assume that F is a compact group and G is a locally compact group. As before, Hom(F, G) denotes the space of continuous homomorphisms of F into G with uniform convergence topology (which is identical with the so-called compact-open topology). For a topological group L, L_0 denotes the connected component of the identity.

Let $\mathcal{C}$ be a connected component of the space Hom(F, G) and let $\theta \in \mathcal{C}$. Then, since $\theta(F)$ is compact, the subgroup $H = \theta(F)G_0$ is closed in G, and hence is locally compact with H/H_0 compact. The following lemma reduces the conjugacy problem to the case where the image group is compact modulo its connected component of the identity.

1.1. LEMMA. *If $\theta' \in \mathcal{C}$, then $\theta'(F) \subseteq H$.*

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Proof. Let $x \in F$. Then $\{\bar{\theta}(x) \mid \bar{\theta} \in \mathcal{C}\} = A$ is connected and contains $\theta(x)$. Since $\theta(x)G_0$ is the connected component in G containing $\theta(x)$, $A \subseteq \theta(x)G_0$. Hence $\theta'(F) \subseteq \theta(F)G_0 = H$ follows.

The following has been proved in [3].

1.2. Lemma. *If G is a Lie group, then any two elements in $\mathcal{C}$ are conjugate.*

1.3. Lemma. *Let G be a Lie group such that G/G_0 is finite. Let A and B be compact subgroups of G, both of which are contained in a maximal compact subgroup K of G. If $g \in G$ is such that $I_g(A) = B$, then there exists an element $k \in K$ such that $I_g = I_k$ on A.*

Proof. It is well known (see, for example, Hochschild [1, p. 180]) that there exists an exponential manifold factor E of G such that

(i) $kEk^{-1} = E$ for all $k \in K$,

(ii) $E \times K \to G$ sending (e, k) to $e \cdot k$ is an isomorphism of analytic manifolds, and

(iii) for any compact subgroup K' of G, there exists an element $e \in E$ such that $eK'e \subseteq K$.

Thus g may be expressed uniquely as $g = ek$ with $e \in E$ and $k \in K$. Thus, for $a \in A$, $I_g(a) = ekak^{-1}e^{-1} = b \in B$. Then $(b^{-1}eb)(b^{-1}ka) = ek$. Since A and B are contained in K, $b^{-1}ka \in K$, and, by (i) above, $b^{-1}eb \in E$. Thus, by (ii), $b^{-1}eb = e$ and $b^{-1}ka = k$. Hence $I_g(a) = I_k(a)$, for all $a \in A$.

Theorem I. *Let G be a locally compact group and F a compact group. Let $\mathcal{C} \subseteq \operatorname{Hom}(F, G)$ be a connected component. If θ and φ are in $\mathcal{C}$, then they are conjugate.*

Proof. By Lemma (1.1), we may assume that G/G_0 is compact. Thus $\theta(F)$ and $\varphi(F)$ are contained in maximal compact subgroups K_1 and K_2, respectively, of G. Since K_1 and K_2 are conjugate, let $g \in G$ be such that $gK_2g^{-1} = K_1$.

Now let N be a normal compact subgroup of G, such that $L = G/N$ is a Lie group and let $\pi : G \to G/N$ be the natural map. Now we define

$$\pi^* : \operatorname{Hom}(F, G) \to \operatorname{Hom}(F, L)$$

by $\pi^*(\theta) = \pi \circ \theta$, $\theta \in \operatorname{Hom}(F, G)$. Clearly π^* is continuous; hence $\pi^*(\theta)$ and $\pi^*(\varphi)$ are in the same component (namely, the component containing $\pi^*(\mathcal{C})$ in $\operatorname{Hom}(F, L)$). Thus, by (1.2), there exists $l \in G/N$ such that $\theta^* = I_l \circ \varphi^*$ where $\theta^* = \pi^*(\theta)$ and $\varphi^* = \pi^*(\varphi)$. On the other hand, $gK_2 g^{-1} = K_1$ implies that

$$\pi(g)\pi(K_2)\pi(g)^{-1} = \pi(K_1).$$

Now we may write $\theta^* = I_l \circ I_{\pi(g)^{-1}} \circ I_{\pi(g)} \circ \varphi^*$. Let C be a maximal compact subgroup of G/N containing $\theta^*(F)$ and $I_{\pi(g)} \circ \varphi^*(F)$. Then $I_{l \circ \pi(g^{-1})}$ maps

