# EX-HOMOTOPY THEORY I

BY
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This is the first of a series of studies of a generalization of ordinary homotopy theory. The basic notion is simple enough and seems to have occurred more or less simultaneously to others as well as myself. I understand that Heller and Hodgkin, independently, have given applications to the Eilenberg-Moore spectral sequence. Also McLendon [4] has announced a generalization of the Adams spectral sequence on these lines. My own work is directed towards applications of a different type. Some of these are contained in [3], to which the present note is closely related. Others will be given subsequently.

## 1. Basic notions

Let $B$ be a space. By an *ex-space* (over $B$) we mean a triple $(X, \sigma, \rho)$, where $X$ is a space and

$$B \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

are maps such that $\rho\sigma = 1$. Normally it will be sufficient to denote the ex-space by $X$. We refer to $\sigma$ as the *section*, to $\rho$ as the *projection*. Together they constitute an *ex-structure* on the *total space* $X$ over the *base space* $B$. Notice that $B$ can always be regarded as an ex-space over itself, with $\rho = 1 = \sigma$. We refer to this as the *trivial ex-space* over the given base.

We describe an ex-space $X$ as *proper* if $\sigma B$ is a closed subspace of $X$. When this condition is satisfied we can embed $B$ in $X$, by means of $\sigma$, so that $\rho$ constitutes a retraction. Instead of regarding $B$ as a retract of $X$ we regard $X$ as an "extract" of $B$. This change of view opens up the prospect of the following development.

We shall outline a theory which reduces to ordinary homotopy theory when $B$ is a point. The generalization proceeds on formal lines, for the most part; whenever we meet a basepoint, in the ordinary theory, we replace it by $B$, using the section and projection in an appropriate way.

Starting from the given base space we have begun to construct a new category out of the category of topological spaces. The objects in the new category are ex-spaces. We now define the morphisms. Let $X_i$ $(i = 0, 1)$ be an ex-space over $B$ with section $\sigma_i$ and projection $\rho_i$. By an *ex-map* $f : X_0 \to X_1$ we mean an ordinary map such that

$$f\sigma_0 = \sigma_1, \qquad \rho_1 f = \rho_0, \tag{1.1}$$

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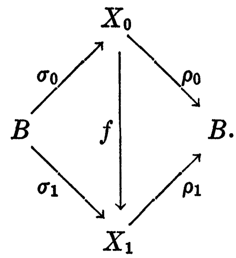
as shown in the following diagram:




By the *trivial ex-map* of $X_0$ to $X_1$ we mean $\sigma_1\rho_0$, which satisfies (1.1) since $\rho_i\sigma_i = 1$. Note that the composition of any ex-map with a trivial ex-map, on either side, is again a trivial ex-map. Thus the category of ex-spaces and ex-maps is a pointed category. The equivalences in the category will be called *ex-homeomorphisms*.

Let $X$ be an ex-space with section $\sigma$ and projection $\rho$. We describe a subspace $X'$ of $X$ as *admissible* if it contains the image of $\sigma$. Under this condition we can regard $X'$ as an ex-space, with section $\sigma'$ obtained by restricting the codomain of $\sigma$, and projection $\rho'$ obtained by restricting the domain of $\rho$. We refer to this as the *relative* ex-structure and to $X'$, with this ex-structure, as a *subspace* of the ex-space $X$. Note that the inclusion map $X' \to X$ is an ex-map.

Let $X''$ be the space obtained from $X$ by identifying points of $X'$ which have the same image under the projection. When $X'$ is admissible we can give $X''$ ex-structure so that the natural projection $X \to X''$ is an ex-map. We denote the ex-space thus obtained by $X/X'$ and refer to $X''$ as the ex-space obtained from $X$ by collapsing $X'$. When $X' = \sigma B$, in particular, we have $X'' = X$.

In our category products are defined as follows. Let $X_i$ $(i = 0, 1)$ be an ex-space over $B$. The *direct product* $X_0 \times X_1$ is the subspace of the ordinary topological product consisting of pairs $(x_0, x_1)$ such that $\rho_0 x_0 = \rho_1 x_1$, with the section $\sigma$ and projection $\rho$ given by

$$\sigma b = (\sigma_0 b, \sigma_1 b) \qquad (b \in B)$$

$$\rho(x_0, x_1) = \rho_0 x_0 = \rho_1 x_1 \qquad (x_i \in X_i).$$

The *inverse product*, or *wedge sum*, $X_0 \vee X_1$ can be defined as the subspace of the direct product consisting of pairs $(x_0, x_1)$ such that $x_0 = \sigma_0 \rho_1 x_1$ or $x_1 = \sigma_1 \rho_0 x_0$. Structural ex-maps

$$X_0 \leftarrow X_0 \times X_1 \rightarrow X_1, \qquad X_0 \rightarrow X_0 \vee X_1 \leftarrow X_1$$

are defined in the obvious way. The *smash product* $X_0 \wedge X_1$ is defined to be the ex-space obtained from $X_0 \times X_1$ by collapsing $X_0 \vee X_1$. Direct, inverse and smash products of ex-maps are similarly defined.

