SUBORDINATE FAMILIES OF ANALYTIC FUNCTIONS[1]

BY

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We shall be concerned with one-parameter families of analytic functions $F(z, t)$, $0 \leq t \leq T$. More specifically, we are interested in conditions for $F(z, t)$ to be subordinate (definition below) to $F(z, 0)$ in $|z| < 1$ for each fixed t. The idea of subordination of a whole family of functions is of frequent occurrence in geometric function theory. In fact we take the view that this idea unifies the theory. For example, if $f(z)$ is univalent in $|z| < 1$ and $f(0) = 0$, then $f(z)$ is starlike if and only if $F(z, t) = (1 - t)f(z)$ is subordinate to $F(z, 0)$ for each t satisfying $0 \leq t \leq 1$. (Throughout the paper, as in this example, we shall have $F(z, 0) \equiv f(z)$.)

The origins of our technique are in the paper [2] of M. S. Robertson, and our Theorem 1, which provides a necessary condition for subordination, is a reformulation of Robertson's theorem. However we are able to replace the original assumption that $f(z)$ is univalent by the simple requirement $f'(0) \neq 0$. One benefit of this modification is the acquisition (after Theorem 2) of a whole class of theorems of the form "Subordination implies univalence and convexity". A member of this class of theorems is the recent result of T. H. MacGregor [1] in which

$$F(z, t) = \frac{1}{t} \int_0^t f(ze^{i\theta})\, d\theta.$$

The four theorems presented here can be described briefly and fairly accurately with the terminology of Differential Calculus. Theorem 1 provides a "first-derivative" criterion necessary for subordination. However if the "first derivative" "vanishes" it is possible and useful to make an assertion concerning the "second derivative"; hence Theorem 2. Theorems 3 and 4 are converses of Theorems 1 and 2 respectively. Thus they provide sufficient conditions for subordination. Theorems "of order greater than 2" can be stated, but it is not clear that they would be useful.

The basic definition of subordination is as follows. If $f(z)$ and $g(z)$ are analytic in $|z| < r$, we say that $g(z)$ is subordinate to $f(z)$ in $|z| < r$ if $g(z) = f(\omega(z))$ for some "Schwarz function" $\omega(z)$. That is, $\omega(z)$ is analytic in $|z| < r$ and $|\omega(z)| \leq |z|$. We shall write $g(z) \prec f(z)$ to express this relationship. A commonly occurring set of conditions sufficient for subordination is that $f(z)$ be univalent, that range $g \subset$ range f, and that $g(0) = f(0)$.

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THEOREM 1. *Let $F(z, t)$ be analytic in $|z| < 1$ for each fixed t in $0 \leq t \leq T$ $(T > 0)$, let*

$$F(z, 0) \equiv f(z), \quad f'(0) \neq 0, \tag{1}$$

and let

$$F(z, t) \prec f(z) \quad (|z| < 1, 0 \leq t \leq T). \tag{2}$$

Finally, let the partial derivative

$$F_2(z, 0) = \lim_{t \to 0^+} (F(z, t) - f(z))/t$$

exist for each z. Then

$$\operatorname{Re}\, (F_2(z, 0)/zf'(z)) \leq 0 \tag{3}$$

whenever the denominator is different from zero.

Proof. For each t there is an analytic function $\omega(z, t)$ such that $|\omega(z, t)| \leq |z|$ and $F(z, t) = f(\omega(z, t)$. In particular $f(z) = f(\omega(z, 0))$. Since f is one-to-one in a neighborhood of 0, $\omega(z, 0) = z$ for $|z|$ sufficiently small and hence for all z. Next, we claim that $\lim_{t \to 0^+} \omega(z, t) = z$ for each z. If $|z|$ is sufficiently small this follows from the equations

$$\lim_{t \to 0^+} f(\omega(z, t)) = \lim_{t \to 0^+} F(z, t) = f(z)$$

and the existence of an analytic inverse of f that maps a neighborhood of $f(0)$ onto a neighborhood of 0. For $|z|$ not necessarily small we consider the Taylor expansion

$$\omega(z, t) = \sum_{n=1}^{\infty} a_n(t) z^n, \qquad a_n(t) = \frac{1}{2\pi i} \int_{|z|=\delta} \frac{\omega(z, t)}{z^{n+1}}\, dz.$$

If δ is sufficiently small, $\lim_{t \to 0^+} \omega(z, t)/z^{n+1} = 1/z^n$. Hence we can use Lebesgue's bounded convergence theorem to obtain the result

$$\begin{aligned} \lim_{t \to 0^+} a_n(t) &= 1, \quad n = 1 \\ &= 0, \quad n > 1. \end{aligned}$$

Since $|a_n(t)| \leq 1$ for all n and t, we can, therefore, conclude from the Taylor series that $\lim_{t \to 0^+} \omega(z, t) = z$ for all z.

We can now deduce (3): If $f'(z) \neq 0$, it follows from the equation $F(z, t) = f(\omega(z, t))$ that $\omega_2(z, 0)$ exists and that $F_2(z, 0) = f'(z)\omega_2(z, 0)$. Therefore

$$\begin{aligned} \operatorname{Re} \frac{F_2(z, 0)}{zf'(z)} &= \operatorname{Re} \frac{\omega_2(z, 0)}{z} = \operatorname{Re} \lim_{t \to 0^+} \frac{\omega(z, t) - z}{tz} \\ &= \lim_{t \to 0^+} \frac{1}{t} \left[\operatorname{Re} \frac{\omega(z, t)}{z} - 1 \right] \leq 0 \end{aligned}$$

as required.

Remarks. 1. In specific examples $F_2(z, 0)$ is usually analytic. It then follows that (3) holds for all $z\,(|z| < 1)$.

2. An interesting and easy application of Theorem 1 is given by $F(z, t) = tf(cz) + (1 - t)f(z)$, where $|c| \leq 1$.

We obtain the conclusion

$$\operatorname{Re} \frac{f(cz) - f(z)}{zf'(z)} \leq 0.$$

In particular, this holds for all such c if $f(z)$ is univalent and convex. As an application of Theorem 3 we shall show that this inequality is also sufficient for convexity of a univalent function. Thus Theorem 1 has generated a non-standard criterion for convexity. The geometric interpretation of our criterion is that an analytic Jordan curve is convex if and only if the angle between the outer normal at any point and the vector from this point to any other point on or inside the curve is always obtuse.

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that for all* z

$$F_2(z, 0) = aizf'(z), \tag{4}$$

where a is a real constant, and the second derivative $F_{22}(z, 0)$ exists. Then

$$a^2[1 + \operatorname{Re} zf''(z)/f'(z)] + \operatorname{Re} F_{22}(z, 0)/zf^1(z) \leq 0 \tag{5}$$

for all z for which $zf'(z) \neq 0$.

Proof. Again we conclude from the relations $F(z, t) = f(\omega(z, t))$, $\lim_{t\to 0^+} \omega(z, t) = \omega(z, 0) = z$, and $f^1(z) \neq 0$ that $\omega(z, t)$ is as smooth as $F(z, t)$ at $t = 0$. In particular, if $F_{22}(z, 0)$ exists, so does $\omega_{22}(z, 0)$. Therefore

$$\omega(z, t) = z + \omega_2(z, 0)t + \tfrac{1}{2}\omega_{22}(z, 0)t^2 + o(t^2),$$

$$\frac{\omega(k, t)}{z} = 1 + ait + \frac{1}{2}\frac{\omega_{22}(z, 0)}{z} t^2 + o(t^2),$$

$$\left|\frac{\omega(z, t)}{z}\right|^2 = 1 + \left[\operatorname{Re} \frac{\omega_{22}(z, 0)}{z} + a^2\right] t^2 + o(t^2).$$

Consequently

$$\operatorname{Re} \omega_{22}(z, 0)/z + a^2 \leq 0.$$

This and (4) lead directly to inequality (5).

Remark. We did not actually require the existence of $F_{22}(z, 0)$, but only the finite Taylor formula thereby implied. It follows from this remark that Theorems 1 and 2 are meaningful even if $F(z, t_n)$ is defined only for a positive sequence $\{t_n\}$ approaching 0. Indeed, the derivatives $F_2(z, 0)$ and $F_{22}(z, 0)$ can then be understood as the limits

$$F_2(z, 0) = \lim_{n\to\infty} \frac{F(z, t_n) - f(z)}{t_n},$$

$$F_{22}(z, 0) = \lim_{n\to\infty} \frac{F(z, t_n) - f(z) - F_2(z, 0)t_n}{\frac{1}{2}t_n^2},$$

or simply as coefficients in a Taylor formula. The existence of the second limit, for instance, is equivalent to the Taylor formula

$$F(z, t_n) = f(z) + F_2(z, 0)t_n + \tfrac{1}{2}F_{22}(z, 0)t_n^2 + o(t_n^2).$$

This in turn is equivalent to the corresponding formula

$$\omega(z, t_n) = z + \omega_2(z, 0)t_n + \tfrac{1}{2}\omega_{22}(z, 0)t_n^2 + o(t_n^2)$$

(by applying f^{-1} or f), with the usual relations among the various derivatives. Thus all the previous reasoning remains valid.

COROLLARY (MacGregor). *Let $f(z)$ be analytic in $|z| < 1, f^1(0) \neq 0$. If*

$$F(z, t) = \frac{1}{t}\int_0^t f(ze^{i\theta})\,d\theta \prec f(z)$$

for small positive t, or even for a positive sequence $\{t_n\}$ approaching 0, then $f(z)$ is univalent and convex.

Proof. Calculations yield

$$F_2(z, 0) = \tfrac{1}{2}izf^1(z) \quad \text{and} \quad F_{22}(z, 0) = -\tfrac{1}{3}[zf^1(z) + z^2f^{11}(z)].$$

Inequality (5) then reduces to the usual criterion for convexity, and this together with $f^1(0) \neq 0$ gives the desired conclusion.

Remark. Theorem 2 leads to more general results of the form

$$F(z, t) \prec f(z), f^1(0) \neq 0 \quad \Rightarrow \quad 1 + \operatorname{Re}\,(zf^{11}(z)/f^1(z)) \geq 0.$$

For instance, if

$$F_2(z, 0) = aizf^1(z), \quad F_{22}(z, 0) = b[zf^1(z) + z^2f^{11}(z)],$$

where a and b are real with $b \neq -a^2$, then (5) becomes

$$(a^2 + b)[1 + \operatorname{Re}\,(zf^{11}(z)/f^1(z))] \leq 0 \quad (|z| < 1).$$

The second factor is positive for $|z|$ small, so $a^2 + b < 0$. Therefore

$$1 + \operatorname{Re}\,(zf^{11}(z)/f^1(z)) \geq 0$$

as required.

Another special case is $F(z, t) = \frac{1}{2}[f(z) + f(ze^{it})]$, and we again obtain $1 + \operatorname{Re}\,(zf^{11}(z)/f^1(z)) \geq 0$ from the assumptions $f^1(0) \neq 0$ and $F(z, t) \prec f(z)$. On the other hand, if it is known in advance that $f(z)$ is univalent and convex, it is obvious that $F(z, t) \prec f(z)$. Hence Theorem 2 serves the elementary purpose of showing that univalence and convexity imply the usual inequality $1 + \operatorname{Re}\,(zf^{11}(z)/f^1(z)) \geq 0$. The sufficiency of this inequality can be deduced from our Theorem 4.

THEOREM 3. *Let $F(z, t)$ be analytic in $|z| < 1$ for each fixed t in $0 \leq t \leq T$ $(T > 0$, let*

(6) $$F(z, 0) \equiv f(z), \quad F(0, t) \equiv f(0), \quad f^1(z) \neq 0 \qquad (|z| < 1),$$

and let

$$F_2(z, 0) = \lim_{t \to 0^+} \frac{F(z, t) - f(z)}{t} \tag{7}$$

exist uniformly on compact subsets of the disk $|z| < 1$. *If*

$$\operatorname{Re}\,(F_2(z, 0)/zf^1(z)) < 0 \qquad (0 < |z| < 1), \tag{8}$$

then for each r, $0 < r < 1$, *there exists a corresponding* $\delta > 0$ *such that*

$$F(z, t) \prec f(z) \qquad (|z| < r, 0 \leq t \leq \delta). \tag{9}$$

Proof. For any such r, a compactness argument yields a positive number α such that f is one-to-one in any open disk $\{z : |z - z_1| < \alpha\}$ with $|z_1| \leq r$. In fact we can choose α small enough so that f is one-to-one in the union of any two such disks that overlap. The images of these disks cannot get arbitrarily small. That is, there is a positive β such that

$$\{f(z) : |z - z_1| < \alpha\} \supset \{w : |w - f(z_1)| < \beta\}$$

for each such z_1. Finally, since $\lim_{t \to 0^+} F(z, t) = f(z)$ uniformly on compact sets, there are positive numbers δ_1 and γ, $\gamma < \alpha$, such that

$$|F(z, t) - f(z_1)| < \beta$$

if $|z - z_1| < \gamma, 0 \leq t \leq \delta_1$, and $|z_1| \leq r$. We now define $\omega(z, t)$ in each disk $|z - z_1| < \gamma$, and for $0 \leq t \leq \delta_1$ by $\omega(z, t) = f^{-1}(F(z, t))$, where f^{-1} maps

$$\{w : |w - f(z_1)| < \beta\} \quad \text{into} \quad \{z : |z - z_1| < \alpha\}.$$

This definition is consistent in overlapping disks by our choice of α. Hence $\omega(z, t)$ is well defined and analytic in $\{z : |z| < r + \gamma\}$ for each t, $0 \leq t \leq \delta_1$.

We conclude the proof by showing that $\omega(z, t)$ is a Schwarz function for sufficiently small t. First, from the identity $F(0, t) \equiv f(0)$ and the definition of $\omega(z, t)$ we obtain $\omega(0, t) \equiv 0$. Next, we deduce from (7) that

$$\omega_2(z, 0) = \lim_{t \to 0^+} (\omega(z, t) - z)/t$$

exists uniformly in some neighborhood of each point of $\{z : |z| \leq r\}$. By compactness it follows that this limit is uniform for $|z| \leq r$. Therefore

$$\lim_{t \to 0^+} (\omega(z, t) - z)/tz = F_2(z, 0)/zf^1(z)$$

uniformly for $|z| = r$. By (7), $F_2(z, 0)$ is analytic, so (8) implies that there is a positive ε such that

$$\operatorname{Re}\,(F_2(z, 0)/zf^1(z)) \leq -\varepsilon \qquad (|z| = r).$$

Hence if we set

$$(\omega(z, t) - z)/tz = u(z, t) + iv(z, t) = u + iv,$$

then for all small positive t we have $u \leq -\varepsilon/2$ and $u^2 + v^2 \leq M$ for sme constant M, $(|z| = r)$. Thus

$$|\omega(z, t)/z|^2 = (1 + tu)^2 + t^2v^2 \leq 1 - \varepsilon t + Mt^2 \qquad (|z| = r).$$

Therefore there is a $\delta > 0$ such that $|\omega(z, t)| < r$ for $|z| = r$ and $0 < t \leq \delta$, and the proof is complete.

Remarks. 1. A convenient sufficient condition for (7) is that $F_2(z, t)$ be continuous.

2. The example $F(z, t) = z + z^2 - (z + 2z^2)t$ shows that the assumption $f^1(z) \neq 0$ cannot be replaced by $f^1(0) \neq 0$. Indeed, if $r > \frac{1}{2}$ and t is small there does not exist a function $\omega(z, t)$ analytic in $|z| < r$ such that $f(\omega(z, t)) = F(z, t)$. This equation would imply

$$[1 + 2\omega(z, t)]^2 = 4(1 - 2t)z^2 + 4(1 - t)z + 1,$$

a contradiction because the quadratic function has two simple zeros in $|z| < r$.

3. The conclusion (9) of "subordination of subfamilies on subdisks" is all that can be expected in Theorem 3 because this condition, rather than (2), is the only assumption required to prove Theorem 1. The example $F(z, t) = z - zt + zt^2/(1 - z)$ shows that there need not exist a positive δ such that $F(z, t) \prec f(z)$ in $|z| < 1$ for $0 \leq t \leq \delta$.

4. In spite of the limited nature of the conclusion (9), Theorem 3 has various applications. As a first illustration we return to the example $F(z, t) = tf(cz) + (1 - t)f(z)$, $|c| \leq 1$. We shall show via Theorem 3 that the inequality

$$\operatorname{Re} \frac{F_2(z, 0)}{zf^1(z)} = \operatorname{Re} \frac{f(cz) - f(z)}{zf^1(z)} \leq 0$$

discussed earlier is sufficient for the convexity of $f(z)$ if $f(z)$ is univalent, or even under the assumption that $f^1(z)$ is nonvanishing. We assume the inequality for all c with $|c| = 1$. If $f(z)$ is not convex, then the image by f of some closed disk $|z| \leq s$, $0 < s < 1$, is not convex. It follows that there exists z_0 with $|z_0| = s$ and c_0 with $|c_0| = 1$ such that $tf(c_0 z_0) + (1 - t)f(z_0)$ does not belong to the image of the disk $|z| \leq s$ for any t with $0 < t < 1$. Since $c_0 \neq 1$ we deduce that

$$\operatorname{Re} \frac{f(c_0 z) - f(z)}{zf^1(z)} < 0$$

for $|z|$ small, and therefore for $|z| < 1$ generally. Thus we can use Theorem 3 to obtain $\delta > 0$ corresponding to $r > s$ such that

$$tf(c_0 z) + (1 - t)f(z) \prec f(z) \qquad (|z| < r, 0 \leq t \leq \delta).$$

But then $tf(c_0 z_0) + (1 - t)f(z_0) = f(\omega(z_0, t))$, where $|\omega(z_0, t)| \leq |z_0|$, and this contradicts our choice of z_0 and c_0.

Another application is given by $F(z, t) = f(z)e^{-te^{i\alpha}}$, the defining family for spiral-like functions. We suppose $f(0) = 0, f^1(z) \neq 0$ for all $z(|z| < 1)$, and

$$\operatorname{Re} (F_2(z, 0)/zf^1(z)) = \operatorname{Re} (-e^{i\alpha}f(z)/zf^1(z)) < 0.$$

We shall show that

$$F(z, t) \prec f(z) \qquad (|z| < 1, 0 \leq t < \infty).$$

For any r, $0 < r < 1$, our theorem gives $F(z, t) = f(\omega(z, t))$ $(|z| < r, 0 \leq t \leq \delta)$. The definition of $\omega(z, t)$ can be extended to the domain $(z) < r$, $0 \leq t < \infty$ with $\omega(z, t)$ remaining a Schwarz function for each fixed t and still satisfying $f(\omega(z, t)) = F(z, t)$. Indeed, suppose these two properties hold for $|z| < r, 0 \leq t \leq S$, any $S > 0$. We then define

$$\omega(z, t + \delta) = \omega(\omega(z, \delta), t) \qquad (|z| < r, 0 \leq t \leq S).$$

The extended function is still a Schwarz function. Moreover

$$f(\omega(z, t + \delta)) = F(\omega(z, \delta), t) = f(\omega(z, \delta))e^{-te^{i\alpha}} = F(z, \delta)e^{-te^{i\alpha}} = F(z, t + \delta).$$

Thus for each r we have an equation $F(z, t) = f(\omega(z, t))$ $(|z| < r, 0 \leq t < \infty)$. For any fixed t, the Schwarz functions $\omega(z, t)$ vary with r, but since f is one-to-one in a neighborhood of 0, it follows that any two of these functions agree wherever both are defined. Therefore we can write $F(z, t) = f(\omega(z, t))$, $(|z| < 1, 0 \leq t < \infty)$.

The key to the above argument is the fact that $F_2(z, t)$ is a function only of $F(z, t)$. Therefore Theorem 3 may have further, more general applications.

THEOREM 4. *Let $F(z, t)$ be analytic in $|z| < 1$ for each fixed t in $0 \leq t \leq T$ $(T > 0)$, let the initial conditions* (6) *hold, let*

$$F(z, t) = f(z) + F_2(z, 0)t + \tfrac{1}{2}F_{22}(z, 0)t^2 + \varepsilon(z, t), \tag{10}$$

where

$$\lim_{t \to 0^+} \varepsilon(z, t)/t^2 = 0$$

uniformly on compact subsets of the disk $|z| < 1$, and let

$$F_2(z, 0) = aizf^1(z) \qquad (|z| < 1), \tag{11}$$

where a is a real constant. If

$$a^2\left[1 + \operatorname{Re}\frac{zf^{11}(z)}{f^1(z)}\right] + \operatorname{Re}\frac{F_{22}(z, 0)}{zf^1(z)} < 0 \qquad (0 < |z| < 1), \tag{12}$$

then for each r, $0 < r < 1$, there exists a corresponding $\delta > 0$ such that

$$F(z, t) \prec f(z) \qquad (|z| < r, 0 \leq t \leq \delta). \tag{13}$$

Proof. Our hypotheses again imply $\lim_{t \to 0^+} F(z, t) = f(z)$ uniformly on compact sets. Hence, given r we obtain $\omega(z, t)$ with $\omega(0, t) \equiv 0$ exactly as in the proof of Theorem 3. Now, from (10) and the definition of $\omega(z, t)$ we obtain

$$\omega(z, t) = z + \omega_2(z, 0)t + \tfrac{1}{2}\omega_{22}(z, 0)t^2 + \eta(z, t),$$

where $\lim_{t \to 0^+} \eta(z, t)/t^2 = 0$ uniformly on some neighborhood of every point of $\{z : |z| \leq r\}$, and therefore for $|z| \leq r$. By (11),

$$\frac{\omega(z, t)}{z} = 1 + ait + \frac{\omega_{22}(z, 0)}{z}t^2 + \frac{\eta(z, t)}{z}.$$

Since $F_2(z, 0)$ and $F_{22}(z, 0)$ are analytic, so is $\omega_{22}(z, 0)$, and therefore $\omega_{22}(z, 0)$ is bounded on $|z| = r$. Thus

$$\left|\frac{\omega(z, t)}{z}\right|^2 = 1 + \left[\operatorname{Re}\frac{\omega_{22}(z, 0)}{z} + a^2\right]t^2 + \theta(z, t),$$

where $\lim_{t\to 0^+} \theta(z, t)/t^2 = 0$ uniformly on $|z| = r$. By (12),

$$\operatorname{Re}\ (\omega_{22}(z, 0)/z) + a^2 < 0,$$

and therefore $\operatorname{Re}\ (\omega_{22}(z, 0)/z) + a^2 \leq -\varepsilon$ for $|z| = r$ and some $\varepsilon > 0$. Thus we obtain $\delta > 0$ such that $|\omega(z, t)| < r$ for $0 < t \leq \delta$ and $|z| = r$.

Remarks. 1. In applications (10) is virtually automatic. For instance, the continuity of $F_{22}(z, t)$ is sufficient for (10).

2. Theorem 4 can be used to show that the classical inequality $1 + \operatorname{Re}\ (zf^{11}(z)/f^1(z)) > 0$ implies the convexity of a univalent function $f(z)$. Let $F(z, t) = \frac{1}{2}[f(z) + f(ze^{it})]$. Then (6), (10), (11), and (12) all hold. Hence Theorem 4 implies that if $0 \leq s < 1$, there exists $\delta > 0$ such that

$$\tfrac{1}{2}[f(z_0) + f(z_0e^{it})] \in \{f(z) : |z| \leq s\} \qquad (|z_0| = s,\ 0 \leq t \leq \delta).$$

We take for granted that this implies the convexity of $\{f(z) : |z| \leq s\}$. Therefore $f(z)$ is a convex function.

References

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