HOLOMORPHIC FUNCTIONS WITH INFINITELY DIFFERENTIABLE BOUNDARY VALUES

BY
W. P. NOVINGER

1. Introduction

The disc algebra A–those functions holomorphic in the open unit disc $U = \{z: |z| < 1\}$ and continuous on its closure $\bar{U}$–has been extensively studied, and the cumulative knowledge of its structure is almost complete. Considerably less is known, however, about subalgebras of A which are obtained by prescribing various smoothness conditions at the boundary of U. In this paper we shall be concerned with the algebras $A^{(p)}$ of functions f, holomorphic in U and such that $f^{(p)}$ (the p^{th} derivative of f) has a continuous extension to $\bar{U}$, and particularly with the algebra $A^{(\infty)} = \bigcap_{p=1}^{\infty} A^{(p)}$. Denote by $C^{(p)}$ the space of p-times continuously differentiable functions (differentiation is with respect to $e^{i\theta}$) on the unit circle $T = \{z: |z| = 1\}$, and normed by

$$Q_p(f) = \sum_{k=0}^{p} (1/k\,!)\, \|f^{(k)}\|_\infty$$

if $1 \leqq p < \infty$; and given the topology Γ which is generated by the family of norms $\{Q_p: 1 \leqq p < \infty\}$ if $p = \infty$. In a manner exactly analogous with the disc algebra A, the space $A^{(p)}$ may be identified with the subalgebra of $C^{(p)}$, consisting of those functions whose negative Fourier coefficients are zero. In Section 2, we extend Wermer's maximality theorem to this setting; that is, $A^{(p)} (1 \leqq p \leqq \infty)$ is a maximal closed subalgebra of $C^{(p)}$. In Section 3 we take $A^{(\infty)}$ (without topology) and observe that a result of Silov to the effect that $C^{(\infty)}$ is not a Banach algebra under any norm, applies to $A^{(\infty)}$ as well. The proof of this result makes use of a theorem due to Singer and Wermer which states that a semisimple commutative Banach algebra admits no nontrivial continuous derivations. However, when $A^{(\infty)}$ is equipped with the topology Γ, the situation is quite different and a simple characterization of the continuous derivations of $(A^{(\infty)}, \Gamma)$ is obtained. Section 4 is devoted to the problem of characterizing those subsets of T which are zero sets for functions in the classes $A^{(p)}$. L. Carleson [2, p. 325–329] provided the answer to this problem for $1 \leqq p < \infty$; our contribution is the solution for $p = \infty$.

2. Maximality

We shall view $A^{(p)}$ $(1 \leqq p \leqq \infty)$ as the closed subalgebra of $C^{(p)}$, consisting of those functions $f \in C^{(p)}$ whose k^{th} Fourier coefficients,

$$\hat{f}(k) = \int_{-\pi}^{\pi} f(e^{it})\, e^{-ikt}\, dt/2\pi,$$

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are zero for $k < 0$. The proof of the following maximality theorem follows closely that given by P. Cohen in the case of the disc algebra A.

THEOREM 2.1. *For* $1 \leqq p \leqq \infty$, $A^{(p)}$ *is a maximal closed subalgebra of* $C^{(p)}$.

Proof. Suppose B is a subalgebra of $C^{(p)}$ which contains $A^{(p)}$ properly. Let $f_0(z) = z$ $(z \in T)$. Then as in Cohen's proof (for the details see [3, p. 94]), one obtains a function $g \in B$ such that $\| 1 - f_0 g \|_\infty < 1$. Define the sequence $\{s_n\}$ by $s_n = \sum_{k=0}^{n} (1 - f_0 g)^k$. The statement of the theorem follows if we can show that s_n converges to $(f_0 g)^{-1}$ *in the topology of* $C^{(p)}$; for this would mean that $(f_0 g)^{-1}$ belongs to the closure of B, Cl (B), and thus $f_0^{-1} \in$ Cl (B). Since the trigometric polynomials are dense in $C^{(p)}$, it would then follow that Cl $(B) = C^{(p)}$. That s_n does, in fact, converge in the topology of $C^{(p)}$ is a consequence of the following observation:

LEMMA 2.1. *Let p be a positive integer and $f \in C^{(p)}$. Then there is a p-tuple of non-negative numbers, $(a_{p1}, a_{p2}, \cdots, a_{pp})$, with the property that for every positive integer $k \geqq p$,*

$$\begin{aligned}\| [(1 - f)^k]^{(p)} \|_\infty &\leqq a_{p1} k \| 1 - f \|_\infty^{k-1} + a_{p2} k(k-1) \| 1 - f \|_\infty^{k-2} \\ &\quad + \cdots + a_{pp} k(k-1) \cdots (k-p+1) \| 1 - f \|_\infty^{k-p}.\end{aligned}$$

We will not go into the proof, except to say that the p-tuples are constructed inductively with the inductive step making use of Leibnitz's rule for computing higher order derivatives of a product.