If $X$ is an ex-space over $B$ then the structural ex-maps constitute ex-homeomorphisms $X \times B \to X \to X \vee B$. Moreover $X \wedge B$ is ex-homeomorphic to $B$.

Pull-backs and push-outs are also defined in our category. Thus let $X, X_i$ $(i = 0, 1)$ be ex-spaces, over $B$, and let $f_i : X_i \to X$ be an ex-map. Then the pull-back $X'$ of $(f_0, f_1)$ is defined to be the subspace of the direct product ex-space $X_0 \times X_1$ consisting of pairs $(x_0, x_1)$ such that $f_0 x_0 = f_1 x_1$, where $x_i \in X_i$. The structural ex-maps of the direct product determine, by restriction, ex-maps $g_i : X' \to X_i$, with the appropriate formal properties to complete the pull-back structure. Push-outs are similarly defined, using the inverse rather than the direct product.

The basic procedure we have followed can be applied to any category $\mathcal{C}$. Having chosen an object $B$ of $\mathcal{C}$ we define the corresponding ex-category $\mathcal{C}_B$ of ex-objects and ex-morphisms. The ex-category is pointed, in any case. If $\mathcal{C}$ admits pull-backs and direct products, or push-outs and inverse products, then $\mathcal{C}_B$ does the same. If $\mathcal{C}$ is a pointed category, moreover, then functors from $\mathcal{C}$ into $\mathcal{C}_B$ are defined by taking the direct or inverse product with $B$. Although these functors are not unimportant in the topological case, more interesting functors into the category of ex-spaces can be defined as follows.

Let $G$ be a compact Lie group with identity $e$, and let $A$ be a completely regular space. Suppose that $G$ acts on $A$ as a topological transformation group. Then we describe $A$ as a *G-space*. The image of $x \in A$ under $g \in G$ will be denoted by $g \cdot x$. We describe $A$ as a *pointed* $G$-space if $A$ is a pointed space, in the ordinary sense, such that

$$(1.2) \qquad g \cdot x_0 = x_0 \qquad (g \in G)$$

where $x_0 \in A$ is the basepoint. For example the sphere $S^n$ is a pointed $O(n)$-space, provided the basepoint is taken to be one of the poles.

We adopt the basic terminology of $G$-spaces, as given in [5], with appropriate modifications where the basepoint is concerned. Thus if $A_i$ $(i = 0, 1)$ is a pointed $G$-space then a pointed $G$-map $f : A_0 \to A_1$ means an ordinary pointed map such that

$$(1.3) \qquad f(g \cdot x) = g \cdot f(x) \qquad (g \in G,\ x \in A_0).$$

This condition is satisfied when $f$ is constant. Hence the category of pointed $G$-spaces and pointed $G$-maps is a pointed category. The direct or inverse product of pointed $G$-spaces is defined in the obvious way.

Now let $P$ be a principal $G$-bundle with base space $B$. Given a pointed $G$-space $A$, consider the associated bundle $E$ over $B$ with $A$ as fibre. Since (1.2) is satisfied we have a canonical cross-section of $E$, defined as in (9.3) of [7], and so $E$ determines an ex-space $P_*(A)$ over $B$. Moreover let $f : A_0 \to A_1$ be a pointed $G$-map, where $A_i$ $(i = 0, 1)$ is a pointed $G$-space. Let $E_i$ denote the associated bundle with fibre $A_i$. The bundle map $E_0 \to E_1$ associated with $f$ respects the canonical cross-sections and hence determines an ex-map

$$P_*(f) : P_*(A_0) \to P_*(A_1).$$

Thus $P_*$ constitutes a functor from the category of pointed $G$-spaces to the category of ex-spaces over $B$. Moreover $P_*$ respects the direct and inverse products in the sense that the transform of the product is naturally ex-homeomorphic to the product of the transforms.

## 2. Ex-homotopy

Let $X$, $Y$ be ex-spaces, over $B$. By an *ex-homotopy*

$$f_t : X \to Y \qquad (t \in I)$$

we mean an ordinary homotopy such that $f_t$ is an ex-map for all values of $t$. We write $f_0 \simeq f_1$, when such an ex-homotopy exists. This defines an equivalence relation, on the set of ex-maps of $X$ into $Y$, and we denote by $\pi(X, Y)$ the set of ex-homotopy classes thus obtained. The class of the trivial ex-map is denoted by 0.

Let $Z$ be an ex-space, over $B$. Composition on the left with an ex-map $g : Y \to Z$ determines a function

$$g_* : \pi(X, Y) \to \pi(X, Z),$$

while composition on the right with an ex-map $f : X \to Y$ determines a function

$$f^* : \pi(Y, Z) \to \pi(X, Z).$$

Both functions send 0 into 0.

Let $E$ be an ex-space. We say that a sequence of ex-maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact for $\pi(E, \ )$ if we have image $f_* =$ kernel $g_*$ in the induced sequence

$$\pi(E, X) \xrightarrow{f_*} \pi(E, Y) \xrightarrow{g_*} \pi(E, Z).$$

We say that the sequence of ex-maps is exact for $\pi(\ , E)$ if we have image $g^* =$ kernel $f^*$ in the induced sequence

$$\pi(X, E) \xleftarrow{f^*} \pi(Y, E) \xleftarrow{g^*} \pi(Z, E).$$

Similar definitions are made in the case of longer sequences of ex-maps.

Let $\pi_i$ $(i = 0, 1)$ denote the structural ex-maps of the direct product of ex-spaces $X_i$; thus

$$X_0 \xleftarrow{\pi_0} X_0 \times X_1 \xrightarrow{\pi_1} X_1.$$

An ex-map $h : X \to X_0 \times X_1$ determines, and is determined by, its component ex-maps $(\pi_0 h, \pi_1 h)$, where $\pi_i h : X \to X_i$, and ex-homotopies behave in the same way. Thus we obtain a natural equivalence

$$\pi(X, X_0 \times X_1) \leftrightarrow \pi(X, X_0) \times \pi(X, X_1).$$

Similarly in the case of the inverse product we obtain a natural equivalence

$$\pi(X_0 \vee X_1, X) \leftrightarrow \pi(X_0, X) \times \pi(X_1, X).$$

We say that an ex-map $f : X \to Y$ is an *ex-homotopy equivalence* if there exists an ex-map $g : Y \to X$ such that $fg \simeq 1$, $gf \simeq 1$. When such a pair of ex-maps exists we say that $X$ and $Y$ have the same *ex-homotopy type*. If $X$ has the same ex-homotopy type as the base space $B$, we say that $X$ is *ex-contractible*. For example, take $X$ to be the total space of a vector bundle over $B$, with $\rho$ the fibration and $\sigma$ the zero cross-section. Then $f_t : X \to X$ constitutes an ex-homotopy where

$$f_t(x) = tx \qquad (x \in X,\ t \in I).$$

Since $f_0$ is the trivial ex-map and $f_1$ is the identity this shows that $X$ is ex-contractible.

By a formal modification of the proof of Lemma 3 of [6] we obtain

LEMMA (2.1). *Let $X$ be an ex-space with an admissible sub-space $X'$. Suppose that the identity ex-map on $X$ is ex-homotopic to an ex-map whose restriction to the ex-space $X'$ is trivial. Then the natural ex-map $X \to X/X'$ is an ex-homotopy equivalence.*

In some cases the set $\pi(X, Y)$ can be expressed in terms of ordinary homotopy theory. For example, let $C$ be a space with basepoint $c_0$ and let $TC$ denote the ordinary topological product $B \times C$ with ex-structure given by

$$\rho(b, c) = b, \qquad \sigma b = (b, c_0),$$

where $b \in B$, $c \in C$. Let $X$ be any proper ex-space, over $B$, and let $B$ be embedded in $X$ through the section. Then there is a (1-1)-correspondence between ex-maps $X \to TC$ and ordinary maps

$$(X, B) \to (C, c_0).$$

Furthermore there is a (1–1)-correspondence between such maps and maps

$$(Z, z_0) \to (C, c_0),$$

where $Z$ denotes the space obtained from the space $X$ by collapsing the subspace $B$ to the point $z_0$. Similar remarks apply in the case of homotopies and so $\pi(X, TC)$ is equivalent to the set of homotopy classes of maps of $Z$ into $C$ in the ordinary basepoint-preserving sense. Note that $Z$ can be interpreted as a Thom space when $X$ is the ex-space associated with a sphere-bundle over $B$.

As an example, with arbitrary $C$, take $B = S^n$ and $X = TS^q$. Then $Z$ has the homotopy type of $S^{n+q} \vee S^q$ and so $\pi(TS^q, TC)$ is equivalent to the set

$$\pi_{n+q}(C) \oplus \pi_q(C).$$

A similar analysis can be made of the set $\pi(B \vee C, Y)$, for any ex-space $Y$, where $B \vee C$ denotes the ordinary inverse product with the obvious ex-structure.

Let $G$ be a compact Lie group. The notion of pointed $G$-homotopy between pointed $G$-maps is defined in the obvious way. If $P$ is a principal $G$-bundle over $B$, as in §1, then the functor $P_*$ transforms pointed $G$-homotopies into ex-homotopies. Hence $P_*$ transforms pointed $G$-homotopy equivalences into ex-homotopy equivalences.

