# THE ADJOINT FUNCTOR THEOREM AND THE YONEDA EMBEDDING

BY

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The aim of this note is to show that the problem of whether direct limit preserving functors $T : \mathfrak{A} \to \mathfrak{A}'$ ($\mathfrak{A}$ fixed) have right adjoints is equivalent to the problem of whether the inverse limit preserving Yoneda embedding $Y : \mathfrak{A} \to \mathrm{Cont}\,[\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$, $A \rightsquigarrow [-, A]$, has a left adjoint, where $\mathrm{Cont}\,[\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$ denotes the category of contravariant set valued functors which take direct limits in inverse limits (also called continuous functors). In other words, the problem of constructing right adjoints of functors with domain $\mathfrak{A}$ can be transformed into the problem of constructing the left adjoint of a functor with domain $\mathfrak{A}$. If the conjugate $t^*$ of every covariant and contravariant continuous functor $t : \mathfrak{A} \to \mathfrak{S}$ is again a functor[2], then it follows from this that co-continuous functors $T : \mathfrak{A} \to \mathfrak{A}'$ have right adjoints iff continuous functors $S : \mathfrak{A} \to \mathfrak{A}''$ have left adjoints ($\mathfrak{A}$ fixed, $\mathfrak{A}'$ and $\mathfrak{A}''$ variable). This gives rise to a simple proof of the adjoint functor theorem which, like the proofs of J. Benabou, J. Beck, P. Dedecker, J. Isbell, J. Lambek and others, does not require that $\mathfrak{A}$ is well or co-wellpowered in the sense of P. Freyd [1]. In other words, the subobjects and the quotient objects of an object in $\mathfrak{A}$ need not form a set.

We were led to the above after observing a new proof of Lambek's version [4] of the special adjoint functor theorem. According to Lambek, Freyd's conditions in [1] that $\mathfrak{A}$ has a family of generators and is co-wellpowered can be replaced by requiring that $\mathfrak{A}$ has a small adequate or dense subcategory $\bar{\mathfrak{A}}$ (cf. Isbell [3], Ulmer [5, 1.3]). Adequate or dense means that every object $A \in \mathfrak{A}$ is the direct limit of the canonical diagram of objects $\bar{A} \in \bar{\mathfrak{A}}$ over $A$. (More precisely, the objects of the index category are morphisms $\iota : \bar{A}_\iota \to A$ in $\mathfrak{A}$, where $\bar{A}_\iota \in \bar{\mathfrak{A}}$. A morphism $\alpha : \iota \to \kappa$ is a morphism $\alpha : \bar{A}_\iota \to \bar{A}_\kappa$ in $\bar{\mathfrak{A}}$ with the property $\iota = \kappa \cdot \alpha$.)

Morphism sets, natural transformations and functor categories are denoted by brackets $[-, -]$, comma categories by parentheses $(-, -)$. The category of sets is denoted by $\mathfrak{S}$. The phrase "Let $\mathfrak{A}$ be a category with direct limits" always means that $\mathfrak{A}$ has direct limits over small index categories. However, we sometimes also consider direct limits of functors $F : \mathfrak{D} \to \mathfrak{A}$, where $\mathfrak{D}$ is not necessarily small. Of course we then have to prove that this specific

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[2] Recall that $t^*A = [t, [A, -]]$ for $A \in \mathfrak{A}$. Thus $t^*$ is a functor iff the natural transformations from $t$ to $[A, -]$ form a set for every $A \in \mathfrak{A}$ (likewise for contravariant functors).

limit exists. The terminology "small direct limit" is used when we want to specify that the index category under consideration is small.

We first give a proof of Lembek's version of the adjoint functor theorem only assuming that every object and every morphism in $\mathfrak{A}$ is in some way a direct limit of objects and morphisms in $\bar{\mathfrak{A}}$.[3]

(1) Recall that every set-valued functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ is canonically a direct limit of hom-functors (cf. Gabriel-Zisman [2], Ulmer [5, 1.10]). In general, the canonical index category is not small unless $\mathfrak{A}^{\text{opp}}$ is small. Call a functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ small if it is in some way a small direct limit of hom-functors.

(2) Special Adjoint Functor Theorem. *Let $\bar{\mathfrak{A}}$ be a small subcategory of a category $\mathfrak{A}$ with direct limits such that for every object $A \in \mathfrak{A}$ there is a functor $F(A) : \mathfrak{D}(A) \to \bar{\mathfrak{A}}$ with the property* dir lim $I \cdot F(A) = A$, *where $I : \bar{\mathfrak{A}} \to \mathfrak{A}$ is the inclusion. Assume, moreover, that for every morphism $f : A \to A'$ in $\mathfrak{A}$ there is a functor*

$$H(f) : \mathfrak{D}(A) \to \mathfrak{D}(A')$$

*together with a natural transformation*

$$\psi(f) : F(A) \to F(A') \cdot H(f)$$

*such that the induced morphism* dir lim $F(A) \to$ dir lim $F(A')$ *coincides with* $f : A \to A'$.[4]

*Then every direct limit preserving functor $T : \mathfrak{A} \to \mathfrak{A}'$ has a right adjoint.*

*Proof.* It is well known that it suffices to show that for every $A' \in \mathfrak{A}'$ the continuous functor $[T-, A'] : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$, $A \rightsquigarrow [TA, A']$, is representable. Let $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ be a continuous functor and let $t \cdot I = \text{dir lim } [-, \bar{A}_\nu]$ be the canonical representation of $t \cdot I : \bar{\mathfrak{A}}^{\text{opp}} \to \mathfrak{S}$ as a direct limit of representable functors, where $I : \bar{\mathfrak{A}} \to \mathfrak{A}$ denotes the inclusion. Let $t_0 : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ be any continuous functor. The assumptions made on $\mathfrak{A}$ clearly imply that the map $[t, t_0] \to [t \cdot I, t_0 \cdot I]$, given by restricting natural transformations, is a bijection. Thus it follows that

$$(3)\quad \begin{aligned} [t, t_0] &\cong [t \cdot I, t_0 \cdot I] = [\text{dir lim } [-, \bar{A}_\nu], t_0 \cdot I] \cong \text{inv lim } [[-, \bar{A}_\nu], t_0 \cdot I] \\ &\cong \text{inv lim } t_0 I \bar{A}_\nu \cong t_0 \text{ dir lim } I \bar{A}_\nu \cong [[-, \text{dir lim } I \bar{A}_\nu], t_0] \end{aligned}$$

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[3] Our condition is weaker than Lambek's. For instance, finite sets do not form a co-adequate subcategory of all sets, but every set is an inverse limit of finite sets.

[4] This implies that every object $A \in \mathfrak{A}$ is the cokernel of a pair of maps $\bigoplus \bar{A}_\gamma \to \bigoplus \bar{A}_\mu$, where $\bar{A}_\gamma$, $\bar{A}_\mu \in \bar{\mathfrak{A}}$, and that every morphism $f : A \to A'$ gives rise in an obvious way to a commutative diagram. The conditions in (2) can be replaced by this weaker assumption, provided the category $\bar{\mathfrak{A}}$ is closed under finite sums and its objects are "finitely generated", i.e. every morphism from an object $\bar{A} \in \bar{\mathfrak{A}}$ in an infinite sum $\bigoplus \bar{A}_\lambda$ factors through a finite subsum, $\bar{A}_\lambda \in \bar{\mathfrak{A}}$.

Hence $t \cong [-, \text{dir lim } I\tilde{A}_\nu]$ is valid, which shows that every continuous functor is representable, Q.E.D.

(4) If in (3) $t_0$ is a hom-functor $[-, A]$, where $A \in \mathfrak{A}$, then we obtain a bijection $[t, [-, A]] \cong [\text{dir lim } I\tilde{A}_\nu, A]$ which can be viewed as an adjunction bijection. In other words, the functor

$$\text{Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}] \to \mathfrak{A}, \quad t \rightsquigarrow \text{dir lim } I\tilde{A}_\nu,$$

is left adjoint to the Yoneda embedding $Y : \mathfrak{A} \to \text{Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$, $A \rightsquigarrow [-, A]$. This suggests investigating what the existence of the left adjoint of the Yoneda embedding implies in general. We will show that, roughly speaking, its existence is equivalent to the validity of the adjoint functor theorem in $\mathfrak{A}$.

(5) Call a functor $S : \mathfrak{A} \to \mathfrak{A}'$ supercontinuous if it preserves all existing inverse limits in $\mathfrak{A}$. (The index categories need not be small.) By

$$\text{S. Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$$

we denote the category of supercontinuous set-valued functors on $\mathfrak{A}^{\text{opp}}$.

A functor $S : \mathfrak{A} \to \mathfrak{A}'$ is called continuous if it preserves all existing small inverse limits. (Note that we do not assume that $\mathfrak{A}$ has small inverse limits.) It is well known that every functor $S : \mathfrak{A} \to \mathfrak{A}'$ which has a left adjoint is supercontinuous. Likewise hom-functors are supercontinuous. However not every continuous functor is supercontinuous. The counterexamples appear artificial and we think that in practice the two notions coincide.

(6) LEMMA (Lambek [4]). *The Yoneda embeddings*

$$Y : \mathfrak{A} \to \text{S. Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}] \quad \textit{and} \quad Y : \mathfrak{A} \to \text{Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$$

*are supercocontinuous and cocontinuous respectively.*

*Proof* (Sketch) Let $A = \text{dir lim } A_\nu$ be an arbitrary direct limit in $\mathfrak{A}$ From the bijections below it follows that $[-,A] = \text{dir lim } [-, A_\nu]$ holds in S. Cont $[\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$. For every supercontinuous functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ the equations

$$[[-, A], t] \cong tA = \text{inv lim } tA_\nu \cong \text{inv lim } [[-, A_\nu], t] \cong [\text{dir lim } [-, A_\nu], t]$$

are valid. The first half can be proved in the same way.

(7) COROLLARY. *Let $\mathfrak{A}$ be a category with direct limits. Then every small continuous functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ is representable.*

To see this, let $t = \text{dir lim } [-, A_\iota]$ be a small direct limit. The Yoneda embedding $Y : \mathfrak{A} \to \text{Cont } [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$ preserves small direct limits. Hence

$$t = \text{dir lim } [-, A_\iota] = \text{dir lim } YA_\iota \cong Y \text{ dir lim } A_\iota = [-, \text{dir lim } A_\iota].$$

Later we will show that the adjoint functor theorem follows from this.

(8) THEOREM. *Let* $\mathfrak{A}$ *be a category. The following are equivalent:*
(a) *Every supercocontinuous functor* $T : \mathfrak{A} \to \mathfrak{A}'$ *has a right adjoint.*[5]
(b) *Every supercontinuous functor* $\mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ *is representable.*
(c) *The Yoneda embedding* $Y : \mathfrak{A} \to \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$, $A \rightsquigarrow [-, A]$ *has a left adjoint* $L : \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}] \to \mathfrak{A}$.
(d) *The Yoneda embedding* $Y$ *has a right adjoint* $R : \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}] \to \mathfrak{A}$.

*Proof.* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) and (b) $\Leftrightarrow$ (d) are trivial, the latter because $tA \cong [[-, A], t] \cong [A, Rt]$ for every $A \in \mathfrak{A}$ and $t \in \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$. To prove (c) $\Rightarrow$ (b) recall that for every $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ there is a direct limit representation $t = \operatorname{dir\,lim}\ [-, A_\nu]$. Since the Yoneda embedding is full and faithful, it is obvious that $L : \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}] \to \mathfrak{A}$ maps a hom-functor $[-, A_\nu]$ onto $A_\nu$. Thus the composite

$$Y \cdot L : \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}] \to \mathfrak{A} \to \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$$

maps hom-functors identically onto themselves. Since both $L$ and $Y$ are supercocontinuous (for the latter see (6)), it follows from $t = \operatorname{dir\,lim}\ [-, A_\nu]$ that $t \cong Y \cdot L(t) = [-, L(t)]$. Hence $L$ and $Y$ are equivalences, Q.E.D.

(9) COROLLARY. *Let* $\mathfrak{A}$ *be a category such that for all super continuous functors*

$$t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S} \quad \text{and} \quad s : \mathfrak{A} \to \mathfrak{S}$$

*the natural transformations* $[t, [-, A]]$ *and* $[s, [A, -]]$ *form a set for every* $A \in \mathfrak{A}$. *Then the following are equivalent:*
(a) *Every supercocontinuous functor* $T : \mathfrak{A} \to \mathfrak{A}'$ *has a right adjoint* ($\mathfrak{A}'$ *variable*).
(b) *Every supercontinuous functor* $S : \mathfrak{A} \to \mathfrak{A}''$ *has a left adjoint* ($\mathfrak{A}''$ *variable*).

This corollary follows from (8) modulo some harmless set theoretical difficulties. The categories $\mathfrak{A}'$ and $\mathfrak{A}''$ should be allowed to have classes of hom-sets. However, $[TA, A']$ and $[A'', SA]$ have to be sets for every $A \in \mathfrak{A}$, $A' \in \mathfrak{A}'$ and $A'' \in \mathfrak{A}''$. Note that (a) and (b) imply that the Yoneda embeddings

$$\mathfrak{A} \to \mathrm{S.\ Cont}\ [\mathfrak{A}, \mathfrak{S}]^{\mathrm{opp}}, A \rightsquigarrow [A, -] \quad \text{and} \quad \mathfrak{A} \to \mathrm{S.\ Cont}\ [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}], A \rightsquigarrow [-, A]$$

have adjoints.

(10) *Remark.* A large class of categories satisfies either (9a) or (9b) but

[5] One can state the theorem for a fixed functor $T : \mathfrak{A} \to \mathfrak{A}'$. Thereby the adjoints in (c) and (d) have to be replaced by partial adjoints which are only defined on supercontinuous functors $[T-, A'] : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$, where $A' \in \mathfrak{A}'$. Likewise the assertion (b) has to be restricted to functors $[T-, A'] : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$, where $A' \in \mathfrak{A}'$. However the theorem is most probably false if "super(co)continuous" is replaced by "(co)continuous" unless (co)continuous functors $\mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ are small (cf. (10) and footnote 8)).

relatively seldom both. Assume that a category $\mathfrak{A}$ has the above property (9a) or equivalently that every supercontinuous functor $\mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ is representable. This implies that the dual of every supercontinuous functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ is again a functor. Thus it follows from the above that $\mathfrak{A}$ satisfies (9b) iff the dual of every supercontinuous functor $s : \mathfrak{A} \to \mathfrak{S}$ is again a functor. In other words, if in a category $\mathfrak{A}$ the adjoint functor theorem holds on one side, then it holds on the other side iff the duals of supercontinuous set valued functors on $\mathfrak{A}$ (both covariant and contravariant) are again functors. This connection can also be made visible directly. Let $T : \mathfrak{A} \to \mathfrak{A}'$ be a supercocontinuous functor and denote by $(T, A')$ the comma category associated with $A' \in \mathfrak{A}'$. Recall that its objects $\iota, \kappa, \cdots$ are pairs $(A_\iota, \varphi_\iota)$, where $A_\iota \in \mathfrak{A}$ and $\varphi_\iota$ is a morphism $TA_\iota \to A'$. P. Freyed [1] showed that the direct limit of

$$F_{A'} : (T, A') \to \mathfrak{A}, \quad \iota \rightsquigarrow A_\iota,$$

is the value of the right adjoint $S : \mathfrak{A}' \to \mathfrak{A}$ at $A'$ (provided $S$ exists).[6] The index category of the canonical representation of $[T\text{-}, A'] : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ as a direct limit of representable functors is isomorphic with $(T, A')$ because the objects of the former are natural transformations $[-, A_\iota] \to [T-, A']$. Therefore we can write $[T-, A'] = \text{dir lim } [-, A_\iota]$, and it follows for every $A \in \mathfrak{A}$ that

$$\begin{aligned}[][T-, A'], [-, A]] = [\text{dir lim } [-, A_\iota], [-, A]] &\cong \text{inv lim } [[-, A_\iota], [-, A]] \\ &\cong \text{inv lim } [A_\iota, A] \cong [F_{A'}, \text{const}_A]\end{aligned}$$

holds, where $\text{const}_A : (T, A') \to \mathfrak{A}$ denotes the constant functor $\iota \rightsquigarrow A$. Clearly a necessary condition for the existence of Freyd's limit dir lim $A_\iota$ is that the natural transformation from $F_{A'}$ to $\text{const}_A$ form a set for every $A \in \mathfrak{A}$ because $[\text{dir lim } A_\iota, A] \cong [F_{A'}, \text{const}_A]$ must hold. For some categories $\mathfrak{A}$ this condition is also sufficient, and thus a supercocontinuous functor $T : \mathfrak{A} \to \mathfrak{A}'$ has a right adjoint iff the natural transformations $[[T-, A'], [-, A]]$ form a set for every $A' \in \mathfrak{A}'$, $A \in \mathfrak{A}$.

To obtain the adjoint functor theorem from (7), we need some more terminology.

(11) With Isbell [3] we call a functor $t : \mathfrak{A}^{\text{opp}} \to \mathfrak{S}$ proper if there is a set $\{A_\lambda\}_{\lambda \in \Lambda}$ of objects in $\mathfrak{A}$ together with an epimorphic natural transformation $\oplus_\lambda [-, A_\lambda] \to t$. The family $\{A_\lambda\}$ is called a support of $t$. For example, let $T : \mathfrak{A} \to \mathfrak{A}'$ be a cocontinuous functor. Then $T$ satisfies Freyd's [1] solution set condition iff for every $A' \in \mathfrak{A}'$ the functor $\mathfrak{A}^{\text{opp}} \to \mathfrak{S}$, $A \rightsquigarrow [TA, A']$, is proper (and the solution sets are supports for the functors $[T-, A']$). Clearly a small functor (cf. (1)) is proper, but the converse is not true. The counterexamples are artificial, and we think that in practice the two notions coincide

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[6] Note that in this situation Freyd's existence proof of $S$ only works if $T$ is supercocontinuous. This is because we do not assume the existence of a solution set (but merely the existence of dir lim $A_\iota$) and because $T$ has to preserve the limit dir lim $A_\iota$.

(cf. (12) below). In general, a functor $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ is not proper (or small) and the counterexamples are closely related with the failure of the adjoint functor theorem in $\mathfrak{A}$.[7]

(12) LEMMA. *Let $\mathfrak{A}^{\mathrm{opp}}$ be a category with pullbacks and $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ a proper functor which preserves pullbacks. Then $t$ is small.*

*Proof* (Sketch). Let $\bar{\mathfrak{A}}$ be the full small subcategory of $\mathfrak{A}$ generated by a support of $t$. With the composite

$$\bar{\mathfrak{A}} \xrightarrow{I} \mathfrak{A} \xrightarrow{Y} [\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$$

there is associated the comma category $(Y \cdot I, t) = \mathfrak{D}$, the objects of which are natural transformations $\iota : [-, \bar{A}_\iota] \to t$ and the morphisms $\iota \to \kappa$ are morphisms $\xi : \bar{A}_\iota \to \bar{A}_\kappa$ in $\bar{\mathfrak{A}}$ such that $[-, \xi] \cdot \kappa = \iota$, where $\bar{A}_\iota, \bar{A}_\kappa \in \bar{\mathfrak{A}}$. It is well known that $t$ is the direct limit of the canonical diagram in $[\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$ associated with the comma category $(Y, t)$ (cf. Gabriel-Zisman [2], Ulmer [5, 1.3, 1.10)]. Thus it suffices to show that the inclusion $(Y \cdot I, t) \to (Y, t)$ is cofinal. Since $t$ is proper, one readily checks by means of the Yoneda lemma $[[-, X], t] \cong tX$ that every natural transformation $[-, X] \to t$ can be decomposed into

$$[-, X] \to [-, \bar{A}_\lambda] \to t,$$

where $\bar{A}_\lambda \in \bar{\mathfrak{A}}$, $X \in \mathfrak{A}$. Hence every object of $(Y, t)$ is dominated by an object of $(Y \cdot I, t)$. Since $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ preserves pullbacks and is proper, every diagram in $D$ of the form

$$\begin{tikzcd}
& {[-,\bar{A}_\iota]} \arrow[dr, "\iota"] \arrow[rrrd, dashed] & & & \\
{[-,A]} \arrow[ur, "{[-,\alpha]}"] \arrow[dr, "{[-,\beta]}"'] \arrow[rr] & & t & {[-,\bar{A}_\lambda]} \arrow[l, dashed, "\lambda"'] & {[-,X]} \arrow[l, dashed] \\
& {[-,\bar{A}_\kappa]} \arrow[ur, "\kappa"'] \arrow[rrru, dashed] & & &
\end{tikzcd}$$

can be completed as indicated, where $A \in \mathfrak{A}$, $\bar{A}_\iota, \bar{A}_\kappa \in \bar{\mathfrak{A}}$ and $X$ is the pushout of

$$\begin{tikzcd}
& \bar{A}_\iota \\
A \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\
& \bar{A}_\kappa
\end{tikzcd}$$

This completes the proof that the inclusion $(Y \cdot I, t) \to (Y, t)$ is cofinal, Q.E.D.

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[7] If $\mathfrak{A}$ has direct limits, then it follows from (13) below that every small continuous functor $\mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ is representable. Thus if a cocontinuous functor $T : \mathfrak{A} \to \mathfrak{A}'$ does not have a right adjoint, the continuous functors $[T-, A'] : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ cannot be small, $A' \in \mathfrak{A}'$.

We are now in a position to obtain Freyd's version of the adjoint functor theorem from (7).

(13) ADJOINT FUNCTOR THEOREM. *Let A be a category with direct limits. The following are equivalent:*[8]

(a) *Every cocontinuous functor $T$ with domain $\mathfrak{A}$ has a right adjoint.*

(b) *Every continuous functor $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ is representable.*

(c) *Every continuous functor $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ is small.*

(d) *Every continuous functor $t : \mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ is proper.*

(e) *For every cocontinuous functor $T$ with domain $\mathfrak{A}$, Freyd's solution set condition holds.*

*Proof.* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial. For (d) $\Leftrightarrow$ (e) see (11). (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) follow from (12) and (7) respectively.

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[8] We leave it to the reader to state the theorem for a single functor $T : \mathfrak{A} \to \mathfrak{A}'$ (cf. footnote 5)).

One can deduce (c) $\Rightarrow$ (b) also from (8) by constructing the left adjoint of the Yoneda embedding $Y : \mathfrak{A} \to \mathrm{Cont}\,[\mathfrak{A}^{\mathrm{opp}}, \mathfrak{S}]$. This can be done because $A$ has direct limits and continuous functors $\mathfrak{A}^{\mathrm{opp}} \to \mathfrak{S}$ are small.