MOMENTS OF RANDOM WALK HAVING INFINITE VARIANCE AND THE EXISTENCE OF CERTAIN OPTIMAL STOPPING RULES FOR S_n/n

BY

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Let $X_1, X_2, \cdots$ be independent identically distributed random variables with mean 0, $S_n = X_1 + \cdots + X_n$. If $\int |X_1|^q < \infty$ for some $q \geq 2$ the asymptotic behavior of the distributions of S_n is known to be very regular. One crude indicant of this is the fact that $\sup_n \| S_n \|_q / \| S_n \|_1 < \infty$. This is proved in [1]. If $\int |X_1|^q < \infty$ for some q between 1 and 2 but $\int |X_1|^2 = \infty$, the situation considered here, the behavior in this norm sense of the distributions of S_n can be much worse. An example will be given to show that, for any q between 1 and 2, $\int |X_1|^q$ can be finite but $\lim \| S_n \|_q / \| S_n \|_1 = \infty$. However, for any such example it is proved that $\int |X_1|^{q+\varepsilon} = \infty$, $\varepsilon > 0$. That is, if $1 < \alpha < \beta < 2$ and $\int |X_1|^\beta < \infty$ then $\liminf \| S_n \|_\alpha / \| S_n \|_1 < \infty$. The lim sup need not be finite and an example is given to show this.

Using this small amount of regularity which does exist it is then proved that if some absolute moment of X_1 higher than the first is finite then an optimal stopping rule exists for S_n/n, verifying a conjecture made by Dvoretzky in [4]. The existence of such a rule when $\operatorname{Var} X_1 < \infty$ has been shown by Dvoretzky in [4] and by Teicher and Wolfowitz in [9]. Some results if $\operatorname{Var} X_1 = \infty$ appear in [11]. It was very helpful to see a copy of Mary Thompson's thesis, [10], before its publication.

1. Moments of S_n

In what follows $p < q$ will be numbers between 1 and 2, X will be a random variable satisfying $E|X|^q < \infty$, $EX = 0$, $\operatorname{Var} X = \infty$, and $X_1, X_2, \cdots$ will be independent random variables each having the distribution of X. $C_1, C_2, \cdots$ will be positive constants depending only on p and q.

The idea of the following lemma is well known.

LEMMA 1. *Let f be a nonnegative random variable and $1 \leq a < b < \infty$. Then*

$$P(f > \|f\|_a/2) \geq 2^{b/(a-b)} (\|f\|_a / \|f\|_b)^{ab/(b-a)}.$$

Also,

$$P(f > \|f\|_a/2) \geq \|f\|_a^a / 2 \|f\|_\infty^a .$$

Proof. Let $E = \{f > \|f\|_a/2\}$. Using Holder's inequality,

$$\|f\|_a^a/2 \leq \int_E f^a \leq \|f^a\|_{b/a} \| I(E) \|_{b/(b-a)} = \|f\|_b^a P(E)^{(b-a)/b},$$

Received September 14, 1970.

[1] Research partially supported by a National Science Foundation grant.

proving the first inequality, while

$$\|f\|_a^a/2 \le \int_E f^a \le \|f\|_\infty^a P(E),$$

proving the second.

For $s \ge 0$ define

$$U(s) = U^X(s) = EX^2I(|X| \le s) \quad \text{and}$$
$$V_\mu(s) = V_\mu^X(s) = E(|X|^\mu I(|X| > s).$$

LEMMA 2. *There are arbitrarily large numbers t such that*

$$t^{2-p}V_p(t) \le C_1U(t). \tag{2.1}$$

Proof. Assume without loss of generality that $V_p(0) = 1$. First suppose that the distribution function F of $|X|$ is continuous. Let $a_0, a_1, \cdots$ be a sequence of numbers satisfying $V_p(a_{n-1}) - V_p(a_n) = 2^{-n}$. Then for infinitely many n

$$a_n \le 2^{1/(q-p)}a_{n-1} \tag{2.2}$$

since if not there is an $n_0 \ge 1$ such that $a_n > 2^{1/(q-p)}a_{n-1}$ if $n \ge n_0$, implying $a_{n_0+k} > 2^{k/(q-p)}a_{n_0}$ so that

$$\begin{aligned} V_q(a_{n_0}) &= \textstyle\sum_{k=0}^{\infty} (V_q(a_{n_0+k}) - V_q(a_{n_0+k+1})) \\ &\ge \textstyle\sum_{k=0}^{\infty} (a_{n_0+k})^{q-p}(V_p(a_{n_0+k}) - V_p(a_{n_0+k+1})) \\ &\ge \textstyle\sum (a_{n_0}2^{k/(q-p)})^{q-p}2^{-(n_0+k+1)} \\ &= \infty, \end{aligned}$$

a contradiction since $V_q(a_{n_0}) \le V_q(0) = E|X|^q < \infty$.

Now if (2.2) holds,

$$\begin{aligned} U(a_n) &\ge U(a_n) - U(a_{n-1}) \\ &\ge a_{n-1}^{2-p}(V_p(a_{n-1}) - V_p(a_n)) \\ &\ge (2^{-1/(q-p)}a_n)^{2-p}2^{-n}, \end{aligned}$$

while $V_p(a_n) = 2^{-n+1}$ so that $a_n^{2-p}V_p(a_n) \le 2^{(q+2-2p)/(q-p)}U(a_n)$, completing the proof in this case.

Dropping the assumption that $|X|$ have a continuous distribution function, let Y have a uniform distribution on [0, 1] and be independent of X and let $Z = |X| + Y$. Then Z has a continuous distribution function and thus there are arbitrarily large numbers t such that

$$t^{2-p}V_p^Z(t) \le 2^{(q+2-2p)/(q-p)}U^Z(t).$$

If t is so large that $U^X(t) > 2E|X| + 1$ then $2U^X(t) \ge U^Z(t)$ because

$$Z^2I(|Z| \le t) \le (|X| + 1)^2I(|X| \le t).$$

Since $V_p^Z(t) \ge V_p^X(t)$, C_1 may be taken to be $2^{1+(q+2-2p)/(q-p)}$.

The principle tool in the proof of the following lemma is the existence of positive constants k_μ and K_μ depending only on μ such that for $\mu \geq 1$ and all n,

$$k_\mu \| (\textstyle\sum_1^n X_i^2)^{1/2} \|_\mu \leq \| S_n \|_\mu \leq K_\mu \| (\textstyle\sum_1^n X_i^2)^{1/2} \|_\mu . \tag{2.3}$$

See Theorem 5 of [6].

LEMMA 3. *If* $U(t) > 0$ *define* $N = N(t)$ *to be the largest integer such that* $NU(t) \leq t^2$. *Then if* t *is so large that* $0 < U(t) \leq t^2/2$, $\| S_N \|_1 \geq C_2 t$.

Proof. Let $Z_n = X_n I(|X_n| \leq t)$, $Y_n = X_n - Z_n$.
Using the left-hand side of (2.3) with $\mu = 1$,

$$\begin{aligned} \| S_N \|_1 &\geq k_1 \| (\textstyle\sum_1^N X_i^2)^{1/2} \|_1 \\ &\geq k_1 \| (\textstyle\sum_1^N Z_i^2)^{1/2} \|_1 \\ &\geq (tk_1/3) P(\textstyle\sum_1^N Z_i^2 \geq t^2/8) \end{aligned}$$

Let v be the first time n that $\sum_1^n Z_i^2 > 2t^2$. Then

$$P(v \leq N) \leq E(\textstyle\sum_1^N Z_i^2)/2t^2 = NU(t)/2t^2 \leq \tfrac{1}{2}.$$

Thus, if $g = \sum_{i=1}^{\min(v,N)} Z_i^2$,

$$\begin{aligned} Eg &= E(\textstyle\sum_1^N Z_i^2 I(v \geq i)) = \sum_1^N E(Z_i^2) P(v \geq i) \geq \sum_1^N EZ_i^2/2 \\ &= NU(t)/2 \geq t^2/4, \end{aligned}$$

the last inequality since $U(t) < t^2/2$ implies $N > t^2/2U(t)$. Also,

$$\| g \|_\infty < \textstyle\sum_1^{v-1} Z_i^2 + Z_v^2 \leq 2t^2 + t^2 = 3t^2,$$

and using Lemma 1,

$$P(\textstyle\sum_1^N Z_i^2 \geq t^2/8) = P(g > t^2/8) > C_3,$$

finishing the proof.

LEMMA 4. *Let* t *satisfy* (2.1), $U(t) > 0$ *and* N *be as in Lemma 3. Then* $\| S_N \|_p \leq C_4 t$.

Proof. Defining Y_n and Z_n as in Lemma 3, and using the righthand side of (2.3),

$$\begin{aligned} \| S_N \|_p &\leq K_p \| (\textstyle\sum_1^N X_i^2)^{1/2} \|_p \\ &\leq K_p (\| (\textstyle\sum_1^N Z_i^2)^{1/2} \|_p + \| (\textstyle\sum_1^N Y_i^2)^{1/2} \|_p) \\ &\leq K_p (\| (\textstyle\sum_1^N Z_i^2)^{1/2} \|_2 + (\textstyle\sum_1^N E|Y_i|^p)^{1/p}) \\ &\leq K_p ((NU(t))^{1/2} + (NC_1 t^{p-2} U(t))^{1/p}) \\ &\leq C_4 t. \end{aligned}$$

Since infinitely large t satisfy (2.1) and $N(t) \to \infty$ as $t \to \infty$, Lemmas 3 and 4 imply

THEOREM 1. *$\| S_n \|_p \leq C_5 \| S_n \|_1$ for infinitely many n.*

COROLLARY 1. *$P(S_n > C_6 \| S_n \|_p) > C_6$ for infinitely many n.*

Proof. $ES_n^+ = E|S_n|/2$, while $\| S_n^+ \|_p \leq \| S_n \|_p$, and the result follows from Lemma 1 and Theorem 1.

LEMMA 5. *Let $1 < \mu < 2$. There is a positive constant Δ_μ depending only on μ such that if $0 < U(t) \leq t^2/2$ and $t^{2-\mu} V_\mu(Kt) > KU(t)$, $K > 1$, then $\| S_N \|_\mu \geq \Delta_\mu K^{1/\mu} t$.*

Proof. Let $W_i = X_i I(|X_i| > Kt)$, and $A = \{W_i = 0, 1 \leq i \leq N\}$. If $P(A) \leq \frac{1}{2}$, then

$$\begin{aligned} \| S_N \|_\mu &\geq k_\mu \| (\textstyle\sum_1^N W_i^2)^{/12} \|_\mu \\ &\geq k_\mu \| \sup_{1 \leq i \leq N} | W_i | \|_\mu \\ &\geq k_\mu \| KtI(\tilde{A}) \|_\mu \\ &\geq k_\mu 2^{-1/\mu} Kt. \end{aligned}$$

If $P(A) > \frac{1}{2}$, since $N > t^2/2U(t)$,

$$\begin{aligned} \| S_N \|_\mu &\geq k_\mu \| \sup_{1 \leq i \leq N} | W_i | \|_\mu \\ &\geq k_\mu [\textstyle\sum_{j=1}^N E(| W_i |^\mu I(W_i = 0, i < j))]^{1/\mu} \\ &\geq k_\mu [\textstyle\sum_{j=1}^N E| W_i |^\mu P(W_i = 0, i < j)]^{1/\mu} \\ &\geq k_\mu [NP(A) V_\mu(Kt)]^{1/\mu} \\ &\geq k_\mu 2^{-2/\mu} K^{1/\mu} t, \end{aligned}$$

completing the proof.