## 3. The suspension functor

Let $X$ be a proper ex-space, over $B$. We embed $B$ in $X$, by means of the section $\sigma$, so that the projection $\rho$ constitutes a retraction of $X$ on $B$. By the *suspension* of $X$, in the ex-category, we mean the ex-space $(SX, \sigma', \rho')$ defined as follows. Consider the ordinary cylinder $X \times I$, and write

$$\pi(x, t) = \rho x \qquad (x \in X, t \in I).$$

Then $SX$ is obtained from $X \times I$ by identifying points of $B \times I \cup X \times \dot{I}$ which have the same image under $\pi$. The section $\sigma'$ is given by $\sigma' b = (b, t)$, for any $t$, and the projection $\rho'$ is induced by $\pi$. Note that $SX$ is a proper ex-space.

It is easy to check that $SX$ is ex-homeomorphic to the smash product $X \wedge TS^1$, in the ex-category, where $T$ is the functor defined at the end of §2. Also the suspension of the wedge-sum is ex-homeomorphic to the wedge-sum of the suspensions.

Suspension of ex-maps and ex-homotopies is similarly defined. We denote by

$$S_* : \pi(X_0, X_1) \to \pi(SX_0, SX_1)$$

the function thus obtained.[1]

Now form the wedge sum of $SC$ with itself and consider the structural ex-maps

$$SX \xrightarrow{u_0} SX \vee SX \xleftarrow{u_1} SX.$$

There is an ex-map $m : SX \to SX \vee SX$, defined by

$$\begin{aligned} m(x, t) &= u_0(x, 2t) \qquad (0 \le t \le \tfrac{1}{2}), \\ &= u_1(x, 2t - 1) \quad (\tfrac{1}{2} \le t \le 1). \end{aligned}$$

Using the natural identification

$$\pi(SX \vee SX, Y) = \pi(SX, Y) \times \pi(SX, Y),$$

where $Y$ is any ex-space, we can regard the induced function $m^*$ as a binary operation on the set $\pi(SX, Y)$. We call this operation *track addition* and

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[1] In §7 of [3] a suspension theorem, of the Freudenthal type, is proved for $S_*$ under certain conditions.

normally write

$$m^*(\alpha_0, \alpha_1) = \alpha_0 + \alpha_1 \qquad (\alpha_i \in \pi(SX, Y))$$

without meaning to suggest that the operation is commutative. The formal properties of $m$ are the same as in the ordinary theory. Thus the ex-homotopy-associativity property

$$(1 \vee m)m \simeq (m \vee 1)m$$

is established, by the same argument as in the ordinary case, and it follows that track addition is associative. Similarly for the other properties, so that we finally obtain

THEOREM (3.1). *Under track addition the set $\pi(SX, Y)$ forms a group. If $X$ has the same ex-homotopy type as $SX'$, for some ex-space $X'$, then the group is abelian.*

Note that if $f : Y \to Z$ is an ex-map then the induced function

$$f_* : \pi(SX, Y) \to \pi(SX, Z)$$

is a homomorphism of track groups.

If $A$ is an ex-space over a point, i.e. a pointed space, then $SA$ is the usual (reduced) suspension. Suppose that $A$ is a pointed $G$-space, where $G$ is a compact Lie group. Then $SA$ is also a pointed $G$-space, with the action of $g \in G$ on $SA$ defined to be the suspension of the action of $g$ on $A$. Let $P$ be a principal $G$-bundle over $B$, where $B$ is regular and locally compact. I assert that $SP_*(A)$ is naturally ex-homeomorphic to $P_*(SA)$. In other words, the suspension functor commutes with $P_*$, to within a natural equivalence in the category of ex-spaces. Some other propositions of this type occur in what follows and can be proved in a similar way.[2]

To prove the assertion consider the associated bundle $E$ over $B$ with fibre $A$. As shown in (3.2) of [7], we can construct $E$ as an identification space of a disjoint union $W$ of spaces

$$W_j = (j) \times V_j \times A \qquad (j \in J)$$

where $J$ is an indexing set and $\{V_j\}$ is a covering of the base space $B$. The identifications are given in terms of the maps

$$g_{ij} : V_i \cap V_j \to G \qquad (i, j \in J).$$

If $x \in V_i \cap V_j$ and $y, z \in A$ we identify $(i, x, y)$ with $(j, x, z)$ when $g_{ij}(x) \cdot y = z$. Let $E'$ denote the bundle constructed by making corresponding identifications in the disjoint union $W'$ of the spaces

$$W'_j = (j) \times V_j \times SA \qquad (j \in J).$$

[2] I have not been able to establish a "continuous functor" lemma, on the lines of (1.2) of [1].

The product of the identity on $(j) \times V_j$ and the identification map

$$A \times I \to SA$$

constitutes a map $W_j \times I \to W'_j$. We form the union $f$ of these product maps, as shown in the following diagram, where $p$, $p'$ are the natural maps of the bundle construction and $g$ is induced by $f$:

$$\begin{array}{ccc} W \times I & \xrightarrow{p \times 1} & E \times I \\ \downarrow f & & \downarrow g \\ W' & \xrightarrow[p']{} & E'. \end{array}$$

Without change of notation, consider the ex-spaces of $B$ determined by $E$, $E'$, with their canonical cross-sections. The suspension is defined, as above, through making identifications on $E \times I$. It is easy to check that $g$ is consistent with these identifications and induces a bijection $h : SE \to E'$. Moreover, $h$ is an ex-map, and has the relevant naturality properties. Up to this stage the restriction on $B$ is not required.

In the usual version of fibre bundle theory, as in [7], the co-ordinate neighbourhoods are open sets of the base space. But it is also possible to develop a theory where the co-ordinate neighbourhoods are compact subspaces whose interiors form a covering. Every bundle in this theory is a bundle in the usual sense, and the converse is true when the base space is regular and locally compact. By hypothesis, $B$ satisfies this condition and so we can take each of the neighbourhoods $V_j$ to be compact. Then $f$, in our diagram, is an identification map, and it follows at once that $h$ is a homeomorphism. Furthermore, $h$ is an ex-homeomorphism since $h$ is an ex-map. This proves our assertion, since

$$E = P_*(A), \qquad E' = P_*(SA).$$

## 4. Well-based ex-spaces

Given an ex-space $(X, \sigma, \rho)$, over $B$, we construct an ex-space $(\hat{X}, \hat{\sigma}, \hat{\rho})$, over $B$, as follows. We form $\hat{X}$ from the union of $X$ and the cylinder $B \times I$ by identifying $\sigma b \in X$ with $(b, 0)$ for all $b \in B$. The section $\hat{\sigma}$ is given by $\hat{\sigma} b = (b, 1)$. Let $\pi : \hat{X} \to X$ be given on $X$ by the identity and on $B \times I$ by $\pi(b, t) = b$. Then $\pi\hat{\sigma} = \sigma$ and we take $\hat{\rho} = \rho\pi$, so as to make $\pi$ an ex-map. Note that $X$ is not an admissible subspace of $\hat{X}$.

We describe the ex-space $X$ as *well-based* if the ex-map $\pi : \hat{X} \to X$ is an ex-homotopy equivalence. It is a simple exercise to prove

THEOREM (4.1). *For any ex-space $X$, the ex-space $\hat{X}$ is well-based.*

Let $f : X \to Y$ be an ex-map, where $X$, $Y$ are ex-spaces over $B$. Then $\pi\hat{f} = f\pi$, where $\hat{f} : \hat{X} \to \hat{Y}$ is the ex-map given by $f$ on $X$, by the identity on

$B \times I$. Similarly with ex-homotopies, and so we obtain

THEOREM (4.2). *If $X$ is well-based and $Y$ has the same ex-homotopy type as $X$ then $Y$ is well-based.*

Note that the well-based ex-spaces over a point are the well-pointed spaces of ordinary homotopy theory.

One of the uses of the present construction is to facilitate comparisons between ex-spaces with the same total space and projection but with different sections. We prove

THEOREM (4.3). *Let $X$ be a space, let $\rho : X \to B$ be a map, and let $\sigma_t : B \to X$ be a homotopy such that $\rho\sigma_t = 1$. Then $\hat{X}_0$ and $\hat{X}_1$ have the same ex-homotopy type, where $X_t = (X, \sigma_t, \rho)$.*

COROLLARY (4.4). *If $X_0$ and $X_1$ are well-based, in* (4.3), *then $X_0$ and $X_1$ have the same ex-homotopy type.*

Let $h : \hat{X}_0 \to \hat{X}_1$ be given by the identity on $X$ and on $B \times I$ by

$$\begin{aligned} h(b, t) &= \sigma_{2t} b \qquad (0 \leq t \leq \tfrac{1}{2}), \\ &= (b, 2t - 1) \quad (\tfrac{1}{2} \leq t \leq 1). \end{aligned}$$

Let $k : \hat{X}_1 \to \hat{X}_0$ be similarly defined, using $\sigma_{1-t}$ instead of $\sigma_t$. Then $h$ and $k$ are ex-maps and it is easy to check that $kh$ and $hk$ are ex-homotopic to the identity ex-maps. This proves the theorem.

Let $G$ be a compact Lie group and let $A$ be a pointed $G$-space with basepoint $x_0 \in A$. Form $\hat{A}$ from the union of $A$ and $x_0 \times I$, as above, and extend the action of $G$ on $A$ to an action of $G$ on $\hat{A}$ so that points of $x_0 \times I$ are left fixed. The natural projection $\pi : \hat{A} \to A$ is a pointed $G$-map. We say that $A$ is a well-pointed $G$-space if $\pi$ is a pointed $G$-homotopy equivalence.[3] Consider the functor $P_*$ determined by a principal $G$-bundle $P$ over $B$, where $B$ is regular and locally compact. Proceeding in much the same way as in the case of the suspension functor we find that $P_*(\hat{A})$ is ex-homeomorphic to $\hat{E}$, where $E = P_*(A)$, and that the ex-homeomorphism carries the transform of $\pi : \hat{A} \to A$ into $\pi : \hat{E} \to E$. When $A$ is well-pointed, as a $G$-space, the transform of $\pi$ is an ex-homotopy equivalence and so $E$ is well-based, as an ex-space.

