ON THE IMAGE OF $S^p \times S^q$ UNDER MAPPINGS OF DEGREE ONE

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0. Introduction

This paper computes the homotopy type of those closed, connected, orientable, topological $(p+q)$-manifolds which admit a degree 1 mapping from $S^p \times S^q$ for $p, q \geq 1$. The principal result is

THEOREM 2. *Let M be a closed, connected, orientable, topological $(p+q)$-manifold. If M admits a degree 1 mapping $f : S^p \times S^q \to M$, then either M has the homotopy type of S^{p+q}, or f is a homotopy equivalence.*

This theorem is analogous to the following results, which appear in [2, 2.6 and 2.7, pp. 216–217].

PROPOSITION. *Let M be a closed, orientable, topological or piecewise linear n-manifold, $n \geq 5$. If there is a degree 1 map $S^n \to M$, then M is isomorphic to S^n.*

THEOREM. *Let M be an unbounded, orientable, differentiable or piecewise linear n-manifold, $n \geq 5$. If there is a proper degree 1 map $R^n \to M$, then M is isomorphic to R^n.*

1. The degree of a map

If M and N are connected, orientable n-manifolds, then

$$H_c^n(M, \partial M) = H_c^n(N, \partial N) = Z,$$

where Z denotes the infinite cyclic group. (H_c^* denotes the integral singular cohomology based on cochains with compact support.) If μ_M and μ_N are the preferred free generators of the groups above, then the degree of a proper map

$$f : (M, \partial M) \to (N, \partial N)$$

is the integer k satisfying

$$f^*(\mu_N) = k\mu_M .$$

The proof of Theorem 2 requires repeated use of the following fundamental lemma, proved in [2, 2.9 and 2.11, pp. 216–217].

LEMMA 1. *If $f : (M, \partial M) \to (N, \partial N)$ is a proper mapping of degree* 1

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between connected, orientable n-manifolds, then

(a) $f_{\#} : \pi_1(M) \to \pi_1(N)$ *is an epimorphism, and*

(b) $f_* : H_*(M, \partial M) \to H_*(N, \partial N)$ *is a split epimorphism.*

2. The main theorem

It is convenient to write the principal result in the following form, where $1 \leq m \leq n - m$.

THEOREM 2. *Let M be a closed, connected, orientable n-manifold and*

$$f : S^m \times S^{n-m} \to M$$

be a mapping of degree 1. *Then either M has the homotopy type of* S^n, *or f is a homotopy equivalence.*

Remark 3. Both of the possibilities in the conclusion of Theorem 2 actually occur. In $S^m \times S^{n-m}$ collapse $S^m \times s_1$ and $s_2 \times S^{n-m}$ to a point; then the quotient map is a degree one map $S^m \times S^{n-m} \to S^n$.

Proof of Theorem 2. If $m = n - m = 1$, the proof follows easily from the classification theorem for closed, connected 2-manifolds. So assume that $n \geq 3$.

Consider first the case of $2 \leq m < n - m$. Since $\pi_1(S^m \times S^{n-m}) = 0$, it follows from Lemma 1 that $\pi_1(M) = 0$. Moreover, the same result shows that $H_k(M) = 0$ except possibly for $k = 0, m, n - m$, and n. Now Poincaré Duality and the universal coefficient theorem for cohomology give

$$H_0(M) = H_n(M) = Z$$

and

$$H_m(M) = H^{n-m}(M) = \text{Hom}\ (H_{n-m}(M), Z) = H_{n-m}(M)$$

(since $H_{n-m-1}(M)$ is free abelian by Lemma 1), where H_* is the integral singular homology. Furthermore, because Z is indecomposable, Lemma 1 implies that $H_m(M) = 0$ or $H_m(M) = Z$.

If $H_m(M) = 0$, the absolute Hurewicz isomorphism theorem [3, 7.5.5, p. 398] implies that the Hurewicz homomorphism

$$\Phi : \pi_n(M) \to H_n(M)$$

is an isomorphism. Let $\mu_M \in H_n(M)$ and $\nu_n \in H_n(S^n)$ be the preferred generators, and select a map $g : S^n \to M$ representing the class $\Phi^{-1}(\mu_M)$. The definition of Φ shows that $\mu_M = \Phi[g] = g_*(\nu_n)$; hence g is a mapping of degree 1. Thus

$$g_* : H_*(S^n) \to H_*(M)$$

is an epimorphism and hence is an isomorphism, for every epimorphism $Z \to Z$ is an isomorphism. It follows that $g : S^n \to M$ is a homotopy equivalence [3, 7.6.25, p. 406].

When $H_m(M) = Z$, $f_* : H_*(S^m \times S^{n-m}) \to H_*(M)$ is an epimorphism by

Lemma 1, and so f_* is an isomorphism as above. Thus f is a homotopy equivalence.

If $m = n - m \geq 2$, then $H_m(S^m \times S^m) = Z + Z$. Hence the possibilities for $H_m(M)$ given by Lemma 1 are 0, Z, and $Z + Z$. If $H_m(M) = 0$ or $Z + Z$, the arguments above show that $M \approx S^n$ or that f is a homotopy equivalence, respectively. So it suffices to prove that $H_m(M) \neq Z$. Assume to the contrary that $H_m(M) = H^m(M) = Z$, and let α generate $H^m(M)$. Then $\alpha \cup \alpha$ can be shown to generate $H^{2m}(M)$, and hence $f^*(\alpha \cup \alpha)$ must generate $H^{2m}(S^m \times S^m)$. However, $f^*(\alpha \cup \alpha)$ is an even integer.

When $m = 1$, M is not necessarily simply connected. But if $\pi_1(M) = 0$, then $M \approx S^n$ as before. Suppose therefore that $\pi_1(M) \neq 0$. Since $\pi_1(M) = H_1(M)$, it follows that $\pi_1(M) = Z$ and hence that

$$f_\# : \pi_1(S^1 \times S^{n-1}) \to \pi_1(M)$$

is an isomorphism. In order that f be a homotopy equivalence it suffices to prove that

$$f_\# : \pi_k(S^1 \times S^{n-1}) \to \pi_k(M)$$

is an isomorphism for $k \geq 2$. In order to argue as above it is necessary to pass to the universal covering spaces of $S^1 \times S^{n-1}$ and M by means of

THEOREM 4. *Let N and M be compact, connected, orientable n-manifolds, and let $f : N \to M$ be a mapping of degree* 1 *which induces an isomorphism*

$$f_\# : \pi_1(N) \to \pi_1(M).$$

If $q : \tilde{M} \to M$ is the universal covering space of M and P is the fibered product (i.e., the pullback) of f and q, then:

(a) *The induced covering projection $p : P \to N$ is the universal covering space of N.*

(b) *There is a proper map $\tilde{f} : P \to M$ of degree* 1 *such that $q\tilde{f} = fp$.*

Applying Theorem 4 to $f : S^1 \times S^{n-1} \to M$ gives a commutative diagram

$$\begin{array}{ccc} R \times S^{n-1} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S^1 \times S^{n-1} & \xrightarrow{f} & M \end{array}$$

in which $\tilde{M}$ is the universal covering space of M and $\tilde{f}$ is a proper map of degree 1. Now $H_{n-1}(\tilde{M}) \neq 0$, lest $\tilde{M}$ be contractible and M be a space of type $(Z, 1)$. Thus $H_{n-1}(\tilde{M}) = Z$, and hence $\tilde{f}_* : H_*(R \times S^{n-1}) \to H_*(\tilde{M})$ is an isomorphism, as before. Therefore $\tilde{f}_\# : \pi_k(R \times S^{n-1}) \to \pi_k(\tilde{M})$ is an isomorphism for $k \geq 1$, and it follows that $f_\# : \pi_k(S^1 \times S^{n-1}) \to \pi_k(M)$ is an isomorphism for $k \geq 2$, completing the proof of Theorem 2.

Proof of Theorem 4. (a) It can be shown easily that P is path connected. That $\pi_1(P) = 0$ follows from the fact that $\pi_1(\tilde{M}) = 0$.

(b) When the geometric degree [2, p. 372] of f is 1, any map $\tilde{f}$ induced by f is readily shown to have geometric degree 1. In any case there is a map $g : N \to M$ homotopic to f and having geometric degree 1 [2, Theorem 4.1]. Lift a homotopy from g to f to a map $H : P \times I \to \tilde{M}$; the desired map $\tilde{f}$ is defined by $\tilde{f}(x) = H(x, 1)$. Since $\tilde{g}(x) = H(x, 0)$ has geometric degree 1 by a previous comment, the fact that $\tilde{f}$ is a proper map of degree 1 follows from the fact that H is a proper homotopy from $\tilde{g}$ to $\tilde{f}$.

References

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