3. Derivations in $A^{(\infty)}$

DEFINITION. A *derivation* of an algebra B is a linear map $D: B \to B$ which satisfies the product rule

$$D(fg) = fD(g) + D(f)g \qquad (f, g \in B).$$

THEOREM (Singer and Wermer [6, pp. 260–261]). *Let B be a semisimple commutative Banach algebra and D a continuous derivation of B. Then $D(f) = 0$ for all $f \in B$.*

With the aid of this theorem we can deduce

THEOREM 3.1. *There is no norm under which* $A^{(\infty)}$ *is a Banach algebra.*

Proof. Suppose to the contrary that $\| \cdot \|$ is a norm on $A^{(\infty)}$ such that that $(A^{(\infty)}, \| \cdot \|)$ is a Banach algebra. The operator D, defined by $Df = f'$, is clearly a derivation of the algebra $A^{(\infty)}$. We claim that D is continuous, for suppose $\{f_n\}$ is a sequence in $(A^{(\infty)}, \| \cdot \|)$ such that $f_n \to f$ and $Df_n \to g$. Since $\| f_n - f \|_\infty \leqq \| f_n - f \|$ (see [4, cor 3.2.2, p. 121]), it follows that $f_n' \to f'$ uniformly on compact subsets of U. But $f_n' = Df_n \to g$, hence $f' = g$ which says that $Df = g$. By the closed graph theorem D is continuous. But the theorem of Singer and Wermer implies that D is the zero operator and this is clearly false. The theorem thus follows.

We do, however, have non-trivial continuous derivations of the topological algebra $(A^{(\infty)}, \Gamma)$ and these can be characterized as follows:

THEOREM 3.2. *A map* $D: A^{(\infty)} \to A^{(\infty)}$ *is a* Γ*-continuous derivation of* $A^{(\infty)}$ *if and only if there exists a function* $g \in A^{(\infty)}$ *such that*

$$D(f) = gf' \qquad (f \in A^{(\infty)}). \tag{3.1}$$

Proof. Suppose $g \in A^{(\infty)}$ and D is given by (3.1). It is straight forward to verify that D is a derivation of $A^{(\infty)}$, and a calculation shows that if p is a positive integer and $\varepsilon > 0$, then $Q_p(Df) < \varepsilon$ provided $Q_{p+1}(f) < \varepsilon/(p+1)Q_p(g)$. It follows that D is continuous at the zero function and consequently continuous.

Conversely, suppose D is a Γ-continuous derivation of $A^{(\infty)}$. As before, let $f_0(z) = z$ and $\hat{f}(k)$ be the k^{th} Fourier coefficient of f. For functions $f \in A^{(\infty)}$ (or $C^{(\infty)}$), one can use integration by parts to show that if p is a positive integer then $\{k^p \, |\hat{f}(k)|\}_{k=1}^{\infty}$ is a bounded sequence. This order condition on the Fourier coefficients of f implies that

$$f = \sum_{k=0}^{\infty} \hat{f}(k) f_0^k$$

with the series converging to f in the Γ-topology. Hence

$$Df = \sum_{k=0}^{\infty} \hat{f}(k) D(f_0^k) = D(f_0) \sum_{k=1}^{\infty} k\hat{f}(k) f_0^{k-1} = D(f_0) f' \qquad (f \in A^{(\infty)}).$$

Setting $g = D(f_0)$ completes the proof.

4. Zero sets for functions of class $A^{(\infty)}$

Let f be a function in $A^{(p)}$ which is not identically zero, and let $F = Z(f) \cap T$ where $Z(f) = \{z \in \bar{U} : f(z) = 0\}$. Since f is (in particular) continuous and

$$-\infty < \int_{-\pi}^{\pi} \log |f(e^{it})| \, dt \tag{4.1}$$

(see [3, p. 52]), it follows that F is closed and has Lebesgue measure zero. A. Beurling [1, p. 13] observed that F has an additional property: if $\{J_n\}$ denotes the sequence of complementary components of F and ϵ_n = measure of J_n, then it follows from the boundedness of f' and (4.1) that

$$-\infty < \sum \varepsilon_n \log \varepsilon_n . \tag{4.2}$$

Conversely, L. Carleson [2, pp. 325–329] showed that if p is a given positive integer and F is a closed subset of T of measure zero which satisfies condition (4.2), then there exists a function $f_p \in A^{(p)}$ whose zero set is precisely F. Such sets F are called *Carleson sets*, and the remaining sequence of lemmas and theorems culminate with the conclusion that Carleson sets are zero sets for the algebra $A^{(\infty)}$.

DEFINITION 4.1. Let $F \subset T$ be a closed set of measure zero with $\{J_n\}$ and ε_n as above. We say that F belongs to the class $C(s, \alpha, p)$, where $s = \{s_n\}$ is a bounded sequence of positive numbers, α is a number between 0 and 1, and p is a positive integer, provided

(i) $\sum s_n \varepsilon_n^{1-\alpha} < \infty$,

(ii) $\{\varepsilon_n^{-p} \cdot \exp(-s_n \varepsilon_n^{-\alpha})\}$ is a bounded sequence.

THEOREM 4.1. *If $F \in C(s, \alpha, p)$, then F is a Carleson set.*

Proof. Condition (ii) of Definition 4.1 implies that there exists a positive number M_p such that for every n,

$$-s_n \varepsilon_n^{1-\alpha} \leqq \varepsilon_n \log M_p + p\varepsilon_n \log \varepsilon_n . \tag{4.3}$$

Summing both sides of (4.3) and applying condition (i), we find that $-\infty < \sum \varepsilon_n \log \varepsilon_r$. Thus F is a Carleson set.

THEOREM 4.2. *If F is a Carleson set, then there exists a bounded sequence s of positive numbers and a number α between 0 and 1 svch that $F \in \bigcap_{p=1}^{\infty} C(s, \alpha, p)$.*

Proof. The statement of the theorem is obviously true if F is a finite set. Suppose then that F is an infinite closed set of measure zero whose (infinitely many) complementary components satisfy Carleson's condition, $\sum_{n=1}^{\infty} \varepsilon_n \log \varepsilon_n > -\infty$. Since $\varepsilon_n \to 0$, there is a positive integer n_0 such that if $n \geqq n_0$, then $\varepsilon_n < 1$. Define t_n by

$$\begin{aligned} t_n &= -1 && \text{if } n < n_0, \\ &= \left[-\textstyle\sum_{k=n}^{\infty} \varepsilon_k \log \varepsilon_k\right]^{-1/2} && \text{if } n \geqq n_0 \end{aligned}$$

and s_n by

$$\begin{aligned} s_n &= 1 && \text{if } n < n_0, \\ &= -t_n \varepsilon_n^{3/4} \log \varepsilon_n && \text{if } n \geqq n_0 . \end{aligned}$$

Then $\{s_n\}$ is a bounded sequence of positive numbers which satisfies condition (i) for the choice $\alpha = \frac{3}{4}$. It remains to be shown that condition (ii) is satisfied for each positive integer p. Let p be a positive integer. Since $t_r \to +\infty$ and $\varepsilon_n \to 0$, it must be the case that eventually, $(t_n - p) \log \varepsilon_n < 0$. Hence there exists a positive number M_p such that for $n = 1, 2, \cdots$, we have

$$(t_n - p) \log \varepsilon_n < \log M_p .$$

It follows from this and the definition of s_n that

$$\exp(-s_n \varepsilon_n^{-3/4}) \leqq M_p \varepsilon_n^p ,$$

$n = 1, 2, \cdots$. Thus $F \in \bigcap_{p=1}^{\infty} C(s, \alpha, p)$ where $s = \{s_n\}$ is the sequence defined above and $\alpha = 3/4$.

The next two lemmas are estimates which are essential in our proof that Carleson sets are zero sets for $A^{(\infty)}$.

LEMMA 4.1. *Let n be a positive integer and k be a non-negative integer. Then there exists a positive real number $M_0(k, n)$ such that if $0 < r < 1$ and $0 < \delta < \pi/2$, then*

$$\left| \int_{-\delta}^{\delta} \frac{t^k}{(e^{it} - r)^n}\, dt \right| \leqq \frac{M_0(k, n)}{r^n} \cdot \frac{\delta^k}{\delta^{n-1}} .$$

Proof. Suppose first that k and n are positive integers such that $k \geq n$. Then

$$\begin{aligned} \left| \int_{-\delta}^{\delta} \frac{t^k}{(e^{it} - r)^n}\, dt \right| &\leqq \int_{-\delta}^{\delta} \frac{|\, t\,|^k}{|\sin t\,|^n}\, dt \\ &\leqq \frac{\pi^n}{2^{n-1}} \int_0^{\delta} t^{k-n}\, dt \\ &< \frac{M_0(k, n)}{r^n} \frac{\delta^k}{\delta^{n-1}} . \end{aligned}$$

For integers k, n such that $0 \leqq k < n$, we proceed by induction on n. If $n = 1$, then necessarily $k = 0$; so in this case we have

$$\left| \int_{-\delta}^{\delta} \frac{1}{e^{it} - r}\, dt \right| = \frac{1}{r} \left| \log \frac{1 - re^{-i\delta}}{1 - re^{i\delta}} \right| \leqq \frac{1}{r} \pi = \frac{1}{r} M_0(0, 1) \frac{\delta^0}{\delta^{1-1}} .$$

Assume now that n is a positive integer and that for $k = 0, 1, \cdots$ there exist positive numbers $M_0(k, n)$ such that if $0 < r < 1$ and $0 < \delta < \pi/2$, then

$$\left| \int_{-\delta}^{\delta} \frac{t^k}{(e^{it} - r)^n}\, dt \right| \leqq \frac{M_0(k, n)}{r^n} \frac{\delta^k}{\delta^{n-1}} .$$

Now

$$\int_{-\delta}^{\delta} \frac{t^k}{(e^{it} - r)^{n+1}}\, dt = \int_{-\delta}^{\delta} [e^{-it}(1 - re^{-it})^{-n-1}] e^{-int} t^k\, dt, \tag{4.4}$$

so that integration by parts and our inductive hypothesis implies that the modulus of the left hand side of (4.4) is less than

$$\begin{aligned} &\frac{1}{r^{n+1}} \frac{\delta^k}{\delta^n} \left\{ \frac{\pi^n}{n2^{n-1}} + \frac{k}{n} M_0(k - 1, n) + \frac{\pi}{2} M_0(k, n) \right\} \\ &\qquad = \frac{1}{r^{n+1}} \frac{\delta^k}{\delta^n} M_0(k, n + 1), \text{ say.} \end{aligned}$$

The statement of the lemma now follows by induction.

The next lemma follows from an integration by parts and the previous one.

LEMMA 4.2. *Let k be a non-negative integer and n be a positive integer $\geqq 2$. Then there exists a positive number $M(k, n)$ such that if $0 < r < 1$ and $0 < \delta < \pi/2$, then*

$$\left| \int_{-\delta}^{\delta} \frac{e^{it}}{(e^{it} - r)^n} t^k\, dt \right| \leqq \frac{M(k, n)}{r^{n-1}} \frac{\delta^k}{\delta^{n-1}} .$$

THEOREM 4.3. *If F is a Carleson set, then there exists an (outer function) $f \in A^{(\infty)}$ whose zero set is F.*

Proof. Let F be a Carleson set which, for convenience, we assume to contain -1. In addition, we assume that F is an infinite subset of T; otherwise the proof of the theorem is trivial. Let $E = \{t \in [-\pi, \pi]: e^{it} \in F\}$. Since E is closed and $-\pi, \pi \in E$, it follows that $[-\pi, \pi] \sim E = \bigcup_{n=1}^{\infty} (a_n, b_n)$ where $(a_m, b_m) \cap (a_n, b_n) = \emptyset$ if $m \neq n$. Moreover, since F is a Carleson set,

$$-\infty < \sum_{n=1}^{\infty} (b_n - a_n) \log (b_n - a_n).$$

Employ Theorem 4.2 to obtain a bounded sequence $s = \{s_n\}_{n=1}^{\infty}$ of positive numbers and a number α, $0 < \alpha < 1$, such that

(i) $\sum_{n=1}^{\infty} s_n (b_n - s_n)^{1-\alpha} < \infty$,
(ii) for each positive integer p,

$$\{(b_n - a_n)^{-p} \cdot \exp [-s_n (b_n - a_n)^{-\alpha}]\}$$

is a bounded sequence.

Let h be the extended real-valued function on $[-\pi, \pi]$ defined by

$$\begin{aligned} h(t) &= -\infty && \text{if } t \in E, \\ &= -s_n/(t - a_n)^{\alpha} + -s_n/(b_n - t)^{\alpha} && \text{if } t \in (a_n, b_n). \end{aligned}$$

Then $h(t) < 0$ if $t \in [-\pi, \pi]$; h is infinitely differentiable as a function on $[-\pi, \pi] \sim E$; and from (i) it follows that $h \in L^1[-\pi, \pi]$. The function g defined by

$$g(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} h(t) \frac{dt}{2\pi} \qquad (z \in U).$$

is holomorphic in U and

$$\text{(4.5)} \qquad \operatorname{Re} g(z) = \int_{-\pi}^{\pi} P(z, t) h(t) \frac{dt}{2\pi} < 0 \qquad (P(z, t) \text{ is Poisson's kernel}).$$

Finally, define f by

$$\text{(4.6)} \qquad f(z) = \exp \{g(z)\} \qquad (z \in U).$$

The first step is to extend f to $\bar{U}$ by setting

$$\text{(4.7)} \qquad f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \qquad (\theta \in [-\pi, \pi]).$$

Now if $\theta \notin E$, then $|f(e^{i\theta})| = \exp \{h(\theta)\} \neq 0$. Suppose, on the other hand, that $\theta \in E$. If $t \in E$, then $h(t) = -\infty$; if $t \notin E$, say $t \in (a_n, b_n)$, then

$$\text{(4.8)} \quad h(t) = -s_n/(t - a_n)^{\alpha} + -s_n/(b_n - t)^{\alpha} \leqq -2^{\alpha+1} s_n/(b_n - a_n)^{\alpha}.$$

Condition (ii) implies (in particular) that

$$\text{(4.9)} \qquad \lim_{n \to \infty} s_n/(b_n - a_n)^{\alpha} = +\infty.$$

From (4.8) and (4.9) it clearly follows that

$$\lim_{t\to\theta} h(t) = -\infty \qquad (\theta \in E),$$

which in turn implies that

$$\lim_{r\to 1} \int_{-\pi}^{\pi} P(re^{i\theta}, t)h(t)\, \frac{dt}{2\pi} = -\infty$$

[3, p. 41, exercise 12]. Consequently (4.7) is an extension of f to $\bar{U}$ such that $F = \{z \in \bar{U}: f(z) = 0\}$.

In order to show that $f \in A^{(\infty)}$ we are going to show that $f^{(p)} \in H^{\infty}, p = 1, 2, \cdots,$ and thus conclude that $f \in A^{(\infty)}$.

Following Carleson [2], we put $\delta_n(\theta) = (1/8)(\theta - a_n)(b_n - \theta)$. The pertinent properties of $\delta_n(\theta)$ are as follows: If $a_n < \theta < b_n$, then

(a) $[\theta - \delta_n(\theta), \theta + \delta_n(\theta)] \subset (a_n, b_n)$,
(b) $0 < \delta_n(\theta) < \pi/2$,
(c) $(\theta - \delta_n(\theta)) - a_n > (1/8)(\theta - a_n), b_n - (\theta + \delta_n(\theta)) > (1/8)(b_n - \theta)$.

The above properties (a), (b), (c) are easily verified and we omit the proof.

We have $f'(z) = f(z)g'(z)$ where

$$g'(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - z)^2}\, h(t)\, dt \qquad (z \in U)$$

In fact, for each positive integer p,

$$g^{(p)}(z) = \frac{p!}{\pi} \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - z)^{p+1}}\, h(t)\, dt \qquad (z \in U). \tag{4.11}$$

Suppose that $z = re^{i\theta}$ and $\theta \notin E$, say $\theta \in (a_n, b_n)$. Put

$$I_n(\theta) = [-\pi, \theta - \delta_n(\theta)) \cup (\theta + \delta_n(\theta), \pi].$$

Then

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}}\, h(t)\, dt \\ &= \int_{I_n(\theta)} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}}\, h(t)\, dt + \int_{\theta-\delta_n(\theta)}^{\theta+\delta_n(\theta)} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}}\, h(t)\, dt. \end{aligned} \tag{4.12}$$

For the modulus of the first term on the right hand side of (4.12) we have the inequality

$$\left| \int_{I_n(\theta)} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}}\, h(t)\, dt \right| \leqq \frac{\pi^{p+1} \| h \|_1}{2^{p+1}[\delta_n(\theta)]^{p+1}}; \tag{4.13}$$

it remains to consider the second integral. For $t \in [\theta - \delta_n(\theta), \theta + \delta_n(\theta)]$, h has the expansion

$$\begin{aligned} h(t) = h(\theta) + h'(\theta)(t - \theta) + \cdots + (h^{(p)}(\theta)/p!)(t - \theta)^p \\ + (1/p!) \int_{\theta}^{t} (t - x)^p h^{(p+1)}(x)\, dx; \end{aligned}$$

hence,

$$\left| \int_{\theta-\delta_n(\theta)}^{\theta+\delta_n(\theta)} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}} h(t)\, dt \right| \leqq |h(\theta)| \left| \int_{-\delta_n(\theta)}^{\delta_n(\theta)} \frac{e^{it}}{(e^{it} - r)^{p+1}} dt \right|$$

$$+ |h'(\theta)| \left| \int_{-\delta_n(\theta)}^{\delta_n(\theta)} \frac{te^{it}}{(e^{it} - r)^{p+1}} dt \right| \tag{4.14}$$

$$+ \cdots + \frac{|h^{(p)}(\theta)|}{p\,!} \left| \int_{-\delta_n(\theta)}^{\delta_n(\theta)} \frac{t^p e^{it}}{(e^{it} - r)^{p+1}} dt \right|$$

$$+ \sup_{\theta-\delta_n(\theta) \leqq t \leqq \theta+\delta_n(\theta)} |h^{(p+1)}(t)| \cdot \frac{1}{p\,!} \int_{-\delta_n(\theta)}^{\delta_n(\theta)} \frac{|t|^{p+1}}{|e^{it} - r|^{p+1}} dt.$$

Since $0 < \delta_n(\theta) < \pi/2$ (property (b) of $\delta_n(\theta)$), Lemma 4.2 can be applied to the first $p + 1$ integrals on the right hand side of the above inequality (4.14), while property (c) of $\delta_n(\theta)$ together with the proof of the first part of Lemma 4.1 can be used on the last term. If the results are collected, the following fact is obtained: there exists a constant K_0 with the property that if $0 < r < 1$ and $\theta \in (a_n, b_n)$, then

$$\left| \int_{\theta-\delta_n(\theta)}^{\theta+\delta_n(\theta)} \frac{e^{it}}{(e^{it} - re^{i\theta})^{p+1}} h(t)\, dt \right| \leqq \frac{s_n}{r^p} \cdot \frac{K_0}{[\delta_n(\theta)]^{\alpha+p}}. \tag{4.15}$$

Combining the earlier result (4.13) with (4.15) and the fact that $\{s_n\}$ is a bounded sequence, we obtain a constant K_p with this property—if $0 < r < 1$ and $\theta \notin E$, say $\theta \in (a_n, b_n)$, then

$$|g^{(p)}(re^{i\theta})| \leqq (1/r^p)K_p/[\delta_n(\theta)]^{p+1}. \tag{4.16}$$

(In obtaining K_p we also use the fact that $p + \alpha < p + 1$). From (4.16) with $p = 1$, we obtain

$$|f'(re^{i\theta})| \leqq (K_1/r)\, |f(re^{i\theta})|/[\delta_n(\theta)]^2. \tag{4.17}$$

A result like (4.17) is needed for each positive integer p; explicitly we need the following: if p is a positive integer, then there exists a constant N_p such that if $\theta \notin E$, say $\theta \in (a_n, b_n)$, and $0 < r < 1$, then

$$|f^{(p)}(re^{i\theta})| \leqq (N_p/r^p)\, |f(re^{i\theta})|/[\delta_n(\theta)]^{2p}. \tag{4.18}$$

If we put $N_0 = 1$, then (4.18) holds for $p = 0$ as well; and (4.18) is just (4.17) for $p = 1$ and $N_1 = K_1$. Assume then that k is a positive integer and constants N_p, $0 \leqq p \leqq k$, exist such that (4.18) is true. Leibnitz's rule says that

$$f^{(k+1)} = \sum_{p=0}^{k} \binom{k}{p} f^{(p)} g^{(k-p+1)}$$

and so

$$|f^{(k+1)}(re^{i\theta})| \leqq \frac{1}{r^{k+1}} \frac{|f(re^{i\theta})|}{[\delta_n(\theta)]^{2k+2}} \sum_{p=0}^{k} \binom{k}{p} \frac{N_p K_{k-p+1}}{[\delta_n(\theta)]^{p-k}} \tag{4.19}$$

by (4.16) and our inductive hypothesis. Examination of (4.19) shows that

N_{k+1} can be chosen so that (4.18) holds with $p = k + 1$; thus by induction, (4.18) holds for all non-negative integers p.

Our next result, (4.20), follows readily from the last one. Let $a_n < \theta_1 \leqq \theta_2 < b_n$. Since h is continuous at each point of $[\theta_1, \theta_2]$,

$$|f(re^{i\theta})| \to \exp\{h(\theta)\} = |f(e^{i\theta})|$$

uniformly on $[\theta_1, \theta_2]$ (cf. [3, p. 18]). But $\exp\{h(\theta)\}$ has a positive lower bound on $[\theta_1, \theta_2]$; so there exists $r_1 = r_1(\theta_1, \theta_2)$ such that if $r_1 \leqq r < 1$ and $\theta \in [\theta_1, \theta_2]$, then $|f(re^{i\theta})| \leqq 2\exp\{h(\theta)\}$. Thus $r_1 \leqq r < 1$ and $\theta \in [\theta_1, \theta_2]$ implies

$$|f^{(p)}(re^{i\theta})| \leqq (M_p/r^p)\,|f(e^{i\theta})|/[\delta_n(\theta)]^{2p}, \tag{4.20}$$

where $M_p = 2N_p$, $p = 0, 1, 2, \cdots$.

We have now reached a point in the proof where the full strength of condition (ii) will be used.

Claim. *If k is a non-negative integer and ϕ is defined on $[-\pi, \pi] \sim E = \bigcup_{n=1}^{\infty} (a_n, b_n)$ by*

$$\phi(t) = \frac{\exp\left\{\dfrac{-s_n}{(t-a_n)^\alpha} + \dfrac{-s_n}{(b_n-t)^\alpha}\right\}}{[(t-a_n)(b_n-t)]^k} \qquad (t \in (a_n, b_n)),$$

then ϕ is a bounded function.

Proof of Claim. ϕ is a positive function and for each n, $\lim_{t\to a_n^+} \phi(t) = \lim_{t\to b_n^-} \phi(t) = 0$; hence there exists $t_n \in (a_n, b_n)$ such that $\phi(t_n) = \sup\{\phi(t): a_n < t < b_n\}$. Put

$$\phi_n(t) = \exp\{-s_n/(t-a_n)^\alpha\}/(t-a_n)^k \qquad (a_n < t < b_n).$$

A calculation shows that $\phi_n'(t) > 0$ if and only if $s_n/(t-a_n)^\alpha > k/\alpha$. However $a_n < t < b_n$ implies $s_n/(t-a_n)^\alpha > s_n/(b_n-a_n)^\alpha$ and $s_n/(b_n-a_n) \to +\infty$ with n; consequently,

$$\sup_{a_n<t<b_n} \phi_n(t) = \exp\{-s_n/(b_n-a_n)^\alpha/(b_n-a_n)^k\}$$

for all but finitely many values of n. Thus for sufficiently large n and $t \in (a_n, b_n)$,

$$\begin{aligned}\phi(t) &= \phi_n(t)\phi_n(a_n+b_n-t)\\ &< \exp\{-s_n/(b_n-a_n)^\alpha\}/(b_n-a_n)^{2k};\end{aligned}$$

so by (ii), ϕ is a bounded function.

We are finally in position to show that $f^{(p)} \in H^\infty$ for each nonnegative integer p; we have already seen that this is the case when $p = 0$. So assume that p is a non-negative integer such that $f^{(p)} \in H^\infty$. We begin by combining the preceding claim (with $k = 2p + 2$) and (4.20) to obtain a constant K with the following property: if $\theta_1, \theta_2 \notin E$ and θ_1, θ_2 belong to the same complementary component, say (a_n, b_n), then there exists $r_1 = r_1(\theta_1, \theta_2)$ such that