$I_{\pi(g)}(\varphi^*(F))$ onto $\theta^*(F)$; thus, by (1.3), there exists $c \in C$ such that $I_{I \circ \pi(g^{-1})} = I_c$ on $I_{\pi(g)}(\varphi^*(F))$. Thus, for each $x \in F$,

$$\theta^*(x) = c\cdot\pi(g)\cdot\varphi^*(x)\pi(g)^{-1}c^{-1},$$

or what amounts to the same thing, $\theta(x) = kg\varphi(x)g^{-1}k^{-1}\alpha(x)$, for all $x \in F$ and for some $k \in \pi^{-1}(C)$ and $\alpha(x) \in N$.

Now since G/G_0 is assumed to be compact, for each neighborhood U of 1 in G, there exists a compact normal subgroup N_U of G such that G/N_U is a Lie group and that $N_U \subseteq U$.

Then by what we have shown in the previous paragraph,

$$\theta(x) = k_U\cdot g\varphi(x)g^{-1}k_U^{-1}\alpha_U(x) \qquad \text{for } x \in F$$

with all k_U contained in a maximal compact subgroup K of G, and $\alpha_U(x) \in N_U$.

Thus we obtain a net $\{k_U\}$, U neighborhoods of 1 in K. Since K is compact, there exists a subnet $k_{U(i)}$ converging to an element $k \in K$. Thus

$$\theta(x) = \lim_i \left(k_{U(i)}\, g\varphi(x)g^{-1}k_{U(i)}^{-1}\, \alpha_{U(i)}(x)\right) = kg\varphi(x)g^{-1}k^{-1},$$

since $\lim_i \alpha_{U(i)} = 1$. Hence the proof is complete.

2. The proof of Theorem II

Throughout this section, the following are assumed. G is locally compact, and compact modulo G_0 and F is a compact subgroup of G. We also maintain notation previously introduced.

2.1. LEMMA (Montgomery and Zippin [4]). *Let G be a Lie group and F a compact subgroup of G. Then there exists an open set O in G such that $F \subseteq O$ with the following property:*

If H is a compact subgroup of G and $H \subseteq O$, then there is an element $g \in G$ such that $gHg^{-1} \subset F$. Moreover, given any neighborhood W of the identity of G, O can be so chosen that for any $H \subseteq O$, g can be selected in W.

THEOREM II. *Let G be a locally compact topological group such that G/G_0 is compact, and F a compact subgroup of G. If $\{g_\lambda : \lambda \in \Lambda\}$ is a net in G such that $I_{g_\lambda} \mid F$, the restriction of I_{g_λ} to F, converges in* Hom(F, G) *to an element $\theta \in$* Hom(F, G), *then there exists an element $h \in G$ such that θ is identical with I_h on F.*

Proof. Since G/G_0 is compact, G can be approximated by Lie groups. That is, for each neighborhood U of 1, there exists a compact normal subgroup N such that $N \subseteq U$ and that G/N is a Lie group. In this case, G has a maximal compact subgroup and all such are conjugate. Let F and $\theta(F)$ be contained in maximal compact subgroups K' and K, respectively. Thus there exists an element $g \in G$ such that $I_g(K') = K$. Let N be a compact normal subgroup such that G/N is a Lie group and let $\pi : G \to G/N$ be the natural map. Then $\pi(K)$ is a maximal compact subgroup of G/L. Since $\lim_\lambda (I_{g_\lambda} \mid F) = \theta$ and since N is normal in G, $\theta(N) \subseteq N$ and thus θ induces a homomorphism

$\bar{\theta} : \pi(F) \to G/N$. It is clear that $I_{\pi(g_\lambda)} \mid \pi(F)$ converge to $\bar{\theta}$ in $\operatorname{Hom}(\pi(F), G/N)$. Moreover, $I_{\pi(g_\lambda)} \circ \pi = \pi \circ I_{g_\lambda}$, for $\lambda \in \Lambda$ implies that $\bar{\theta} \circ \pi \mid F = \pi \circ \theta \mid F$. By (2.1), there exists an open subset O in L such that $\pi \circ \theta(F) = \bar{\theta}\pi(F)$ is contained in O and that the property described in (2.1) holds.