## 5. Ex-cofibrations

Let $f : X \to Y$ be an ex-map, where $X$, $Y$ are ex-spaces over $B$. We say that $f$ is an *ex-cofibration* if $f$ has the following *ex-homotopy extension property*. Let $Z$ be an ex-space, let $h : Y \to Z$ be an ex-map, and let $g_t : X \to Z$ be an ex-homotopy such that $g_0 = hf$. Then there exists an ex-homotopy $h_t : Y \to Z$ such that $h_0 = h$ and $g_t = h_t f$. When $f$ has this property we define the *ex-cofibre* of the ex-cofibration to be the push-out of $f : X \to Y$ and $\rho : X \to B$.

[3] This is the case, for example, if $A$ is a differentiable $G$-space (i.e. a paracompact differentiable manifold with $G$ acting differentiably).

An important special case is when $Y$ contains an admissible subspace $Y'$ such that the inclusion $Y' \subset Y$ is an ex-cofibration. In that case we prefer to say that the pair $(Y, Y')$ has the ex-homotopy extension property. A necessary and sufficient condition for this property is for $Y \times O \cup Y' \times I$ to be an ex-retract of $Y \times I$. Note that $Y/Y'$ is the ex-cofibre in this case. Just as in Theorem 2 of [6] we obtain

THEOREM (5.1). *Suppose that $Y'$ is ex-contractible, and that $(Y, Y')$ has the ex-homotopy extension property. Then the natural projection $Y \to Y/Y'$ is an ex-homotopy equivalence.*

In ordinary homotopy theory complexes play a special role, but it seems doubtful whether there is a satisfactory generalisation to ex-homotopy theory. We need a reasonably comprehensive class of ex-spaces with various desirable properties. After some experiment I propose the following, which is adequate for the type of application I have in mind.

Let $A$ be a $G$-space, where $G$ is a compact Lie group. We recall that a subspace $A'$ of $A$ is described as *invariant* if $gA' \subset A'$ for all $g \in G$. When this condition is satisfied we regard $A'$ as a $G$-space with the induced action. As in (1.6.1) of [5] we say that $A$ is a *G-ANR* (absolute neighbourhood retract) if, given a normal $G$-space $Y$ and a $G$-map $f$ of a closed invariant subspace $F$ of $Y$ into $A$, there exists an extension of $F$ to a $G$-map of an invariant neighbourhood of $F$. It has been shown (see p. 27 of [5] for references) that every compact differentiable $G$-space is a *G-ANR*. Note that the (finite) product of *G-ANR*'s is again a *G-ANR*.

Let $A$ be a normal pointed $G$-space which is a *G-ANR*. Let $A'$ be a closed invariant pointed subspace of $A$. Make $A \times I$ into a $G$-space, with action given by

$$g\cdot (x, t) = (g\cdot x, t) \qquad (g \in G,\ x \in A,\ t \in I).$$

I assert that $A \times 0 \cup A' \times I$ is a $G$-retract of $A \times I$. For since $A \times I$ is a *G-ANR* there exists an invariant neighbourhood $U$ of $A \times 0 \cup A' \times I$ in $A \times I$, such that $A \times 0 \cup A' \times I$ is a $G$-retract of $U$. Let $V$ be an invariant neighbourhood of $A'$ in $A$ such that $V \times I \subset U$. By (1.1.7) of [5] there exists an invariant map $p : A \to I$ such that $p(x) = 0$ if $x \notin V$ and $p(x) = 1$ if $x \in A'$. If $f$ is a $G$-retraction of $U$ on $A \times 0 \cup A' \times I$ then $f'$ is a $G$-retraction of $A \times I$, where

$$f'(x, t) = f(x, tp(x)) \qquad (x \in A,\ t \in I).$$

This proves the assertion.

Now consider the ex-spaces $E = P_*(A)$, $E' = P_*(A')$, where $P$ is a principal $G$-bundle over $B$. Since $A \times 0 \cup A' \times I$ is a $G$-retract of $A \times I$ it follows that $E \times 0 \cup E' \times I$ is an ex-retract of $E \times I$, and so the pair $(E, E')$ has the ex-homotopy extension property. Suppose, furthermore, that $A'$ is $G$-contractible. Then $E'$ is ex-contractible and hence the natural projection of

$E$ onto $E/E'$ is an ex-homotopy equivalence, by (5.1). We shall use this in the next section.

## 6. Spaces over $B$

By a space over $B$ we mean an ordinary space $X$ with a map $\rho : X \to B$. Usually it is sufficient to denote the pair $(X, \rho)$ by $X$ alone. By a section of $(X, \rho)$ we mean a map $\sigma : B \to X$ such that $\rho\sigma = 1$. An ex-space over $B$ can be regarded as a space over $B$ with a section. Maps and homotopies over $B$ are defined in the obvious way.

The suspension $(\tilde{S}X, \rho')$ of a proper space $(X, \rho)$ over $B$ is defined as follows. The space $\tilde{S}X$ is formed from the union of $X \times I$ and $B \times \dot{I}$ by identifying $(x, t)$ with $(\rho x, t)$ for $x \in X$, $t \in \dot{I}$. The map $\rho'$ is given by

$$\rho'(x, t) = \rho x \qquad (x \in X, t \in I),$$

$$\rho'(b, t) = b \qquad (b \in B, t \in \dot{I}).$$

We have $\rho'\sigma_t = 1$, where $\sigma_t : B \to \tilde{S}X$ is defined by $\sigma_t b = (b, t)$ for $t = 0, 1$. The ex-space $(\tilde{S}X, \sigma_t, \rho')$ will be denoted by $\tilde{S}_t X$. Note that $\tilde{S}_0 X$ is ex-homeomorphic to $\tilde{S}_1 X$.

Now suppose that $X$ admits a section $\sigma$, with $\rho\sigma = 1$, and so constitutes an ex-space. The definition of $\sigma_t$ can be extended to all $t \in I$, by $\sigma_t b = (\sigma b, t)$. Thus a family $\tilde{S}_t X$ of ex-spaces is derived from $X$. When $0 < t < 1$ it is easy to see, by reparametrization, that $\tilde{S}_t X$ is independent of the choice of $t$, to within an ex-homeomorphism.

For any $t \in I$, the identity function on $X \times I$ determines an ex-map

$$\varphi : \tilde{S}_t X \to SX.$$

Now $B \times I$ constitutes an admissible subspace of $\tilde{S}_t X$, and if the pair

$$(\tilde{S}_t X, B \times I)$$

has the ex-homotopy extension property it follows at once from (5.1) that $\varphi$ is an ex-homotopy equivalence. When this is the case we can replace $\tilde{S}_t X$ by $SX$, for many purposes.

When the base space is a point $\tilde{S}A$ is the unreduced suspension of the space $A$. Suppose that $A$ is a $G$-space, and regard $\tilde{S}A$ as a $G$-space in the obvious way. Suppose that $A$ is a compact differentiable pointed $G$-space. Then it follows[4] from (2.7.9) of [5] that $\tilde{S}A$ is a $G$-$ANR$, and hence that the natural projection $\varphi : \tilde{S}_t A \to SA$ is a $G$-homotopy equivalence.

Let $P$ be a principal $G$-bundle over $B$, where $B$ is regular and locally compact. If $A$ satisfies the above conditions, so that $\varphi$ is a $G$-homotopy equivalence, then the transform

$$P_*(\varphi) : P_* \tilde{S}_t(A) \to P_* S(A)$$

[4] It seems reasonable to conjecture that $\tilde{S}A$ is a $G$-$ANR$ if $A$ is a $G$-$ANR$.

lent to $SP_*(A)$, under an ex-homomorphism. A similar argument shows that is an ex-homotopy equivalence. We have shown in §3 that $P_* S(A)$ is equiva- $P_* \tilde{S}_t(A)$ is equivalent to $\tilde{S}_t P_*(A)$ and that $P_*(\varphi)$ is equivalent to

$$\varphi : \tilde{S}_t P_*(A) \rightarrow SP_*(A).$$

Hence $\varphi$ is an ex-homotopy equivalence. Of course this can also be deduced from (5.1).

It is convenient to take $t = \frac{1}{2}$ as standard, and henceforth $\tilde{S}X$ will mean the ex-space with this section. When $\varphi$ is an ex-homotopy equivalence we identify $\pi(SX, Y)$ with $\pi(\tilde{S}X, Y)$ under $\varphi^*$, so that the track addition defined in §3 is transferred to the latter set. Thus $\pi(\tilde{S}X, Y)$ obtains a natural group structure, which is abelian if $X = \tilde{S}X'$ for some ex-space $X'$.

In the applications we usually begin with a euclidean bundle $V$ over $B$, and take $P$ to be the associated principal bundle. Consider the Whitney sum $V \oplus r$ of $V$ and the trivial $r$-plane bundle, where $r = 1, 2, \cdots$. If we denote the associated sphere-bundle by square brackets then $[V \oplus r]$ can be identified with the iterated suspension $\tilde{S}_r[V]$. Thus

$$\pi([V \oplus r], Y)$$

constitutes a group for $r \geq 1$, an abelian group for $r \geq 2$. In the sequel to this note we shall study the structure of these ex-homotopy groups in some detail.

