ON THE RELATIVE STRENGTH OF KARAMATA MATRICES

BY
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1. Introduction and definitions

If $A = (a_{nk})$ is an infinite matrix and $S = \{S_k\}$ is a sequence then A is *applicable* to S if all of the series

$$y_n = \sum_{k=0}^{\infty} a_{nk} S_k, \qquad n = 0, 1, 2, \cdots$$

converge. If, in addition, $\{y_n\}$ is a convergent sequence, then A *sums* S to $\lim y_n$. Whenever A sums every convergent sequence S to $\lim S_n$ then A is *regular*. If A and B are two matrices and A sums every sequence that B sums then A is *stronger* than B.

Many useful matrices in the theory of summability are obtained from non-constant functions $f(z)$ that are analytic in a neighborhood of the origin by setting

$$[f(z)]^n = \sum_{k=0}^{\infty} f_{nk} z^k, \; n = 1, 2, \cdots, \qquad f_{00} = 1, \qquad f_{0k} = 0, \; k = 1, 2, \cdots.$$

The matrix (f_{nk}) is said to be generated by $f(z)$. For example $f(z) = 1 - r + rz$ generates the Euler E_r method [1] and $f(z) = (1 - r)/(1 - rz)$ generates a method studied by W. Meyer-König [3] and P. Vermes [6].

A natural generalization of these methods is the matrix generated by

$$f(z) = (\alpha + (1 - \alpha - \beta)z)/(1 - \beta z).$$

Such a method is called a Karamata matrix and will be denoted by $K[\alpha, \beta]$. B. Bajsanski [2] has studied these matrices and determined conditions that they be regular when α and β are real. He has also investigated the relative strength of different Karamata matrices.

The conditions for regularity of $K[\alpha, \beta]$ have been generalized to complex values of α and β [5], and Bajsanski's theorem about the relative strength of Karamata matrices is a corollary of a more powerful result in his paper. It is then reasonable to hope that more specialized techniques together with more information on regularity will yield other theorems about the relative strength of Karamata matrices. Results of this type will be found in Section 3. Section 2 is devoted to some preparatory theorems, and Section 4 to closing remarks.

2. Some preparatory theorems

The Weierstrass theorem on uniformly convergent series of analytic functions will be a primary tool, and when reference is made to Weierstrass' theorem it is this theorem which is being cited.

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THEOREM 2.1. *If $f(z)$ and $g(z)$ generate matrices F and G, respectively, and if $f(z)$ is analytic on an open disc which contains $g(0)$, then $h(z) = f(g(z))$ generates a matrix H and $H = FG$.*

Proof. Since $f(z)$ is analytic at $g(0)$ then $h(z)$ is analytic in a neighborhood of the origin and hence generates a matrix. If $f(z)$ is analytic in $\{z : |z| < R\}$ then since $|g(0)| < R$, there is a neighborhood N of the origin where $|g(z)| \leq R_1 < R$. Applying Weierstrass' theorem

$$\sum_{k=0}^{\infty} h_{nk} z^k = [h(z)]^n = [f(g(z))]^n = \sum_{p=0}^{\infty} f_{np}[g(z)]^p = \sum_{k=0}^{\infty} z^k \sum_{p=0}^{\infty} f_{np}\, g_{pk}$$

in N. So $H = FG$.

COROLLARY 2.2. *If $|\alpha\delta| < 1$ then $K[\gamma, \delta]K[\alpha, \beta] = K[\eta, \mu]$ where*

$$\eta = (\gamma + \alpha(1 - \gamma - \delta))/(1 - \alpha\delta) \quad \text{and} \quad \mu = (\beta + \delta(1 - \alpha - \beta))/(1 - \alpha\delta).$$

Proof. This result is a direct consequence of Theorem 2.1.

THEOREM 2.3. *If $K[\alpha, \beta] = (f_{nk})$ then*

$$\sum_{n=0}^{\infty} f_{nk}\, t^n = \frac{(1 - \alpha)(1 - \beta)t}{(1 - \alpha t)^2}\left(\frac{\beta + (1 - \alpha - \beta)t}{1 - \alpha t}\right)^{k-1}, \qquad k = 1, 2, \cdots$$

$$\sum_{n=0}^{\infty} f_{no}\, t^n = 1/(1 - \alpha t),$$

Proof. Let $f(z) = (\alpha + (1 - \alpha - \beta)z)/(1 - \beta z)$. If $0 < R < 1/|\beta|$ then there is a $\rho > 0$ such that if $|t| \leq \rho$ and $|z| \leq R$ then $|tf(z)| \leq M < 1$. Fix $|t| \leq \rho$ and let

$$\phi_t(z) = 1/(1 - tf(z)) = \sum_{n=0}^{\infty} t^n[f(z)]^n.$$

Since this series converges uniformly in $|z| \leq R$, we may apply the Weierstrass theorem and write

$$\phi_t(z) = \sum_{k=0}^{\infty} z^k \sum_{n=0}^{\infty} f_{nk}\, t^n.$$

But

$$\frac{1}{1 - tf(z)} = \frac{1 - \beta z}{1 - \alpha t}\, \frac{1}{1 - \left(\dfrac{\beta + (1 - \alpha - \beta)t}{1 - \alpha t}\right) z}$$

and for sufficiently small $|z|$ and $|t|$ this may be expanded into the series

$$\sum_{k=0}^{\infty} \frac{1 - \beta z}{1 - \alpha t}\left(\frac{\beta + (1 - \alpha - \beta)t}{1 - \alpha t}\right)^k z^k.$$

Equating coefficients of the two power series for $\phi_t(z)$ then completes the proof.

THEOREM 2.4. (a) *If $\beta \neq 0$ then $K[\alpha, \beta]$ is applicable to $\{S_n\}$ if and only if $S_n = O(n^{-p}\beta^{-n})$, $p = 0, 1, 2, \cdots$.*

(b) *If* $\{S_n\}$ *is summed by* $K[\alpha, 0]$ *then*

$$S_n = O\left(\left(\frac{1 + |\alpha|}{|1 - \alpha|}\right)^n\right)$$

Proof. (a) By Corollary 2.2., $K[\alpha, 0]K[0, \beta] = K[\alpha, \beta]$. The matrix $K[\alpha, 0]$ is normal, i.e. all of its terms which lie above the diagonal are zero and those terms on the diagonal are not zero. If $K[\alpha, 0] = (a_{nk})$, $K[0, \beta] = (b_{nk})$ and $K[\alpha, \beta] = (c_{nk})$ then

$$b_{nk} = \binom{k-1}{n-1}\beta^{k-n}(1 - \beta)^n, \qquad n, k = 1, 2, \cdots$$

and

$$\sum_{k=0}^{\infty} c_{nk}\, S_k = \sum_{k=0}^{\infty} S_k \sum_{p=0}^{n} a_{np}\, b_{pk}\,.$$

Since $a_{nn} \neq 0$, mathematical induction may be used to establish that $\sum c_{nk}\, S_k$ converges for each n if and only if $\sum b_{pk}\, S_k$ converges for each p. But it is known [4] that a necessary and sufficient condition that all these latter series converge is that $S_n = O(n^{-p}\beta^{-n})$, $p = 0, 1, 2, \cdots$.

(b) It is well-known [1] that if $K[\alpha, 0] = (a_{nk})$ and $\sigma_n = \sum_{k=0}^{n} a_{nk}\, S_k$, $n = 0, 1, 2, \cdots$ then

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\left(-\frac{\alpha}{1 - \alpha}\right)^k \left(\frac{1}{1 - \alpha}\right)^{n-k} \sigma_k\,.$$

Since $\{\sigma_k\}$ converges and hence is bounded, then

$$|S_n| = O\left(\sum_{k=0}^{n} \binom{n}{k}\left|\frac{\alpha}{1 - \alpha}\right|^k \left|\frac{1}{1 - \alpha}\right|^{n-k}\right) = O\left(\left(\frac{1 + |\alpha|}{|1 - \alpha|}\right)^n\right).$$

COROLLARY 2.5. *If* $K[\alpha, \beta]$ *sums the sequence* $\{S_n\}$ *then* $\sum S_k\, t^k$ *converges in some neighborhood of the origin.*

Proof. This follows from the estimates obtained in Theorem 2.4.

3. The principal results

THEOREM 3.1. *Suppose* $K[\alpha, \beta]$ *and* $K[\eta, \mu]$ *are regular Karamata matrices, and that there is a regular Karamata matrix* $K[\gamma, \delta]$ *such that*

$$K[\gamma, \delta]K[\alpha, \beta] = K[\eta, \mu].$$

If $K[\alpha, \beta]$ *sums the sequence* $\{S_n\}$ *to* L *and* $\sum S_k\, z^k$ *is analytic at* β, *and if* $K[\eta, \mu]$ *is applicable to* $\{S_n\}$ *then* $K[\eta, \mu]$ *also sums* $\{S_n\}$ *to* L.

Proof. Without loss of generality assume that $S_0 = 0$. For if not, let $S'_n = S_n - S_0$ and $L' = L - S_0$. Since $K[\alpha, \beta]$ is regular, then $|\beta| < 1$ [5] so that $K[\alpha, \beta]$ sums the sequence $\{S'_n\}$ to L'. Moreover $\sum S'_k\, t^k$ is analytic at β, and since $K[\eta, \mu]$ is applicable to $\{S_n\}$, it is also applicable to $\{S'_n\}$. Throughout the proof let $K[\alpha, \beta] = (a_{nk})$, $K[\gamma, \delta] = (c_{nk})$, and $K(\eta, \mu] = (b_{nk})$. The estimates of Theorem 2.4 show that $\sum S_k\, t^k$ and all of its derivatives converge uniformly in a closed disc about the origin whose radius is $|\beta|$ if $\beta \neq 0$ and is $|1 - \alpha|/2(1 + |\alpha|)$ if $\beta = 0$. (We exclude from our consideration the case where $\beta = 0$ and $\alpha = 1$.) Thus if

$$a(\omega) = (\beta + (1 - \alpha - \beta)\omega)/(1 - \alpha\omega)$$

then $\sum S_k[a(\omega)]^k$ converges uniformly together with its derivatives in a closed region which contains the origin. By Theorem 2.3 if

$$H(\omega) = \frac{(1-\alpha)(1-\beta)}{(1-\alpha\omega)^2}\,\omega \sum_{k=1}^{\infty} S_k[a(\omega)]^{k-1}$$

then

$$H(\omega) = \sum_{k=1}^{\infty} S_k \sum_{p=0}^{\infty} a_{pk}\,\omega^p = \sum_{k=1}^{\infty} S_k\, a_k(\omega).$$

If Weierstrass' theorem is applied to $H(\omega)$ it follows that

$$H^{(n)}(\omega) = \sum_{k=1}^{\infty} S_k\, a_k^{(n)}(\omega)$$

and the uniform convergence of this series implies that

$$\lim_{\omega \to 0} \frac{H^{(n)}(\omega)}{n!} = \sum_{k=1}^{\infty} S_k \frac{a_k^{(n)}(0)}{n!} = \sum_{k=1}^{\infty} a_{nk} S_k .$$

But since $\sum S_k t^k$ is analytic at β and $a(0) = \beta$ then $H(\omega)$ is analytic at the origin. So

$$H(\omega) = \sum_{l=1}^{\infty} \omega^l \sum_{k=1}^{\infty} a_{lk} S_k = \sum_{l=1}^{\infty} \sigma_l\, \omega^l.$$

Since $\{\sigma_l\}$ is bounded, $H(\omega)$ is analytic at least where $|\omega| < 1$. Let

$$G(z) = \frac{(1-\gamma)(1-\delta)zH(c(z))}{(1-\gamma z)(\delta + (1-\gamma-\delta)z)} \quad \text{where} \quad c(z) = \frac{\delta + (1-\gamma-\delta)z}{1-\gamma z}.$$

Now $K[\gamma, \delta]$ is regular, so $|\delta| < 1$ [5] and if $\delta = 0$, then $c(0) = 0$. Since $H(0) = 0$ then $G(z)$ is analytic in a neighborhood of the origin. By Theorem 2.3

$$G(z) = \sum_{k=1}^{\infty} S_k \sum_{l=0}^{\infty} a_{lk} \sum_{j=0}^{\infty} c_{jl}\, z^j$$

and for sufficiently small $|z|$,

$$\sum_{l=0}^{\infty} a_{lk} \sum_{j=0}^{\infty} c_{jl}\, z^j = \sum_{j=0}^{\infty} z^j \sum_{l=0}^{\infty} c_{jl}\, a_{lk} = \sum_{j=0}^{\infty} b_{jk}\, z^j.$$

Using Theorem 2.3 again shows that

$$G(z) = \frac{(1-\eta)(1-\mu)z}{(1-\eta z)^2} \sum_{k=1}^{\infty} S_k[b(z)]^{k-1} \quad \text{where} \quad b(z) = \frac{\mu + (1-\eta-\mu)z}{1-\eta z}.$$

Now by hypothesis $K[\eta, \mu]$ is applicable to $\{S_n\}$, so the argument used earlier may be duplicated on $G(z)$ to show that

$$G^{(n)}(0)/n\,! = \sum_{k=1}^{\infty} b_{nk}\, S_k .$$

But

$$G(z) = \frac{(1-\gamma)(1-\delta)z}{(1-\gamma z)(\delta + (1-\gamma-\delta)z)} \sum_{l=1}^{\infty} \sigma_l[c(z)]^l$$

when $|c(z)| < 1$, so by Theorem 2.3 $G(z) = \sum_{l=1}^{\infty} \sigma_l \sum_{j=0}^{\infty} c_{jl} z^j$ and it follows that

$$G^{(n)}(0)/n! = \sum_{l=1}^{\infty} c_{nl} \sigma_l,$$

so that

$$\sum_{k=1}^{\infty} b_{nk} S_k = \sum_{l=1}^{\infty} c_{nl} \sigma_l.$$

Since $K[\gamma, \delta]$ is regular and $\{\sigma_l\}$ converges to L, then $K[\eta, \mu]$ sums $\{S_n\}$ to L.

COROLLARY 3.2. *Suppose that* $|\alpha(\mu - \beta)| < |1 - \alpha - \beta + \alpha\mu|$ *and*

$$|1 - \alpha - \beta + \alpha\mu|^2 - |\eta(1 - \beta) - \alpha(1 - \mu)|^2 > (1 - \alpha)(1 - \bar{\beta})(1 - \bar{\eta})(1 - \mu) > 0$$

and that $K[\alpha, \beta]$ *sums* $\{S_n\}$ *to* L. *If* $\sum S_k t^k$ *is analytic at* β *and* $K[\eta, \mu]$ *is applicable to* $\{S_n\}$ *then* $K[\eta, \mu]$ *sums* $\{S_n\}$ *to* L.

Proof. The conditions on α, β, μ, η are those needed to use Corollary 2.2 and to insure that $K[\gamma, \delta]$, as given in Theorem 3.1, be regular [5].

4. Closing remarks

In Theorem 3.1 it was necessary to assume that $\sum S_n t^n$ be analytic at β. To see this we use the fact [4] that if $0 < \beta < 1$ then there is an infinite-dimensional linear space of sequences $S = \{S_n\}$ such that $K[0, \beta]S = 0$. It is readily seen that if S is such a sequence and if $\sum S_n t^n$ is analytic at β then $S_n = 0$ for each n. Now let $\gamma = \eta$ and $\delta = -\beta/(1 - \beta)$ so that $\mu = 0$. Let $S \neq 0$ be a sequence for which $K[0, \beta]S = 0$ and let $0 < \beta < 1/3$ so that there is a number η satisfying

$$\beta/(1 - \beta) < \eta < (1 - \beta)/(1 + \beta)$$

The left-hand side of this inequality implies that $K[\gamma, \delta]$ is regular but the right-hand side implies that $(1 - \eta)/(1 + \eta) > \beta$. Consequently if

$$S_n = O((1 + \eta)/(1 - \eta))^n$$

then $\sum S_n t^n$ is analytic at β. So by Theorem 2.4 $K[\eta, 0]$ cannot sum $\{S_n\}$

It is also necessary to assume that $K[\eta, \mu]$ was applicable to the sequences in question. That the other hypotheses of the theorem do not always guarantee this may be seen by considering the case where $0 < \alpha < 1$, $0 < \delta < 1$ and

$$A = K[\alpha, 0], B = K[\alpha(1 - \delta)/(1 - \alpha\delta), \delta(1 - \alpha)/(1 - \alpha\delta)], C = K[0, \delta].$$

Then C is a regular matrix [5] and by Corollary 2.2, $CA = B$. But if $S_n = r^n((-1 - \alpha)/(1 - \alpha))^n$ and $0 < r < 1$ then $\{S_n\}$ is assumed to zero by $K[\alpha, 0]$ while if r and δ are chosen near enough to 1, then

$$r\left(\frac{1 + \alpha}{1 - \alpha}\right)\delta\left(\frac{1 - \alpha}{1 - \alpha\delta}\right) > 1$$

so that by Theorem 2.4, B is not applicable to $\{S_n\}$.

However, there are relatively common situations where the applicability of $K[\alpha, \beta]$ to $\{S_n\}$ implies the applicability of $K[\eta, \mu]$. It may be seen that if z satisfies

$$\text{(I)} \qquad |z| < 1/|\beta|$$

and

$$\text{(II)} \qquad |(\alpha + (1 - \alpha - \beta)z)/(1 - \beta z)| < 1$$

then $K[\alpha, \beta]$ sums $\{z^n\}$ to zero. If $|\alpha| < 1$, then an easy application of Weierstrass' theorem implies that $K[\eta, \mu]$ also sums $\{z^n\}$ to zero. But if there are values of z which satisfy (II) but not (I) and if $|\mu| > |\beta|$ then there are values of z which satisfy (II) and $|z\mu| > 1 > |\beta z|$. Then $K[\alpha, \beta]$ sums $\{z^n\}$ to zero but $K[\eta, \mu]$ is not applicable. Consequently $|\mu| \leq |\beta|$ and so by Theorem 2.4, $K[\eta, \mu]$ is applicable to every sequence to which $K[\alpha, \beta]$ applies. Conditions on $K[\alpha, \beta]$ that there be a z satisfying (II) but not (I) are complicated in general, but in the case where α and β are real and $K[\alpha, \beta]$ is regular, these conditions become

$$(1 + \beta) < |\beta| (1 - \beta - 2\alpha).$$

References

1. R. P. Agnew, *Euler transformations*, Amer. J. Math., vol. 66 (1944), pp. 313–338.
2. B. Bajsanski, *Sur une classe generale de procedes de summations du type d'Euler-Borel.* Publ. Inst. Math. (Beograd), T. X (1956), vol. 131–152.
3. W. Meyer-König, *Untersuchungen über einige verwandte Limitierungsverfahren.* Math. Zeitschrift, vol. 52 (1949), pp. 257–304.
4. W. Meyer-König and K. Zeller, *Über das Taylorsche Summierungsverfahren,* Math. Zeitschrift, vol. 60 (1954), pp. 348–352.
5. W. T. Sledd, *Regularity conditions for Karamata matrices*, J. London Math. Soc., vol. 38 (1963), pp. 105–107.
6. P. Vermes, *Series to series transformations and analytic by matrix methods*, Amer. J. Math., vol. 71 (1949), pp. 541–562.

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