Since $\bar{I}_{\pi(g_\lambda)} \mid \pi(F)$ converges to $\bar{\theta}$, we may assume that $\bar{I}_{\pi(g_\lambda)} \circ \pi(F) \subseteq O$ for all $\lambda \in \Lambda$. Hence, if W is a compact neighborhood of 1 in G, then there exists $w_\lambda \in W$ such that

$$\bar{I}_{\pi(w_\lambda)} \circ \bar{I}_{\pi(g_\lambda)} \circ \pi(F) \subseteq \pi\theta(F).$$

We note that

$$\bar{I}_{\pi(w_\lambda)} \circ \bar{I}_{\pi(g_\lambda)} \circ \pi = \bar{I}_{\pi(w_\lambda)} \circ \bar{I}_{\pi(g_\lambda)} \circ \bar{I}_{\pi(g^{-1})} \circ \bar{I}_{\pi(g)} \circ \pi.$$

Since $\bar{I}_{\pi(g)} \circ \pi(F) = \pi \circ I_g(F) \subseteq \pi(K)$, there exists $k_\lambda \in K$ such that

$$\bar{I}_{\pi(k_\lambda)} = \bar{I}_{\pi(w_\lambda g_\lambda g^{-1})} \quad \text{on } \pi(F)$$

by (1.3).

Since W and K are compact, there exist subnets of $\{w_\lambda : \lambda \in \Lambda\}$ and of $\{k_\lambda : \lambda \in \Lambda\}$ which converge to $w \in W$ and to $k \in K$, respectively. Thus we may assume that $\lim_\lambda w_\lambda = w$ and $\lim_\lambda k_\lambda = k$. Thus passing to the limit, we have

$$\bar{I}_{\pi(k)} = \bar{I}_{\pi(w)} \circ \bar{\theta} \quad \text{on } \pi(F).$$

Hence, for each $x \in F$, $kxk^{-1} = w\theta(x)w^{-1}\alpha(x)$ for some $k \in K$, $\alpha(x) \in N$ and $w \in W$.

Now let U be any neighborhood of 1 in G. Thus if $N_U \subseteq U$ is a compact normal subgroup of G such that G/N_U is a Lie group, then there exists $k_U \in K$, $\alpha_U : F \to N_U$ and $w_U \in W$ such that, for each $x \in F$,

$$k_U\, x k_U^{-1} = w_U\, \theta(x) w_U^{-1} \alpha_U(x).$$

Since the k_U form a net when U varies over all neighborhood of 1, the k_U converge to an element $k \in K$ using the compactness of K. Similarly $w = \lim_U w_U$ exists and belongs to W. Hence $kxk^{-1} = w\theta(x)w^{-1}$ holds, since $\lim_U \alpha_U(x) = 1$.

Thus $\theta(x) = w^{-1}kx(w^{-1}k)^{-1}$ and thus $\theta = I_{w^{-1}k}$ on F, which completes the proof of the Theorem II.

Example. Let T_1, T_2 be the countable product of the circle group, with elements in T_1, T_2 denoted by $\langle x_i \rangle$, $\langle y_i \rangle$, respectively. Let A be the discrete abelian group generated by countable generators, $\{a_1, a_2, \cdots\}$. For each k, regard a_k as an automorphism of $T_1 \times T_2$ defined by

$$a_k(\langle x_i \rangle, \langle y_i \rangle) = (\langle x_i, \cdots, x_k\, y_k, x_{k+1}, \cdots \rangle, \langle y_1, y_2, \cdots \rangle).$$

Let G be the semi-direct product of $T_1 \times T_2$ and A. Let $F = (\langle 1_i \rangle, \langle y_i \rangle, \langle 0 \rangle)$, θ be the inclusion $F \to G$, and define the automorphism (α, θ) of $T_1 \times T_2$ by

$$(\alpha, \theta)(\langle 1_i \rangle, \langle y_i \rangle, \langle 0 \rangle) = (\langle y_i \rangle, \langle y_i \rangle, \langle 0 \rangle).$$

Then $(\alpha, \theta) = \lim_{n\to\infty} I_{a_1 a_2 \cdots a_n} \circ \theta$. But $(\alpha, \theta) \neq I_x \circ \theta$ for any $x \in G$.

Thus this example shows that the condition that G/G_0 be compact in Theorem II is necessary.

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CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO