A CHARACTERIZATION OF MONOTONE FUNCTIONS

BY
DONALD ORNSTEIN[1]

The purpose of this note is to prove the following theorem:

THEOREM. *Let $f(x)$ be a real-valued function of a real variable satisfying the following.*

(a) *$f(x)$ is approximately continuous, i.e., for each x_0 and $\varepsilon > 0$ the set of x such that $|f(x_0) - f(x)| < \varepsilon$ has density 1 at x_0;*

(b) *For each x_0, let E be the set of x, such that $f(x) - f(x_0) \geq 0$. Then*

$$\limsup_{|h|\to 0} m[E \cap (x_0, x_0 + |h|)]/|h| \neq 0$$

where $m(C)$ is Lebesgue measure of C.
Then $f(x)$ is monotone increasing and continuous.

One may be tempted to weaken (b) as follows: (b′) for each x_0 the set of x such that $(f(x) - f(x_0))/(x - x_0) \geq 0$ does not have 0 density at x_0. In this case, however, the conclusion is false, even if we assume $f(x)$ to be continuous. (We will describe such an example at the end of this note.)

Condition (b) may be replaced by the following weaker condition:

$$\limsup_{x\to x_0} (f(x) - f(x_0))/(x - x_0) \geqq 0, \qquad x > x_0$$

neglecting any set of values of x that has density 0 at x_0. This follows from our theorem because if $f(x)$ were not monotone we could add a linear function with positive slope to $f(x)$ in such a way that the result is still not monotone but condition (b) is satisfied.

Without loss of generality we will assume $f(x)$ to be defined only on the unit interval. We will now prove Theorem 1.

LEMMA. *Let A be a measurable set in the unit interval, I, of measure $\gamma > 0$, and r a real number > 1. Assume that $2r\gamma < 1$. Let U be the union of all the intervals J in I such that $m(A \cap J)/m(J) > r\gamma$. Then $m(U) < 2/r$.*

Proof. Pick a finite subset S of the intervals which make up U, such that the measure of their union is within ε of the measure of U.

If there is an interval in S which is contained in the union of the remaining intervals delete it from S. Call the new collection S_1. Delete from S_1 an interval (if there is any) that is in the union of the remaining interval in S_1. Call the result S_2. We will eventually get a collection S' so that no interval in S' is in the union of the remaining intervals and the union of the intervals of S' = the union of the intervals of S. It is easy to see that no point is in

Received September 22, 1968.

[1] This research is supported in part by a National Science Foundation grant.

more than 2 intervals of S'. If l is the sum of the lengths of the intervals of S', then $lr\gamma < 2\gamma$ hence $l < r/2$.

Proof of Theorem 1. We will assume that $f(x)$ satisfies the hypothesis of the theorem but that there is an x_1 and x_2 with $x_1 < x_2$ and that $f(x_1) > f(x_2)$. Pick a y such that $f(x_1) > y > f(x_2)$ and let A be the set of x such that $f(x) \geq y$. We will pick a nested sequence of closed intervals, I_n, (I_{n+1} is in the interior of I_n) whose lengths tend to 0 satisfying:

(1) Let A_n be the set of x such that $f(x) < y - 1/n$. Then $m(A_n \cap I_n)/m(I_n) < 1/2$.

(2) If R is an interval in I_n with left end point in the interior of I_{n+1} and right end point in A and not in I_{n+1}, then $m(A \cap R)/m(R) < 1/2^{n-1}$.

This will give us a contradiction as follows: Let $x_0 = \bigcap_{n=1}^{\infty} I_n$. (1) and (a) imply that $f(x_0) \geq y$, (2) implies that the part of A that lies to the right of x_0 has density 0 at x_0. (I.e., it is impossible to have an $\alpha > 0$ and a sequence of intervals K_n whose lengths tend to 0 and whose left end point is x_0 and the density of A in K_n is greater than α. Arguing by contradiction, we could, without loss of generality, assume that the right end point of K_n is in A. For each K_n we can find an $I_{l(n)}$ such that $K_n \subset I_{l(n)}$ and $K_n \not\subset I_{l(r)+1}$. (2) then implies that $\alpha < 1/2^{l(n)-1}$, but $\lim_{n\to\infty} l(n) = \infty$.)

Our proof will now be finished when we have constructed our sequence I_n satisfying (1) and (2). We will do this inductively and in order to go from I_n to I_{n+1} we will construct a sequence of I_n that, in addition to (1) and (2), satisfies:

(3) $m(A \cap I_n)/m(I_n) < 1/2^{n+2}$;

(4) the left end point of I_n is in A.

We will demand that I_1 satisfy only (3) and (4). Since the density of A at x_2 is 0, it is easy to see that we can pick an interval I_1 with right end point x_2, and left end point in A, and $m(A \cap I_1)/m(I) < 1/2^3$.

Assume, now, that we have picked I_n with properties (3) and (4). We will construct I_{n+1} with properties (1), (2), (3) and (4) and $m(I_{n+1}) \leq \frac{1}{2}m(I_n)$.

Let T be the collection of subintervals L, of I_n with right end point in A, and such that $m(A \cap L)/\mathrm{m}(L) > 1/2^{n-1}$. Let $\bar{T}$ be the set covered by T. Note that $\bar{T}$ covers A except for a set of measure 0.

Let M be a finite subcollection of T such that $m(\bar{T}) - m(\bar{M}) < (1/2^{n+4})m(B)$ where B is the part of A that lies in the left half of I_n. ($\bar{M}$ is the set covered by M.) Note that B must have non-zero measure because of (4) and (b).

We will next pick r subintervals of I_n J_i, $1 \leq i \leq r$, with disjoint interiors such that $\bigcup_{i=1}^{r} J_i \supset \bar{M} \cap I_n^l$ (I_n^l is the left half of I_n), $m(J_i) = 2\mathrm{m}(\bar{M} \cap J_i)$ and the left end point of each J_i is also the left end point of some interval in M. We will pick the J_i as follows: Of all the points that are left end points of some interval in M, pick the one farthest to the left. Call it p_1. (p_1 lies in the left half of I_n since $\bar{M}$ covers part of B.) Let J_1 be the smallest interval whose left end point is p_1 and such that $m(\bar{M} \cap J_1) \leq \frac{1}{2}m(J_1)$. (To see that $J_1 \subset I_n$, we first note that our lemma implies that $m(\bar{T}) \leq \frac{1}{4}m(I_n)$ $m(\bar{M}) \leq m(\bar{T})$, so $m(J_1) \leq 1/2$. Since the left end point of J_1 is in the left half of I_n, $J_1 \subset I_n$.) Note that if J_1 intersects the interior of an interval in M, then it covers the interval completely. Of all the points that are end points of some interval in M not covered by J_1, let p_2 be the one farthest to the left. If $p_2 \notin I_n^l$ we are finished because of the preceding remark. Other wise, we let J_2 be the smallest interval whose left end point is p_2 and such that $m(\bar{M} \cap J_2) \leq \frac{1}{2}m(J_2)$. A continuation of this process yields $J_1 \cdots J_r$.

Let J be the part of $\bigcup_{i=1}^r J_1$ that does not lie in $\bar{M}$ (i.e., $J = \bigcup_{i=1}^r J_i - \bar{M}$). J is the union of a finite number of disjoint intervals, each of which has its left end point in A (because the right end points of the intervals in M are in A). J is in the interior of I_n and has length $< \frac{1}{2}m(I_n)$.

The density of $\bar{T}$ in $J < 1/2^{n+3}$. This is an immediate consequence of the following inequalities: $m(\bar{T} - \bar{M}) \leq (1/2^{n+4})m(B)$ and

$$m(J) \geq (1 - 1/2^{n+4})m(B).$$

(The second inequality comes from the fact that

$$m(\textstyle\bigcup_{i=1}^r J_i \cap \bar{M}) \geq (1 - 1/2^{n+4})m(B).)$$

Now pick one of the intervals making up J in which the density of $\bar{T} < 1/2^{n+3}$. It is clear that we can move the right end point a little to the left so that the density of $\bar{T}$ in this new interval is still $< 1/2^{n+3}$ and the right end point is not in $\bar{T}$. Call this interval H^1.

If we let $I_{n+1} = H^1$, then I_{n+1} would satisfy (3) and (4). (2) would hold because the right end point of H^1 is not in $\bar{T}$.

If the density A_{n+1} in $H^1 < 1/2$, then we will let $I_{n+1} = H^1$. If not, we will pick an interval $H^2 \subset H^1$ precisely as we picked H^1 except that H^1 will play the role of I_n. Then pick $H^3 \subset H^2$ with H^2 playing the role of I_n.

We will now show that for some integer t, the density of A_{n+1} in $H^t < 1/2$. Assume the above statement is false. Then H^t, $t = 1, 2, \cdots$ determine a nested sequence of intervals and a point x_0. By (a), $f(x_0) \leq y - 1/(n+1)$. However x_0 is contained in another nested sequence of intervals, $J_{i(t)}^t$. (When we are determining H^{t+1}, we will let M^t, J^t, etc. denote what corresponds to M, J, etc. $J_{i(t)}^t$ will be the interval in $\bigcup_{i=1}^{r(t)} J_i^{(t)}$ in which H^{t+1} lies.) The density of A in $J_{i(t)}^t > 2/2^{n-1}$. (a), together with the above fact shows that $f(x_0) \geq y$, giving a contradiction.

Pick a t such that the density of A_{n+1} in $H^t < 1/2$, and let $I_{n+1} = H^t$. I_{n+1} obviously satisfies (1), (3) and (4). To show (2): suppose there were an interval R, with left end point in $I_{n+1} = H^t$ and right end point in A but not in H^t, and $m(A \cap R)/m(R) > 1/2^{n-1}$. Pick H^s so that $R \subset H^s$ and $R \not\subset H^{s+1}$ (let $H^0 = I_n$). But then R will be an interval in T^s and that means that the right end point of H^{s+1} is in $\bar{T}^s$, contradicting the construction of H^{s+1}. (This was insured by the very last step in picking H^{s+1}.)

Description of a continuous function satisfying (b') *which is not monotone.* We will first describe a function $g(x)$ $0 \leq x \leq 1$ as follows: Pick six points $0 < p_1 < p_2 < p_3 < p_4 < p_5 < p_6 < 1$. Let $f(0) = 1, f(p_1) = 1 + 1/3$, $f(p_2) = 1/3, f(p_3) = 1 + 1/3, f(p_4) = -1/3, f(p_5) = 2/3, f(p_6) = -1/3$, $f(1) = 0$. Choose $g(x)$ to be linear on each of the intervals $(0, p_1)$, (p_2, p_2) $\cdots$ (p_5, p_6), $(p_6, 1)$.

It is easily checked that there is an $\alpha > 0$ such that for each x_0 $(0 \leq x_0 \leq 1)$ the set of x $(0 \leq x \leq 1)$ such that $(g(x) - g(x_0))/(x - x_0) > 0$ has measure $> \alpha$.

We will now define a sequence of functions g_n. Let $g_1 = g$ and assume that we have defined g_n such that $g(0) = 1, g(1) = 0$ and the graph of g_n consists of a finite number of straight line segments. To get g_{n+1} we simply replace each line segment of the graph of g_n, having negative slope, with a linear transformation of the graph of g. (I.e., suppose g_n is linear on (a, b) but not in any larger interval and has negative slope in (a, b). Let $T(x) = cx + d$ and $T'(x) = C' x + d'$. Determine T so that $T(a) = 0$, $T(b) = 1$ and T' so that $T'(0) = g_n(a)$, $T'(1) = g_n(b)$ and replace g_n on (a, b) by $T'[g(T(x)]$ on (a, b).)

It is easily checked that g_n converges pointwise to a continuous function, g_∞ and that g_∞ satisfies (b') and $g_\infty(0) = 1$ and $g_\infty(1) = 0$.

STANFORD UNIVERSITY
STANFORD, CALIFORNIA