ON THE INVERSE FUNCTION THEOREM IN COMMUTATIVE BANACH ALGEBRAS

BY

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Introduction

Let A be a complex commutative Banach algebra, and D a domain in A. An analytic isomorphism of D is an injective, L-analytic (i.e. analytic in the sense of Lorch [4]) mapping $f : D \to A$ so that $f(D)$ is also a domain, and f^{-1} is L-analytic on $f(D)$. It is known that if $f : D \to A$ is L-analytic and $f'(a_0)$ is invertible, then there is some open neighborhood U of a_0 in D so that $f \mid U$ is an analytic isomorphism of U. This result sets the classical inverse function theorem for analytic functions of a complex variable in the Lorch theory of analytic functions of an A-variable. It is an immediate consequence of the remarks of Arens and Calderon [2, p. 214] on the inversion of a power series with coefficients in A, and was first explicitly given by Mibu [5, p. 333].

The central goal of this paper is to prove the following two theorems, which are both related to, and corollaries of, the above inverse function theorem.

THEOREM 1. *If $f : D \to A$ is L-analytic and injective, $f(D)$ is a domain, and f^{-1} is continuous on $f(D)$, then f is an analytic isomorphism of D.*

THEOREM 2. *Suppose $A = C(X)$, where X is a compact Hausdorff space. If $f : D \to A$ is L-analytic and injective, then either f is an analytic isomorphism of D, or there is some fixed $x \in X$ so that $f(g)(x)$ is identically constant, all $g \in D$.*

In a preliminary section, we discuss the quotient function f_F (which may or may not exist) and the general quotient (possibly multiple-valued) function f^F (which always exists) of an L-analytic $f : D \to A$ by a maximal ideal F of A. Both f_F and f^F will be used in the proofs of Theorems 1 and 2, and are of interest in their own right. In this regard, we will prove that if D is star-shaped, then f_F exists, and then give an example where f_F does not exist even though D is simply connected.

The author would like to raise the following questions.

(a) Can the hypothesis that f^{-1} be continuous be removed from Theorem 1?

(b) Can Theorem 2 be generalized to other Banach algebras?

Notation and terminology

1. A will denote a complex, commutative Banach algebra with identity.
2. D will denote a domain in A, i.e. an open, connected subset of A.
3. D is simply connected iff each loop in D is homotopic to a point in D.

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D is star-shaped iff there is some $a_0 \in D$ so that the line segment connecting a_0 and a is contained in D for all $a \in D$.

4. As is usual, we identify the maximal ideals M of A with the associated complex homomorphisms $F : A \to C$. $\mathfrak{M}$ = the maximal ideal space of A.

5. As is usual, we identify the complex number 1 with the identity element of A. Thus C is considered to be a subset of A.

6. z will be used to denote complex numbers and complex variables, while a will be used to denote elements of A and A-variables.

7. If A-domains and C-domains are under consideration at the same time, the C-domains will be called complex domains, while the A-domains will simply be called domains.

8. Except when preceded by "general", "function" will have the same meaning as "single-valued function".

9. The composition of two functions g and h will be denoted by $g \circ h$.

10. If $a_0 \in A$, and R is a non-negative number, $B(a_0 : R)$ denotes the open norm ball in A of radius R about a_0. For simplicity, we will use B_R in place of $B(O : R)$.

11. If $z_0 \in C$, and R is a non-negative number, $K(z_0 : R)$ and $\bar{K}(z_0 : R)$ respectively denote the open and closed discs of radius R about z_0 in C. For simplicity, we will use K_R in place of $K(O : R)$. $C(z_0 : R)$ will denote the circumference of $\bar{K}(z_0 : R)$.

12. If K is an open disc in C, dK will denote the boundary of K.

1. Quotient functions

Suppose that $f : D \to A$ is L-analytic, and F is a maximal ideal of A. If there is a (necessarily unique) complex analytic function g defined on the complex domain $F(D)$ so that $g \circ F = F \circ f$ on D, we say g is the quotient function of f with respect to F, and write $g = f_F$. (This definition first appears in [3, p. 16].)

When D is a norm ball $B(a_0 : R)$, f_F exists. In fact, if f is given on $B(a_0 : R)$ by the Taylor series $\sum_n \alpha_n (a - a_0)^n$, then f_F is defined on

$$F(D) = K(F(a_0) : R)$$

by

$$f_F(z) = \sum_n F(\alpha_n)(z - F(a_0))^n.$$

For general D, however, we need the following construction. For each $a \in D$, let B_a be the largest norm ball with center at a which is contained in D, and let f_a be the restriction to B_a of f. Define f^F, the quotient general function of f with respect to F, to be the set of all function elements [1, p. 209] $(f_{aF}, F(B_a))$, where a varies over D.

Notice that if $\beta : [0, 1] \to D$ is a curve in D starting at $\beta(0) = a$ and ending at $\beta(1) = b$, then the function element $(f_{bF}, F(B_b))$ is obtained by analytic continuation of the function element $(f_{aF}, F(B_a))$ along the curve $F \circ \beta$ in $F(D)$. Thus f^F is a general analytic function [1, p. 210].

We can now obtain the following lemma, which will be useful in the proof of Theorem 2.

LEMMA 1.1. *Let $f : D \to A$ be L-analytic, F a maximal ideal and c a complex constant. If there is a norm ball B contained in D, so that $F(f(a)) = c$, all $a \in B$, then $F(f(a)) = c$, all $a \in D$.*

Proof. Let b be the center of B. Then since $f_{bF} \circ F = F \circ f_b$ on B_b, f_{bF} is identically c. Since f^F is a general analytic function, f_{aF} is identically c for all $a \in D$. But then for $a \in D$,

$$F(f(a)) = f_{aF}(F(a)) = c.$$

We turn to showing that f_F exists when D is star-shaped.

LEMMA 1.2. *Let $f : D \to A$ be L-analytic, F a maximal ideal, suppose that D is star-shaped. If a, b, $\in D$ and $F(a) = F(b)$, then $F(f(a)) = F(f(b))$.*

Proof. For $x, y \in A$, let $L_{x,y} : [0, 1] \to A$ *be defined by*

$$L_{x,y}(t) = (1 - t)x + ty.$$

Choose $\alpha' \in D$ so that range $L(\alpha' : \alpha)$ is contained in D, all $\alpha \in A$. Now $(f_{aF}, F(B_a))$ and $(f_{bF}, F(B_b))$ are both obtained by analytic continuation of the function element $(f_{\alpha' F}, F(B_{\alpha'}))$ along the curve

$$F \circ L(\alpha' : a) = F \circ L(\alpha' : b).$$

Thus $f_{aF} = f_{bF}$ in a neighborhood of $F(a) = F(b)$, so

$$F(f(a)) = f_{aF}(F(a)) = f_{bF}(F(b)) = F(f(b)).$$

THEOREM 1.3. *If $f : D \to A$ is L-analytic, D is star-shaped, and F is a maximal ideal, then f_F exists.*

Proof. It follows from 1.2 that f^F defines a single-valued function g on $F(D)$: g is easily seen to be f_F.

We present an example which shows that the hypothesis that D be star-shaped in 1.3 cannot be replaced by the hypothesis that D be simply connected. Let $A = C \times C$, with pointwise algebraic operations and the sup norm. Fix ε so that $0 < \varepsilon < \pi/4$. Set

$$D_1 = \{(z, w) : 0 < \operatorname{Re} z < 2, w \neq 0, -\varepsilon < \operatorname{Arg} w < \pi - \varepsilon\},$$

$$D_2 = \{(z, w) : 1 < \operatorname{Re} z < 3, w \neq 0, \pi - 2\varepsilon < \operatorname{Arg} w < 2\pi - 2\varepsilon\},$$

$$D_3 = \{(z, w) : 2 < \operatorname{Re} z < 4, w \neq 0, 2\pi - 3\varepsilon < \operatorname{Arg} w < 2\pi + \varepsilon\}.$$

Clearly each D_i is convex, since $D_1 \cap D_2 \neq \emptyset$, $D_2 \cap D_3 \neq \emptyset$ and $D_1 \cap D_3 = \emptyset$, $D = D_1 \cup D_2 \cup D_3$ is simply connected.

Define, for $i = 1, 2, 3$, $f_i : D_i \to A$ via

$$f_1(z, w) = (z, \text{Log } w) \quad \text{where Log } 1 = 0,$$

$$f_2(z, w) = (z, \text{Log } w) \quad \text{where Log } (-1) = \pi i,$$

$$f_3(z, w) = (z, \text{Log } w) \quad \text{where Log } 1 = 2\pi i.$$

Define $f : D \to A$ by $f(z, w) = f_i(z, w)$ if $(z, w) \in D_i$. Let F be the maximal ideal of A defined by $F(z, w) = w$. Suppose there is a function $f_F : F(D) \to C$ which satisfies the quotienting relation $f_F \circ F = F \circ f$ on D. Then

$$0 = F(f(1, 1)) = f_F(1) = F(f(3, 1)) = 2\pi i;$$

clearly no such f_F can exist.

2. The proof of Theorem 1

Suppose that $f : D \to A$ is L-analytic and injective, $f(D)$ is a domain, and f^{-1} is continuous on $f(D)$. In view of the inverse function theorem, to prove Theorem 1 it is sufficient to show that $f'(a)$ is invertible for all a in D. Two translations enable us to assume without loss of generality that $0 \in D$ and $f(0) = 0$, and to reduce the problem of showing $f'(a)$ invertible for all $a \in D$ to that of showing $f'(0)$ invertible.

Choose some $\delta > 0$ so that $B_\delta \subset D$. For each maximal ideal F let $f_F : K_\delta \to C$ be the quotient function of $f \mid B_\delta$ with respect to F. Obviously $f_F(0) = 0$ but f_F is not identically zero because of the quotienting relation $f_F \circ F = F \circ f$ and the openness of f. Since $(f_F)'(0) = F(f'(0))$, to prove $f'(0)$ invertible it is sufficient to prove $(f_F)'(0) \neq 0$, all $F \in \mathfrak{M}$.

Fix a maximal ideal F. Choose positive numbers ε and μ so that $\mu < \varepsilon < \delta$ and

(1) $f_F(K_\varepsilon) \subset f(B_\delta)$ and $f_F(K_\mu) \subset f(B_\varepsilon)$,
(2) $f_F(z) \neq 0$ when $0 < |z| < \varepsilon$, and
(3) $(f_F)'(z) \neq 0$ when $0 < |z| < \varepsilon$.

Define $h_F : K_\varepsilon \to K_\delta$ by $h_F = F \circ f^{-1} \circ f_F$. Clearly h_F is continuous, and maps K_μ into K_ε. The two crucial properties (4) and (5) of h_F are directly obtained via the quotienting relation $f_F \circ F = F \circ f$.

(4) $h_F(h_F(z)) = h_F(z)$ when $|z| < \mu$, and
(5) $f_F(h_F(z)) = f_F(z)$ when $|z| < \varepsilon$.

Now by (2), (5) and $f_F(0) = 0$, we see that

(6) $h_F(z) \neq 0$ when $0 < |z| < \varepsilon$.

Set

$$S = \{z : h_F(z) = z \text{ and } 0 < |z| < \varepsilon\}.$$

Obviously S is closed in $K_\varepsilon \sim \{0\}$. $h_F(\mu/2) \in S$ via (4) and (6), so S is non-empty. But S is also open, in view of (5) and the local conformality (via (3)) of f_F at each z where $0 < |z| < \varepsilon$. Thus $S = K_\varepsilon \sim \{0\}$, i.e.

$$h_F(z) = z \quad \text{when } 0 < |z| < \varepsilon.$$

But since h_F is injective on $K_\varepsilon \sim \{0\}$, so is f_F. Therefore by classical function theory, $(f_F)'(0) \neq 0$, *Q.E.D.*

3. The proof of Theorem 2 (beginning)

The setting of this section is the realm of classical function theory; the prime tool is Rouché's theorem. No mention will be made of abstract function theory. The goal is to prove Lemmas 3.2 and 3.3, from which Theorem 2 will be directly obtained in Section 4.

Let H be the metrizable space of complex-valued analytic functions defined on the unit disc K_1, with the topology of uniform convergence on compacta.

DEFINITION. A fundamental pair is an ordered pair (h, K), where $h \in H$, K is an open complex disc whose closure $\bar{K}$ is contained in K_1, and there is some (unique) complex number λ so that

(1) $h'(\lambda) = 0$,
(2) $z \in \bar{K}$ and $h'(z) = 0$ implies $z = \lambda$, and
(3) $z \in \bar{K}$ and $h(z) = h(\lambda)$ implies $z = \lambda$.

λ is called the analytic center of (h, K).

The order J of a fundamental pair (h, K) with analytic center λ is defined to be the order of the zero of $h(z) - h(\lambda)$ at $z = \lambda$. Clearly $J - 1$ is the order of the zero of $h'(z)$ at $z = \lambda$, and $J \geq 2$.

DEFINITION. Let (h, K) be a fundamental pair with analytic center λ. A non-negative number μ is free iff

$$\mu < \inf \{ |h(z) - h(\lambda)| : z \in dK \}.$$

The following two basic remarks can be proved by standard winding number, local conformality, and piecing together arguments, and are thus left to the reader.

Remark 1. If (h, K) is a fundamental pair with analytic center λ and order J, and μ is free, then $h(z) - (h(\lambda) + \mu)$ has exactly J zeros (counting multiplicity) in K and none on dK. When $\mu > 0$, condition (2) above thus guarantees that $h(z) - (h(\lambda) + \mu)$ has J distinct zeros in K, each of multiplicity 1.

Remark 2. Suppose that (h, K) is a fundamental pair with analytic center λ and order J, and that $\mu > 0$ is free. Let ζ be one of the J distinct points of K which h maps onto $h(\lambda) + \mu$. Then there is a unique curve

$\beta : [0, \mu] \to K$ which ends at ζ (i.e. $\beta(\mu) = \zeta$), and satisfies

$$h(\beta(t)) = h(\lambda) + t, \qquad 0 \le t \le \mu.$$

Furthermore, if $\zeta' \neq \zeta$ is another point of K which h maps into $h(\lambda) + \mu$, and $\beta' : [0, \mu] \to K$ is the unique curve which ends at ζ' and satisfies

$$h(\beta'(t) = h(\lambda) + t, \qquad 0 \le t \le \mu,$$

then

$$(\text{range } \beta) \cap (\text{range } \beta') = \{\lambda\}.$$

DEFINITION. Suppose that (h, K) is a fundamental pair with analytic center λ and order J, and that $\mu > 0$ is free. Let $\zeta_1, \cdots, \zeta_J$ be an enumeration of the J distinct points of K which h maps into $h(\lambda) + \mu$. For each i, $1 \le i \le J$, let $\beta_i : [0, \mu] \to K$ be the unique curve which ends at ζ_i and satisfies

$$h(\beta_i(t)) = h(\lambda) + t, \qquad 0 \le t \le \mu.$$

The set of curves $\Delta = \{\beta_1, \cdots, \beta_J\}$ is called the μ-system of (h, K). Note that it follows from Remarks 1 and 2 that

$$K \cap h^{-1}(h(\lambda) + [0, \mu]) = \textstyle\bigcup_{i=1}^{J} \text{range } \beta_i .$$

We need the following technical extension of Rouché's theorem.

LEMMA 3.1. *Suppose that $h \in H$, h is not identically zero, and that $Z \subset U \subset K_1$, where Z is compact and U is open. Then there is an open neighborhood N of h in H, and an open set V in C, so that when $g \in N$,*

$$h(Z) \subset V \subset g(U).$$

Proof. For each $w \in Z$, choose $\delta_w > 0$ so that $\bar{K}(w : \delta_w) \subset U$, and $h(z) - h(w)$ has no zeros z on the boundary $C(w : \delta_w)$. Set

$$p_w = \inf \{ | h(z) - h(w) | : z \in C(w : \delta_w)\}$$

and

$$N_w = \{g : g \in H, | g(z) - h(z) | < p_w/2, \text{ all } z \in C(w : \delta_w)\}.$$

By Rouché's theorem, each $g \in N_w$ assumes the value $h(w)$ on $K(w : \delta_w)$. But

$$p_w/2 \le \inf \{ | g(z) - h(w) | : z \in C(w : \delta_w)\}.$$

Thus for all $g \in N_w$,

$$g(K(w : \delta_w)) \supset K(h(w) : p_w/2).$$

Now choose $w_1, \cdots, w_n \in Z$ so that the $K(h(w_i) : p_{w_i}/2)$, $i = 1, \cdots, n$, are an open cover of $h(Z)$. Set

$$N = \textstyle\bigcap_{i=1}^{n} N_{w_i} \quad \text{and} \quad V = \bigcup_{i=1}^{n} K(h(w_i) : p_{w_i}/2).$$

If $g \in N$, and $1 \le i \le n$,

$$g(U) \supset g(K(w_i : p_{w_i})) \supset K(h(w_i) : p_{w_i}/2),$$

so

$$g(U) \supset V \supset h(Z).$$

LEMMA 3.2. *Let S be a subset of H so that*

(1) *no $h \in S$ is identically constant, and*
(2) *there is some $h \in S$, and $|z| < 1$, so that $h'(z) = 0$.*

Then there is a non-empty, open (in S) subset U of X, a complex disc K, a positive integer J, and a positive number μ so that

(3) *(h, K) is fundamental with order J, all $h \in U$, and*
(4) *μ is (h, K)-free, all $h \in U$.*

Proof. Let Q be the set of all fundamental pairs (h, K), where $h \in S$. Conditions (1) and (2) above guarantee that Q is non-empty. Choose a fundamental pair (h_0, K_0) of Q with minimal order J and analytic center λ_0. By Rouché's theorem and the continuity of the mapping $h \to h'$ of H into H, choose an open neighborhood U_0 of h_0 in S so that when $h \in U_0$, h' has no zeros on dK_0 and exactly $J - 1$ zeros (counting multiplicity) in K_0. Since (h_0, K_0) has minimal order, no zero of h', when $h \in Q$, has order less than $J - 1$. Thus when $h \in U_0$, h' has exactly one zero z_h (of multiplicity $J - 1$) in K_0. Clearly $\lambda_0 = z_{h_0}$.

It follows from Rouché's theorem and the continuity of $h \to h'$ that $h \to z_h$ is a continuous mapping of U_0 into K_0. Furthermore, it is not hard to see that the mapping $h \to h(z_h)$ of U_0 into C is also continuous.

Set

$$p = \inf \{|h_0(z) - h_0(\lambda_0)| : z \in dK_0\}.$$

Since (h_0, K_0) is fundamental, $p > 0$. Choose an open neighborhood U_1 of h_0 in U_0 so that when $h \in U_1$,

(5) $|h(z) - h_0(z)| \leq p/3$, all $z \in dK_0$, and
(6) $|h(z_h) - h(\lambda_0)| \leq p/3$.

(5) and (6) yield that when $h \in U_1$ and $z \in dK_0$,

$$|h(z) - h_0(z) + h_0(\lambda_0) - h(z_h)| < |h_0(z) - h_0(\lambda_0)|.$$

Thus by Rouché's theorem, when $h \in U_1$, $h(z) - h(z_h)$ has no zeros on dK_0, and J zeros (counting multiplicity) on K_0. Since $h(z) - h(z_h)$ has a zero of order J at z_h, $z \in \bar{K}_0$ and $h(z) = h(z_h)$ implies $z = z_h$. Therefore when $h \in U_1$, (h, K_0) is a fundamental pair of order J with analytic center z_h.

(5) and (6) also yield

$$|h(z) - h(z_h)| \geq p/3 \quad \text{when } h \in U_1 \text{ and } z \in dK_0.$$

Fix some positive number $\mu < p/3$. μ is obviously (h, K_0)-free, all $h \in U_1$.

LEMMA 3.3. *Suppose that U is a non-empty subset of H, K a complex disc,* *J a positive integer, and μ a positive number so that*

(1) *(h, K) is fundamental, with order J, all $h \in U$, and*
(2) *μ is (h, K)-free, all $h \in U$.*

Then there is a non-empty, open (in U) subset V of U, so that we can prescribe, for each $h \in U$, an enumeration $\beta_{h1}, \cdots, \beta_{hJ}$ of the μ-system $\Delta_{(h,K)}$ of (h, K), so that for each i, $1 \leq i \leq j$, the mapping $\alpha_i : V \times [0, \mu] \to K$ defined by

$$\alpha_i(h, t) = \beta_{hi}(t)$$

is continuous.

Proof. For each $h \in U$, let λ_h be the analytic center of (h, K), and set $w_h = f(\lambda_h)$. It follows from Rouché's theorem and the continuity of $h \to h'$ that $h \to \lambda_h$ is a continuous mapping of U into K and thus $h \to w_h$ is a continuous mapping of U into C.

Now fix some function $h_0 \in U$, write $\lambda_0 = \lambda_{h_0}$ and $w_0 = w_{h_0}$. Let $\zeta_1, \cdots, \zeta_J$ be a fixed enumeration of the J distinct points of K which h_0 maps into $w_0 + \mu$. Choose a positive number p so that each $\bar{K}(\zeta_i : p)$ is contained in K, and the $K(\zeta_i : p)$ are pairwise disjoint. Choose, via Lemma 3.1, an open neighborhood V of h_0 in U so that when $h \in V$ and $1 \leq i \leq J$,

$$w_h + \mu \in f(K(\zeta_i : p)).$$

For $h \in V$, since the order of (h, K) is J, there is exactly one point ζ_{hi} in each $K(\zeta_i : p)$ which h maps onto $w_h + \mu$: for each i let β_{hi} be the unique element of the μ-system $\Delta_{(h,K)}$ which ends at ζ_{hi}. Define, for $1 \leq i \leq J$, $\alpha_i : V \times [0, \mu] \to K$ by

$$\alpha_i(h, t) = \beta_{hi}(t).$$

Fix $(g, s) \in V \times [0, \mu]$, where $s > 0$, we will now show that each α_i is continuous at (g, s). For each i set

$$Z_i = \alpha_{gi}([s/2, \mu]).$$

Each Z_i is a compact subset of K, the Z_i are pairwise disjoint (by Remark 2), and

$$g(Z_i) = w_g + [s/2, \mu].$$

For each i, set

$$Y_i = \{z : z \in C, \text{dist } (z, Z_i) < r\},$$

where r is a fixed positive number small enough so that the Y_i are pairwise disjoint subsets of K, and

(1) $K(\zeta_{gi} : r) \subset K(\zeta_i : p)$, all i.

By Lemma 3.1, choose a neighborhood W of g in V so that when $h \in V$,

(2) $h(Y_i) \supset w_h + [s/2, \mu]$, and
(3) $w_h + \mu \in h(K(\zeta_{gi} : r))$, all i.

Since each $w \in w_h + [s/2, \mu]$ is taken on by h at exactly J distinct points of K, and the Y_i are disjoint, it follows from (2) that

(4) $h^{-1}(w_h + [s/2, \mu]) \cap K = \bigcup_{i=1}^{J} Y_i, h \in W$.

For $h \in W$ and $i = 1, \cdots, J$ let β'_{hi} denote the restriction of β_{hi} to $[s/2, \mu]$. It follows from (3) that each β'_{hi} ends in some $K(\zeta_{gj} : r)$. But β'_{hi} ends at $\zeta_{hi} \in K(\zeta_i : p)$, thus by (1) and the disjointness of the $K(\zeta_j : p)$, β'_{hi} ends in $K(\zeta_{gi} : r) \subset Y_i$. Now via (4) and the connectedness of range β'_{hi}, we see that

(5) range $\beta'_{hi} \subset Y_i, h \in W, 1 \leq i \leq J$.

Now fix i, set $\xi = \beta_{ig}(s) = \alpha_i(g, s)$; obviously $g(\xi) = w_g + s$. Consider an $\varepsilon > 0$ and small enough so that $K(\xi : \varepsilon) \subset Y_i$. Choose, by 3.1, a neighborhood W_1 of g in W, and an interval I about s in $[s/2, \mu]$ so that

(6) $w_h + t \in h(K(\xi : \varepsilon), h \in W_1$ and $t \in I$.

By (4) and (5) the only point of Y_i which h maps onto $w_h + t$ is $\beta'_{hi}(t) = \beta_{hi}(t)$. Since $K(\xi : \varepsilon) \subset Y_i$, it follows from (6) that $\beta_{hi}(t) \in K(\xi : \varepsilon)$. In other words, when $h \in W_1$ and $t \in I$,

$$| \beta_{hi}(t) - \beta_{gi}(s) | < \varepsilon.$$

Therefore α_i is continuous at (g, s).

The proof, via Rouché's theorem, that each α_i is continuous at $(g, 0)$ is straightforward, and is left to the reader.

4. The proof of Theorem 2 (conclusion)

We return to the proof of Theorem 2 per se. Let $A = C(X)$, where X is a compact Hausdorff space, and suppose that $f : D \to A$ is L-analytic and injective. Lemma 1.1 enables us to reduce Theorem 2 to the special case when D is a norm ball. Two translations and a normalization reduce Theorem 2 further to the special case when D is the unit norm ball B_1 and $f(0) = 0$, we will now prove Theorem 2 for this case. More specifically, we will show that if $f : B_1 \to A = C(X)$ is L-analytic and injective, $f(0) = 0$, f is not an analytic isomorphism, and there is no $x \in X$ so that $f(g)(x) = 0$, all $g \in B_1$, then there are two distinct functions g_1 and g_2 in $C(X)$ so that $f(g_1) = f(g_2)$.

For each $x \in X$, let $f_x : K_1 \to C$ be the quotient function of f with respect to the maximal ideal "evaluation at x". The equations

(1) $f_x(g(x)) = f(g)(x)$, and
(2) $f'(g)(x) = f'_x(g(x)), g \in B_1, x \in X$,

are immediate.

Set $S = \{f_x : x \in X\} \subset H$. Since there is no $x \in X$ so that $f(g)(x) = 0$, all $g \in B_1$, it follows from (1) that no f_x is identically constant. Since f is not an analytic isomorphism, by the inverse function theorem there is some

$g \in B_1$ at which $f'(g)$ is singular. This, together with (2), implies that there is some $x \in X$ and $z \in K_1$ so that $f'_x(z) = 0$. Therefore S satisfies the hypotheses of Lemma 3.2.

Choose, by 3.2 and 3.3, a non-empty open (in S) subset V of S, a complex disc K and a positive integer J so that (h, K) is fundamental with order J for all $h \in V$, a positive number μ which is (h, K)-free for all $h \in V$, and an enumeration $\beta_{h1}, \cdots, \beta_{hJ}$ of the μ-system of each (h, K), $h \in V$, so that for each i, $1 \leq i \leq J$, the mapping $\alpha_i : V \times [0, \mu] \to K$ defined by

$$(3) \quad \alpha_i(h, t) = \beta_{hi}(t)$$

is continuous.

Now fix a point $x_0 \in X$ so that $f_{x_0} \in V$. Choose, by the continuity of the mapping $X \to H$ defined by $x \to f_x$ [3, p. 17] a compact neighborhood Z of x_0 so that $f_x \in V$, all $x \in Z$. Select an open neighborhood W of x_0 whose closure $\bar{W}$ is contained in the interior of Z. By Urysohn's lemma, choose a continuous function $\varphi : X \to [0, \mu]$ so that $\varphi(x_0) = \mu$ and $\varphi(X - W) = 0$. For $i = 1, 2$ define $g_i : Z \to K$ by

$$g_i(x) = \alpha_i(f_x, \varphi(x)).$$

From (3) and Remark 2 of Section 3 we have that

$$(4) \quad \alpha_i(f_x, 0) = \lambda_x, \quad \text{and}$$
$$(5) \quad f_x(g_i(x)) = f_x(\lambda_x) + \varphi(x), x \in Z, i = 1, 2,$$

where λ_x is the analytic center of the fundamental pair (f_x, K). But it follows from (4) that $g_1(x) = g_2(x)$, all $x \in Z - W$. Extend g_1 to a continuous mapping of X into K via Tietze's theorem, then extend g_2 to a continuous mapping of X into K by defining $g_2(x) = g_1(x)$, all $x \in X - K$. Now by (5),

$$f_x(g_1(x)) = f_x(g_2(x)), \qquad \text{all } x \in X,$$

so $f(g_1) = f(g_2)$. But since $g_1(x_0) \neq g_2(x_0)$, $g_1 \neq g_2$. Theorem 2 is proved.

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