Example 1. Let $1 < \mu < 2$. Suppose X_1 has the symmetric distribution which satisfies $V_\mu(t) = \Gamma$, $0 \leq t \leq e$, $V_\mu(t) = \Gamma \log^{-1} t$, $t > e$, where Γ is the normalizing constant. Then using the formulas

$$V_1(t) = \int_{t+}^{\infty} y^{1-\mu}\, dV_\mu(y), \qquad U(t) = \int_0^t y^{2-\mu}\, dV_\mu(y)$$

and the fact that for every a,

$$\lim_{x \to \infty} (d(x^a \log^{-2} x)/dx)/x^{a-1} \log^{-2} x = a,$$

it can be checked that

$$\limsup_{t \to \infty} tV_1(t)/U(t) < \infty$$

so that, applying the analogue of Lemma 4 with p replaced by 1,

$$\limsup_{t \to \infty} \| S_N \|_1 / t < \infty.$$

However $\lim_{t\to\infty} V_\mu(tK)t^{2-\mu}/U(t) = \infty$ for every $K > 1$ so that by Lemma 5,

$$\lim_{t\to\infty} \| S_N \|_\mu / t = \infty.$$

Thus $\lim_{t\to\infty} \| S_N \|_\mu / \| S_N \|_1 = \infty$ implying $\lim_{n\to\infty} \| S_n \|_\mu / \| S_n \|_1 = \infty$.

Example 2. Let $v < \mu$ be arbitrary numbers between 1 and 2. In the following example $\int | X_1 |^{\mu-\varepsilon} < \infty$ for each $\epsilon > 0$ and

$$\limsup \| S_n \|_v / \| S_n \|_1 = \infty.$$

Let α satisfy

$$2(\mu - v)/(2 - v) < \alpha < \mu. \tag{2.4}$$

Let $1 = f(1) < f(2) < \cdots$ satisfy limit $f(n)/f(n-1) = \infty$ and also, for notational convenience, let $\alpha f(n)$ be an integer for every n.

Let X have the symmetric distribution given by $P(| X | = 2^{f(n)}) = \Gamma 2^{-\mu f(n)}$, where Γ is the normalizing constant. Define

$$Z_n = X_n I(| X_n | \geq 2^{f(n)}), \quad Y_n = X_n - Z_n, \quad A_n = \textstyle\sum_1^n Z_n, \ B_n = \sum_1^n Y_n$$

and

$$n' = 2^{\alpha f(n)}.$$

Now $P(A_n \neq 0) \leq \sum_1^{n'} P(Z_n \neq 0) = o(1)$ since $\alpha < \mu$. Thus, for large enough n,

$$\int | A_{n'} |^v \geq \left(\frac{1}{2}\right) \sum_1^n \int | Z_k |^v \geq \left(\frac{1}{2}\right) n' 2^{f(n)v} \Gamma 2^{-\mu f(n)}.$$

Also,

$$\| B_{n'} \|_v \leq \| B_{n'} \|_2 \leq \sqrt{n'} \sup | Y_1 | = \sqrt{n'}\, 2^{f(n-1)}.$$

Thus, since (2.4) holds, $\| B_{n'} \|_v / \| A_{n'} \|_v \to 0$, so $\| S_{n'} \|_v / \| A_{n'} \|_v \to 1$. Since $P(A_{n'} \neq 0) \to 0$ we have $\| A_{n'} \Gamma \|_v / \| A_{n'} \|_1 \to 1$ by Holder's inequality. Thus

$$\begin{aligned} \| S_{n'} \|_1 &\leq \| A_{n'} \|_1 + \| B_{n'} \|_1 \\ &\leq \| A_{n'} \|_1 + \| B_{n'} \|_v \\ &= o(\| A_{n'} \|_v) + o(\| A_{n'} \|_v) \\ &= o(\| S_{n'} \|_v). \end{aligned}$$

2. Existence of an optimal rule for S_n/n

Mary Thompson has proved in [10] that Corollary 2 implies the existence of an optimal rule for S_n/n. The approach used here is new.

Let T be the class of all finite valued stopping rules and let T_∞ be the larger class of all random variables t taking values in $\{1, 2, \cdots, \infty\}$ such that

$$\{t = n\} \in B(n) = \sigma(X_i, i \leq n), \quad n = 1, 2, \cdots.$$

The proof of the following lemma is similar to arguments in [2]. It is also a special case of Theorem 4 of [8]. The proof is included for completeness. Let $A_n = S_n/n$ and $M = \sup_{t \in T} EA_t$.

LEMMA 6. *There is a* $t \in T_\infty$ *such that* $EA_t I(t < \infty) = M$.

Proof. Call $\tau \in T$ regular if $E(A_\tau \mid B(i)) > A_i$ on $\{\tau > i\}$. If u and v are regular then max (u, v) is regular since on $\{\max(u, v) > i\}$,

$$E(A_{\max(u,v)} \mid B(i))$$
$$= E(E(A_v I(v > u) \mid B(u)) + A_u I(v \le u) \mid B(i)) \ge E(A_u \mid B(i)) > A_i.$$

Also,

$$EA_{\max(u,v)} = EE(A_{\max(u,v)} \mid B(u)) \ge EA_\mu$$

and similarly

$$EA_{\max(u,v)} \ge EA_v .$$

Now let $t_n \in T$ satisfy $EA_{t_n} \to M$. If v_n is the first time k such that

$$A_k \ge E(A_{t_n} \mid B(k)),$$

v_n is regular since on $\{i < v_n\}$,

$$E(A_{v_n} \mid B(i)) \ge E(E(A_{t_n} \mid B(i)) > A_i,$$

and also

$$EA_{t_n} = EE(A_{t_n} \mid B_{v_n}) \le EA_{v_n} .$$

Let $\tau_n = \max(v_1, \cdots, v_n)$. Then τ_n is regular and $EA_{\tau_n} \to M$. Let $t = \text{limit } \tau_n$. Since $A_n \to 0$ as $n \to \infty$, $A_t I(t < \infty) = \text{limit } A_{\tau_n}$. Since this convergence is dominated by $\sup |A_n|$, $EA_t I(t < \infty) = \text{limit } EA_{\tau_n} = M$, completing the proof.

THEOREM 2. *An optimal stopping rule exists* for S_n/n.

Proof. Let $t \in T_\infty$ satisfy $E((S_t/t)I(t < \infty)) = M$. It will be shown that $P(t < \infty) = 1$. Let $D_n = \{t \ge n\}$, $D = \text{limit } D_n$ and suppose $P(D) > 0$. For any n,

$$P(S_{n+k} > (C_6/2)\| S_{n+k} \|_p , D_n) > (C_6/2)P(D_n)$$

for infinitely many k since $S_n/\| S_{n+k} \|_p \to 0$ as $k \to \infty$ while $S_{n+k} - S_n$ and D_n are independent. Thus, since n is arbitrary,

$$P(S_j \ge (C_6/2)\| S_j \|_p , D) \ge (C_6/4)P(D)$$

for infinitely many j. Let m be one of these j. Then

$$E((S_t/t)I(D_m)) \ge (C_6^2/8)\| S_m/m \|_p P(D).$$

If this were not true $\tau \in T_\infty$ defined by $\tau = t$ if $t < m$, $\{\tau = m\} = \{S_m > 0\} \cap D_m$,

$\tau = \infty$ elsewhere would satisfy $E(S_\tau/\tau) > E(S_t/t)$. Also

$$\| (S_t/t)I(D_m) \|_p \leq \| \sup_{k \geq m} | S_k/k | \|_p \leq C_7 \| S_m/m \|_p,$$

the last inequality following since $(S_{m+k}/(m + k), k = n, n - 1, \cdots, 0)$ is a martingale so that inequality (3.7) on page 317 of [3] can be used. Thus by Lemma 1,

$$P(t < \infty, D_m) = P((S_t/t)I(D_m) > 0) > C_8 P(D)^{p/(1-p)}.$$

Since m can be made arbitrarily large this gives $P(t < \infty, D) > 0$, a contradiction.

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