$$|f^{(p+1)}(re^{i\theta})| \leqq K/r^{p+1}, \tag{4.21}$$

provided $r_1 \leqq r < 1$ and θ is between θ_1 and θ_2. The next step is to extend $f^{(p)}$ to $\bar{U}$. If $\theta \notin E$ and if $\lim_{r \to 1} f^{(p)}(re^{i\theta})$ exists, we define $f^{(p)}(e^{i\theta})$ to be this limit. This extends $f^{(p)}$ to a dense subset of $T \sim F$ because $f^{(p)} \in H^\infty$ and H^∞ functions have radial limits almost everywhere on T. Suppose now that $\theta_1, \theta_2 \in (a_n, b_n)$ and that $f^{(p)}(e^{i\theta_1})$, and $f^{(p)}(e^{i\theta_2})$ are defined. Since

$$f^{(p)}(re^{i\theta_2}) - f^{(p)}(re^{i\theta_1}) = ir \int_{\theta_1}^{\theta_2} f^{(p+1)}(re^{it})e^{it}\, dt,$$

it follows that

$$|f^{(p)}(re^{i\theta_2}) - f^{(p)}(re^{i\theta_1})| \leqq (K/r^p)\, |\theta_2 - \theta_1|;$$

hence,

$$|f^{(p)}(e^{i\theta_2}) - f^{(p)}(e^{i\theta_1})| \leqq K\, |\theta_2 - \theta_1|. \tag{4.22}$$

This means that $f^{(p)}$ is uniformly continuous on the dense subset of the open arc, (e^{ia_n}, e^{ib_n}), where it is defined; consequently, $f^{(p)}$ has a unique extension to (e^{ia_n}, e^{ib_n}) such that (4.22) holds for *all* points $\theta_1, \theta_2 \in (a_n, b_n)$, $n = 1, 2, \cdots$. We now have $f^{(p)}$ continuous, as a function on $T \sim F$; and

$$|f^{(p)}(e^{i\theta})| \leqq M_p\, |f(e^{i\theta})|/[\delta_n(\theta)]^{2p} \tag{4.23}$$

(see (4.20)) on a dense subset of $T \sim F$. $f^{(p)}$ can therefore be extended to a function on $\bar{U}$ whose restriction to T is continuous if we put $f^{(p)}(e^{i\theta}) = 0$ for $e^{i\theta} \in F$. Thus $f^{(p)}(e^{i\theta})$ is defined for all $\theta \in [-\pi, \pi]$. We want to show now that (4.22) holds for all $\theta_1, \theta_2 \in [-\pi, \pi]$; we have shown this when θ_1, θ_2 belong to the same complementary component. To begin with, it is easily seen that (4.22) holds if θ_1, $\theta_2 \in [a_n, b_n]$; this is because $\lim_{\theta_1 \to a_n^+} f^{(p)}(e^{i\theta_1}) = \lim_{\theta_2 \to b_n^-} f^{(p)}(e^{i\theta_2}) = 0$. Suppose next that $\theta_1 \in [a_k, b_k]$ and $\theta_2 \in [a_n, b_n]$ where $k \neq n$. If $\theta_1 < \theta_2$, then necessarily $b_k \leqq a_n$; so what we have already proved and the triangle inequality yields

$$|f^{(p)}(e^{i\theta_2}) - f^{(p)}(e^{i\theta_1})| \leqq K(\theta_2 - a_n) + K(b_k - \theta_1) \leqq K(\theta_2 - \theta_1) = K\, |\theta_2 - \theta_1|.$$

Finally, if $\theta_1 \in E$ but θ_1 is not an end point of one of the complementary components, choose a sequence $\{t_j\} \subset [-\pi, \pi] \sim E$ such that $t_j \to \theta_1$. If $\theta_2 \in E$, then (4.22) obviously holds; while if $\theta_2 \notin E$, then for every j,

$$|f^{(p)}(e^{i\theta_2}) - f^{(p)}(e^{it_j})| \leqq K\, |\theta_2 - t_j|,$$

and (4.22) follows by letting $j \to \infty$. Hence (4.22) holds for all $\theta_1, \theta_2 \in [-\pi, \pi]$ and consequently there exists a constant M such that

$$|f^{(p)}(e^{i\theta_2}) - f^{(p)}(e^{i\theta_1})| \leqq M\, |e^{i\theta_2} - e^{i\theta_1}| \tag{4.24}$$

for every $\theta_1, \theta_2 \in [-\pi, \pi]$. But (4.24) implies that $f^{(p+1)} \in H^\infty$; thus $f^{(p)} \in H^\infty$ for every positive integer p. We conclude that $f \in A^{(\infty)}$ which completes the proof of the theorem.

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UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA