# ZEROS OF $\zeta'(s)$ AND THE RIEMANN HYPOTHESIS

BY

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## Abstract

It is shown that the Riemann hypothesis implies that the derivative of the Riemann zeta function has no zeros in the open left half of the critical strip. It is also shown, with no hypothesis, that, with the exception of a bounded region where the zeros can be calculated, the closed left half plane contains only real zeros of the derivative. It is further shown that the Riemann hypothesis is equivalent to the condition that $|\zeta(s)|$ increases as Re $s$ moves left from $1/2$ for Im $s$ sufficiently large.

## 1. Introduction

It was shown in [5] and [7] that $\zeta^{(k)}(s)$, the $k$-th derivative of the Riemann zeta function, has certain zero free regions. Write $s = \sigma + it$. In [5], it was shown that for $k \geq 1$, if $\sigma \geq 2 + 7k/4$, then $\zeta^{(k)}(s) \neq 0$; it was also proved that for each $\varepsilon > 0$, there is an $r_k = r_k(\varepsilon)$ such that $\zeta^{(k)}(s) \neq 0$ for $|s| > r_k$, $\sigma < -\varepsilon$ and $|t| > \varepsilon$. In [7], it was shown that there is an $\alpha_k$ such that $\zeta^{(k)}(s)$ has only real zeros for $\sigma \leq \alpha_k$, and exactly one real zero in each open interval $(-1 - 2n, 1 - 2n)$ for $1 - 2n \leq \alpha_k$.

For $k = 1$, the results can be improved. First, in Titchmarsh [8, Theorem 11.5C] it was shown that there is a constant $E$, $2 < E < 3$, so that the real parts of zeros of $\zeta'(s)$ are dense in $[1, E]$, and that $\zeta'(s) \neq 0$ for $\sigma > E$. Second, in Spira [6], it was proved that on the critical line $\sigma = 1/2$, $\zeta'(s) \neq 0$ except possibly at the points where $\zeta(s) = 0$. Also, in [6] a conjecture was made which implies both the Riemann hypothesis and that $\zeta'(s) \neq 0$ for $0 \leq \sigma < 1/2$. In this paper, it is shown that the Riemann hypothesis alone implies that $\zeta'(s) \neq 0$ for $0 < \sigma < 1/2$, and also the situation in the closed left half plane is settled up to the point of a feasible calculation. The method is to show a modified form of the conjecture of [6], using the Riemann hypothesis for results in the critical strip. Results are also given on the increase of $|\zeta(s)|$ as $\sigma$ tends to $-\infty$ from $1/2$.

## 2. Results on the zeros of $\zeta'(s)$

THEOREM 1. *The Riemann hypothesis implies $\zeta'(s) \neq 0$ for $0 < \sigma < 1/2$. If $|s| > 165$, then $\zeta'(s)$ has only real zeros for $\sigma \leq 0$, and exactly one real zero in each open interval $(-1 - 2n, 1 - 2n)$, $n = 1, 2, \cdots$.*

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*Proof.* In the proof, and in Section 3 we reduce the region of uncertainty even further.

In what follows take $\sigma > 1/2$, $|t| \neq 0$. The functional equation gives

$$(1)\qquad \begin{aligned} -\zeta'(1-s)/\zeta(1-s) &= -\log 2\pi - \tfrac{1}{2}\pi \tan(\pi s/2) \\ &\quad + \Gamma'(s)/\Gamma(s) + \zeta'(s)/\zeta(s), \end{aligned}$$

where we are using the Riemann hypothesis. Next, using [8, 2.12.7],

$$(2)\qquad \begin{aligned} \zeta'(s)/\zeta(s) &= \log 2\pi - 1 - \tfrac{1}{2}\gamma - 1/(s-1) \\ &\quad - \tfrac{1}{2}\Gamma'(\tfrac{1}{2}s+1)/\Gamma(\tfrac{1}{2}s+1) + \textstyle\sum_\rho (1/(s-\rho) + 1/\rho), \end{aligned}$$

where the series runs over the complex roots of the zeta function and converges absolutely, we obtain,

$$(3)\qquad \begin{aligned} -\zeta'(1-s)/\zeta(1-s) &= -(1 + \tfrac{1}{2}\gamma + 1/(s-1) + \tfrac{1}{2}\pi \tan(\pi s/2)) \\ &\quad + \Gamma'(s)/\Gamma(s) - \tfrac{1}{2}\Gamma'(\tfrac{1}{2}s+1)/\Gamma(\tfrac{1}{2}s+1) \\ &\quad + \textstyle\sum_\rho (1/(s-\rho) + 1/\rho). \end{aligned}$$

Since for a zero $\rho = \alpha + i\beta$, we have

$$\operatorname{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{\sigma - \alpha}{(\sigma-\alpha)^2 + (t-\beta)^2} + \frac{\alpha}{\alpha^2 + \beta^2} > 0,$$

using the Riemann hypothesis, we get

$$(4)\qquad \begin{aligned} -\operatorname{Re}\zeta'(1-s)/\zeta(1-s) &> \operatorname{Re}(\Gamma'/\Gamma)(s) - \tfrac{1}{2}\operatorname{Re}(\Gamma'/\Gamma)(\tfrac{1}{2}s+1) \\ &\quad - (1 + \tfrac{1}{2}\gamma + \operatorname{Re} 1/(s-1) + \tfrac{1}{2}\pi \operatorname{Re}\tan(\pi s/2)). \end{aligned}$$

Now from [1, Eq. (3)], in the plane cut along the non-positive real axis,

$$(5)\qquad \frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - \frac{1}{12s^2} + 6\int_0^\infty \frac{P_3(x)}{(s+x)^4}\,dx$$

where $P_3(x)$ is a function of period 1 which is equal to

$$x(2x^2 - 3x + 1)/12$$

on [0, 1], and the log is principal. As in [1], $6\,|P_3(x)| \leq 1/8$, so

$$(6)\qquad 6\int_0^\infty \frac{P_3(x)}{(s+x)^4}\,dx \leq \frac{1}{8}\int_0^\infty \frac{dx}{|s+x|^4} \leq \frac{1}{6\,|s|^3},$$

the last inequality coming from [3, (6)]. Thus,

$$(7)\qquad \operatorname{Re}(\Gamma'/\Gamma)(s) \geq \log|s| - 1/2\,|s| - 1/12\,|s|^2 - 1/6\,|s|^3,$$

and

$$(8)\qquad \begin{aligned} \operatorname{Re}(\Gamma'/\Gamma)(\tfrac{1}{2}s+1) &\leq \log|\tfrac{1}{2}s+1| + 1/\,|s+2| \\ &\quad + 1/3\,|s+2|^2 + 4/3\,|s+2|^3. \end{aligned}$$

Next,

$$\text{(9)} \qquad \operatorname{Re} \tan(\pi s/2) \le |\tan(\pi s/2)| \le (1 + e^{-\pi|t|})/(1 - e^{-\pi|t|}).$$

by [3, (14)], and this last function is monotone decreasing with increasing $|t|$. Also,

$$\text{(10)} \qquad \log(2|s|^2/|s+2|) = \log 2 + \log|s| - \log|1 + 2/s|.$$

and

$$\text{(11)} \qquad \log|1 + 2/s| \le \log(1 + 2/|s|) \le 2/|s|.$$

Finally,

$$\text{(12)} \qquad \operatorname{Re} 1/(s-1) \le |1/(s-1)| \le 1/(|s| - 1).$$

Putting (7), (8), (9), (10), (11) and (12) in (4), we obtain for $|s| > 1$ (because of (12)),

$$\begin{aligned} &-2 \operatorname{Re} \zeta'(1-s)/\zeta(1-s) \\ &\quad > \log|s| + \log 2 - 2 - \gamma - 2/|s| \\ \text{(13)} \qquad &\quad - \pi(1 + e^{-\pi|t|})/(1 - e^{-\pi|t|}) + 2/(|s| - 1) \\ &\quad - 1/|s| - 1/6|s|^2 - 1/3|s|^3 \\ &\quad - 1/|s+2| - 1/3|s+2|^2 - 4/3|s+2|^3. \end{aligned}$$

Using now $|t| \ge 2$, $|s| \ge 164$, we obtain easily that the left hand side of (13) is $>0$. Thus, on the Riemann hypothesis, we have $\zeta'(s) \ne 0$ for $0 < \sigma < 1/2$, $|t| \ge 164$. Now the calculation of [5] showed that $\zeta'(s) \ne 0$ for $0 < \sigma < 1/2$, $|t| \le 200$, (although only the region $|t| \le 100$ was reported on), so we have shown the first part of the theorem. Further, since the inequality $|s| \ge 165$ implies the inequality $|1 - s| \ge 164$, we have the second result of the theorem for the closed left half-plane except for a strip of width 4 centered on the negative real axis.

For this strip, we apply the method of [7]. We have by the functional equation

$$\begin{aligned} -\zeta'(1-s)/2(2\pi)^{-s} &= \zeta(s)\Gamma'(s)\cos(\pi s/2) \\ \text{(14)} \qquad &\quad + \Gamma(s)\{\cos(\pi s/2)(\zeta'(s) - (\log 2\pi)\zeta(s)) \\ &\quad - \sin(\pi s/2)((\pi/2)\zeta(s))\} \\ &= f(s) + g(s), \end{aligned}$$

where $f(s)$ is defined by $\zeta(s)\Gamma'(s)\cos(\pi s/2)$. Next, we apply Rouché's theorem to the rectangle with vertices $2n \pm 2i$, $2n + 2 \pm 2i$. Thus,

$$f(s) + g(s)$$

will have exactly the same number of zeros inside the rectangle as $f(s)$, provided $|f(s)| > |g(s)|$ on the boundary. Dividing $|f(s)| > |g(s)|$ by $|\zeta(s)\Gamma(s) \cos(\pi s/2)|$ and applying the triangle inequality, we will have $|f(s)| > |g(s)|$ on the boundary provided

$$(15) \qquad |(\Gamma'/\Gamma)(s)| > |(\zeta'/\zeta)(s)| + \log 2\pi + (\pi/2)\,|\tan(\pi s/2)|.$$

Now for $\sigma > 1$,

$$(\zeta'/\zeta)(s) = -\sum_{n=2}^{\infty} \Lambda(n)/n^s$$

where $\Lambda(n) = \log p$ if $n$ is a power of a prime $p^k$, $k \geq 1$, and $\Lambda(n) = 0$ otherwise. Hence, for $\sigma \geq 2$

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \sum_{n=2}^{\infty} \frac{\log n}{n^\sigma}$$
$$\leq \frac{\log 2}{2^\sigma} + \int_2^\infty \frac{\log x}{x^\sigma}$$
$$= \frac{2 + (\sigma^2 - 1)\log 2}{(\sigma - 1)^2 2^\sigma}.$$

Also, on the boundary

$$|\tan(\pi s/2)| \leq (e^{-2\pi} + 1)/(e^{-2\pi} - 1) < 1.005.$$

Thus, using (7), we have that (15) will hold on the boundary provided

$$(17) \qquad \begin{aligned} \log|s| > 1/2\,|s| + 1/12\,|s|^2 + 1/6\,|s|^3 + \log 2\pi + 1.005\,\pi/2 \\ + (2 + (\sigma^2 - 1)\log 2)/(\sigma - 1)^2 2^\sigma \end{aligned}$$

which holds for $\sigma \geq 32$. Now from (17), we see that the right hand side of (7) is $> 0$, so $\Gamma'(s)$ has no zeros in the rectangle with vertices $2n \pm 2i$, $2n + 2 \pm 2i$. Thus, there is exactly one zero of $\zeta'(s)$ in the reflected rectangle corresponding to the zero of $\cos(\pi s/2)$. As in [7], this zero must lie on the real axis, since non-real zeros occur in conjugate pairs. Thus, the theorem is proved, for the strip of width 4 and indeed for $\sigma \leq -31$.

## 3. Feasibility of the calculation

It is possible to reduce further the region where $\zeta'(s)$ may be zero. Indeed, from

$$|\zeta'(1 - s)/2(2\pi)^{-s}| = |f(s) + g(s)| \geq |f(s)| - |g(s)|$$

we will have $\zeta'(1 - s) \neq 0$ if $|f(s)| > |g(s)|$. For example, using (17), for $\sigma \geq 3$, $|t| \geq 2$, it suffices to take $|s| \geq 38$. Hence $\zeta'(s) \neq 0$ for $|t| \geq 2$, $\sigma \leq -2$, $|s| \geq 39$. Thus, the region of uncertainty naturally falls into three pieces:

I. $-31 \leq \sigma \leq 0$, $-2 \leq t \leq 2$,
II. $\sigma \leq -2$, $t \geq 2$, $|s| \leq 39$,
III. $-2 \leq \sigma \leq 0$, $2 \leq t \leq 165$.

For region III, the ordinary Euler Maclaurin formula could easily be used [4]. This formula can also be used in the neighborhood of the origin. However, when $\sigma$ lies in the more negative regions, the formula becomes difficult to apply. Using the differentiated functional equation (14) is impractical because of the size of $\Gamma$ and $\Gamma'$. However, in the form (1), it becomes reasonable if one uses the asymptotic series for $\Gamma'/\Gamma$ whose first few terms are given by (5). For $s$ near the origin one can perform a number of translations using

$$(\Gamma'/\Gamma)(s) = (\Gamma'/\Gamma)(s+1) - 1/s$$

to get rapid convergence. For $\zeta'(s)$ and $\zeta(s)$ in (1), one would again use the Euler-Maclaurin formula. Thus, the calculation is feasible.

Finally, the author remarks that the theory for the location of zeros of $\Xi^{(k)}(s)$, the $k$-th derivative of the Riemann $\Xi$-function, is much simpler. By a theorem of Laguerre [9, p. 266], if the Riemann hypothesis is true, then all zeros of $\Xi^{(k)}(s)$ are real.

## 4. Results on $|\zeta(s)|$

In Spira [6], it was conjectured that

$$\text{(18)} \quad \operatorname{Re}\,(\zeta'/\zeta)(s) \le -\tfrac{1}{2}\log|t| + \tfrac{1}{2}\log 2\pi + O(1/|t|), \qquad 0 \le \sigma < \tfrac{1}{2},\ |t| \ge t_0,$$

and from (13) above, we have that the Riemann hypothesis implies

$$\text{(19)} \quad \operatorname{Re}\,(\zeta'/\zeta)(s) \le -\tfrac{1}{2}\log|t| + K + O(1/|t|), \qquad 0 \le \sigma < \tfrac{1}{2},\ |t| \ge 164,$$

where

$$K = 1 + \gamma/2 + 1.001\,\pi/2 - \tfrac{1}{2}\log 2$$

which is (18), except for a larger constant.

As remarked after (13) we have

$$\text{(20)} \quad \operatorname{Re}\zeta'(1-s)/\zeta(1-s) < 0, \quad |t| \ge 2,\ \sigma \ge 1,\ |s| \ge 164$$

and on the Riemann hypothesis also for $\frac{1}{2} < \sigma < 1$. Since

$$\operatorname{Re}\zeta'/\zeta(s) = \partial/\partial\sigma \log|\zeta(s)|.$$

we have for the regions indicated that $\log|\zeta(s)|$, and hence $|\zeta(s)|$ is an increasing function of $\sigma$ as $\sigma$ moves leftward, using the Riemann hypothesis in the critical strip. It is also clear that if $|\zeta(s)|$ so increases in the critical strip, then the Riemann hypothesis holds.

The results obtained do not appear to be easily extendable to $\zeta^{(k)}(s)$ for $k > 1$. However, they do appear to go over to the Dirichlet $L$-functions. In these cases, the inequality is assisted by a term $\log k$, and it may be that the inequality (19) will be provable for the entire left half of the critical strip for $k$ sufficiently large (where we consider $L(s)/s$ instead of $L(s)$ in case $L(0) = 0$).

The problem reduces to estimating $L'(0)/L(0)$ as a function of $k$ (see [2, p. 507]).

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