The unreduced suspension is a special case of the join[5] operation, which can be discussed on similar lines. Thus let $X_i$ $(i = 0, 1)$ be a space over $B$. In this category the direct product $X_0 \times X_1$ is the subspace of the ordinary topological product consisting of pairs $(x_0, x_1)$ $(x_i \in X_i)$ such that $\rho_0 x_0 = \rho_1 x_1$, with projection $\rho$ given by $\rho(x_0, x_1) = \rho_i x_i$. Let $X_0 * X_1$ denote the space formed from the union of the cylinder $(X_0 \times X_1) \times I$ and $X_0, X_1$ by identifying $(x_0, x_1, i)$ with $x_i$ for $i = 0, 1$. The join of $X_0$ and $X_1$ in the category of spaces over $B$ is defined to be this space $X_0 * X_1$ with projection $\rho'$ given by

$$\rho'(x_0, x_1, t) = \rho_i x_i \qquad (x_i \in X_i, t \in I).$$

If $X_1 = I \times B$, and $\rho_1$ is right projection, then $X_0 * X_1$ is equivalent to $\tilde{S}X_0$.

Next suppose that $X_1$ admits a section $\sigma_1$, and so constitutes an ex-space. Then we can define a section $\sigma'$ of the join $X_0 * X_1$ by $\sigma' b = (x_0, \sigma_1 b, 1)$, where $b \in B$ and where $x_0 \in X_0$ is arbitrary. Thus $X_0 * X_1$ also constitutes an ex-space.

Finally, suppose that $X_i$ admits a section $\sigma_i$, for $i = 0, 1$, and so constitutes an ex-space. Then we can define a family $\sigma'_t$ of sections of $X_0 * X_1$, for $t \in I$, by

$$\sigma'_t b = (\sigma_0 b, \sigma_1 b, t) \qquad (b \in B).$$

Choose a value of $t$ and regard $X_0 * X_1$ as an ex-space with $\sigma'_t$ as section. In the ex-category, the smash product $X_0 \wedge X_1$ is defined and the identity function

[5] In fibre bundle theory this is known as the fibre-join. Given a pair of euclidean bundles, over $B$, the fibre-join of their associated sphere-bundles is equivalent to the sphere-bundle associated with their Whitney sum.

on the cylinder $(X_0 \times X_1) \times I$ determines an ex-map

$$\varphi : X_0 * X_1 \to S(X_0 \wedge X_1).$$

This natural projection is an ex-homotopy equivalence, under certain conditions. For example, let $G_i$ be a compact Lie group and let $P_i$ be a principal $G_i$-bundle over $B$. Suppose that $X_i = P_{i*}(A_i)$, where $A_i$ is a differentiable $G_i$-space. Then an argument similar to the one used in the case of $\tilde{S}$ shows that $\varphi$ is an ex-homotopy equivalence, provided $B$ is regular and locally compact. For applications in which the join operation plays a major role see the latter part of [3].

## 7. The Puppe sequence

Let $X$ be an ex-space over $B$. By the *cone* on $X$, in the ex-category, we mean the ex-space $(CX, \sigma', \rho')$ defined as follows. Consider the ordinary cylinder $X \times I$ and write

$$\pi(x, t) = \rho x \qquad (x \in X, t \in I).$$

Then $CX$ is obtained from $X \times I$ by identifying points of $B \times I \cup X \times 0$ which have the same image under $\pi$. The section $\sigma'$ is given by $\sigma' b = (b, t)$, for any $t$, and the projection $\rho'$ is induced by $\pi$. It is easy to check that $CX$ is ex-contractible. An ex-map $u : X \to CX$ is given by $ux = (x, 1)$.

Given an ex-map $f : X \to Y$ we define the *ex-map cone* $C_f$ to be the push-out of $(u, f)$. It is simple exercise to check that the structural ex-map $v : Y \to C_f$ is an embedding, in the topological sense, so that $Y$ can be regarded as a subspace of $C_f$. Furthermore, following almost literally the argument in §1 of [6], we obtain

Lemma (7.1). *If $f : X \to Y$ is an ex-map then the structural ex-map $v : Y \to C_f$ is an ex-cofibration, with ex-cofibre $C_f/Y = SX$.*

Let $E$ be any ex-space. It is a simple exercise to show that the sequence

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ v\ } C_v$$

is exact for $\pi(\ \ , E)$. Substitute $u$ for $f$ and consider the corresponding exact sequence

$$Y \to C_f \to C_\varphi .$$

By (7.1), $u$ is an ex-cofibration and so it follows as in Theorem 2 of [6] that $C_v$ has the same ex-homotopy type as the ex-cofibre $SX$ of $v$. Pursuing this argument in exactly the same way as Puppe does in Theorem 5 of [6], we arrive at

Theorem (7.2). *Let $f : X \to Y$ be an ex-map, and let $E$ be any ex-space. Then the sequence*

$$X \to Y \to C_f \to SX \to SY \to \cdots$$

*with appropriate ex-maps, is exact for $\pi(\ \ , E)$.*

Thus we have a generalised form of Puppe's mapping sequence, and following Puppe we can deduce various corollaries. As an example we state

COROLLARY (7.3). *Let* $X_i$ $(i = 0, 1)$ *be a well-based ex-space. Then* $S(X_0 \times X_1)$ *has the same ex-homotopy type as* $SX_0 \vee SX_1 \vee S(X_0 \wedge X_1)$.

The dual notion of ex-fibration will be discussed in the second paper in this series.

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