DIRICHLET SPACES AND RANDOM TIME CHANGE

BY

MARTIN L. SILVERSTEIN[1]

Introduction

We apply the machinery of Dirichlet spaces to the study of processes obtained from a given symmetric Markov process via a random time change. The basic results are collected and presented in their final form in Section 8. These results are applicable in particular to the "processes on the boundary" of K. Sato and T. Ueno in [25] and so yield information about boundary conditions for symmetric Markov processes.

Necessary preliminary machinery is set up in Sections 1 through 4. Some potential theory for regular Dirichlet spaces as developed for special cases by H. Cartan [3 and 4] and in general by A. Beurling and J. Deny [1] and by M. Fukushima [13] is collected in Section 1 and then applied in Section 2 to construct strong Markov processes. Our construction differs from that of Fukushima [13] in that we do not make use of "strongly regular Dirichlet spaces" and the associated "Ray resolvents." The increasing processes for implementing random time changes are constructed in Section 3. Also in that section we introduce an appropriate notion of balayage more or less in the spirit of [13]. A useful form of invariance under time reversal is established in Section 4.

The real work is done in Sections 5 and 6. The time changed process is analyzed in Section 5. The main result is that the corresponding Dirichlet space is, in an appropriate sense, contained in a "universal Dirichlet space" which itself depends only on a complementary "killed" process and topological notions. A converse result is established in Section 6 so that these two sections together effectively classify certain symmetric extensions of the "killed" process.

Probabilistic interpretations for the original Dirichlet norm and also for the "universal Dirichlet norm" are established in Section 7. The results here are incomplete but promise to be useful for applications.

Our results are illustrated with Brownian motion in Section 9. In particular we show that a result in Section 7 gives a general form of Green's identity, similar to one established by Doob [5] for the Martin closure using different techniques.

We plan to treat processes of purely "jump type" in a separate publication.

For related work by other authors we refer to [8], [14], [18], [25] and [26].

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1. Potential Theory of Regular Dirichlet Spaces

The point of view taken in this section goes back at least to H. Cartan [4] for the classical Dirichlet spaces associated with the Laplacian and Brownian motion. The general formulation in terms of regular Dirichlet spaces is due to A. Beurling and J. Deny. (See especially [1].) The first detailed presentation which is adequate for further work in the direction of Markov processes seems to be that of M. Fukushima [13]. We give in this section a slight variation of Fukushima's presentation.

We consider a separable locally compact Hausdorff space **X** together with a Radon measure dx on **X**. (By a *Radon measure* we mean a regular Borel measure which is bounded on compact sets.) We denote by $L^2(dx)$ or $L^2(\mathbf{X}, dx)$ the Hilbert space of square integrable real-valued functions on **X** and by $C_{\text{com}}(\mathbf{X})$ the collection of continuous functions on **X** with compact support. We use the abbreviation [a. e. μ] for "almost everywhere relative to the measure μ" and adopt the usual cavalier attitude towards the distinction between functions and equivalence classes of functions. However all functions on **X** are Borel measurable unless otherwise specified.

1.1. Definition. A symmetric (submarkovian) resolvent on $L^2(dx)$ is a family of bounded symmetric linear operators G_u, $u > 0$ on $L^2(dx)$ satisfying:

1.1.1. $G_u f \geq 0$ [a.e. dx] whenever $f \geq 0$ [a.e. dx] and $uG_u f \leq 1$ [a.e. dx] whenever $f \leq 1$ [a.e. dx].

1.1.2 $G_u - G_v = (v - u)G_u G_v$.

To any symmetric resolvent $\{G_u, u > 0\}$ is associated its *generator* A defined by $(u - A)G_u f = f$ for f in $L_0^2(\mathbf{X})$, the L^2-closure of the common range of the G_u. The operator A is self adjoint and non-positive definite on $L_o^2(\mathbf{X})$ and so $-A$ has a unique nonnegative definite square root $\sqrt{-A}$ which is a self adjoint operator on $L_o^2(\mathbf{X})$.

1.2. Definition. A normalized contraction is a mapping T of the real line into itself satisfying $T0 = 0$ and $|Tx - Ty| \leq |x - y|$.

1.3. Definition. A Dirichlet space relative to $L^2(dx)$ is a pair $(\mathbf{F}, E)$ where:

1.3.1. **F** is a linear (in general not closed) subset of $L^2(dx)$ and E is a bilinear form on **F**.

1.3.2. For each $u > 0$ the vector space **F** is a Hilbert space relative to the inner product

$$E_u(f, g) = E(f, g) + u \int dx\, f(x)\, g(x).$$

1.3.3. If f belongs to **F** and if T is a normalized contraction, then also Tf belongs to **F** and

$$E(Tf, Tf) \leq E(f, f).$$

In general we will say that a bilinear form E^* on **F** is *contractive*, if

$$E^*(Tf, Tf) \leq E^*(f, f) \tag{1.1}$$

for f in **F** and for T a normalized contraction. Of course 1.3.3 states in particular that E itself is contractive on **F**. We will see that other contractive forms on **F** play an important role.

Note that the inner products E_u, $u > 0$ are equivalent and so topological notions in **F** are independent of $u > 0$.

The connection between Dirichlet spaces relative to $L^2(dx)$ and symmetric resolvents on $L^2(dx)$ is spelled out in

PROPOSITION 1.1. *There is a one to one correspondence between Dirichlet spaces* $(\mathbf{F}, E)$ *on* $L^2(dx)$ *and symmetric resolvents* $\{G_u, u > 0\}$ *on* $L^2(dx)$ *given by*

$$\mathbf{F} = \text{domain}\,(\sqrt{-A}), \qquad E(f, g) = \int dx\, \sqrt{-A}f(x)\sqrt{-A}g(x).$$

Moreover

$$E_u(G_u f, g) = \int dx\, f(x)g(x)$$

for f in L^2 (dx) and g in **F** *and*

$$E_u(f, g) = \text{Lim}\, v \int dx\, \{f(x) - vG_{u+v} f(x)\}g(x) \quad (v \uparrow \infty) \tag{1.2}$$

for f, g in **F**. *Indeed the quantity*

$$v \int dx\, \{f(x) - vG_{u+v} f(x)\}f(x)$$

increases for general f in $L^2(dx)$ and has a finite limit as $v \uparrow \infty$ if and only if f belongs to **F**.

This connection between Dirichlet spaces and resolvents in a slightly different form was first noted by Beurling and Deny [1]. A detailed verification of the connection as stated in Proposition 1.1 can be found in [14, Chaper 2]. We remark that the argument there shows in particular that in verifying the contraction property 1.3.3 for Dirichlet spaces it suffices to consider the two special cases $Tx = \max(x, 0)$ and $Tx = \min(x, 1)$. Furthermore, once it is verified that $(\mathbf{F}, E)$ is a Dirichlet space, the contractivity property can be strengthened as follows. We say that g is a normalized contraction of f if $|g(x)| \leq |f(x)|$ and if $|g(x) - g(y)| \leq |f(x) - f(y)|$ for all x, y. Then if f belongs to **F**, so does g and $E(g, g) \leq E(f, f)$.

We note for convenient later reference the following easily verified facts.

1.4.1. For $u > 0$ the operator uG_u is a contraction relative to any E_v norm and $uG_u f \to f$ strongly in **F** for general f in **F** as $u \uparrow \infty$.

1.4.2. For fixed $u > 0$ the functions $G_u f$ form a dense subset of **F** as f runs over a dense subset of $L^2(dx)$.

1.5. Definition. The Dirichlet space (**F**, E) relative to $L^2(dx)$ is regular if:

1.5.1. $\mathbf{F} \cap C_{\text{com}}(\mathbf{X}$ is uniformly dense in $C_{\text{com}}(\mathbf{X})$ and strongly dense in **F**.

1.5.2. The measure dx is everywhere dense. That is, $\int_G dx > 0$ for any nonempty open set G.

Condition 1.5.1 was first introduced and effectively exploited by Beurling and Deny in [1]. We give a construction (essentially the one by Fukushima in [12]) in the appendix which shows that every Dirichlet space **F**, E) such that **F** is dense in $L^2(dx)$ (that is, $L_0^2(dx) = L^2(dx)$) can be replaced by a regular Dirichlet space without changing any of the relevant structure. *In the remainder of this section we assume that* (**F**, E) *is a regular Dirichlet space relative to* $L^2(dx)$.

1.6. Definition. f in **F** is a u-potential, $u > 0$, if $E_u(f, g) \geq 0$ whenever g in **F** and $g \geq 0$ [a. e. dx].

Proposition 1.2 (i) *f in* **F** *is a u-potential if and only if there exists a Radon measure μ such that*

$$E_u(f, g) = \int \mu(dx) g(x)$$

for g in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$

(ii) *f in* **F** *is a u-potential if and only if* $E_u(f+g, f+g) \geq E_u(f, f)$ *whenever g is in* **F** *and* $g \geq 0$ [a. e. dx].

(iii) *If f is a u-potential, then* $f \geq 0$ [a. e. dx].

(iv) *f in* **F** *is a u-potential if and only if* $vG_{u+v} f \leq f$[a. e. dx] *for all* $v > 0$.

(v) *If f, g are both u-potentials, then so is* $\min(f, g)$.

(vi) *If f is a u-potential, then so is* $\min(f, c)$ *for any* $c \geq 0$.

Proof. We begin with (i). The if part is clear from $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ being dense in **F**. To prove "only if" let f be a u-potential and consider the nonnegative linear functional I defined on nonnegative g in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ by $I(g) = E_u(f, g)$. If g_n decrease to 0 pointwise, then they do so uniformly and, after comparing to a fixed nonnegative g in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ which is ≥ 1 on the support of g_1, we easily see that $I(g_n) \downarrow 0$. The existence of the desired Radon measure μ now follows by the Daniell approach to integration (as presented for example in [20, p. 29]). The proof of (i) is complete. The only if part of (ii) is immediate and "if" follows upon replacing g by tg for

$t > 0$ sufficiently small. Conclusion (iii) follows from

$$E_u(|f|, |f|) = E_u(f, f) + E_u(|f| - f, |f| - f) + 2E_u(f, |f| - f)$$
$$\geq E_u(f, f) + E_u(|f| - f, |f| - f)$$

since by contractivity $E_u(f, f) \geq E_u(|f|, |f|)$. "If" in (iv) follows from (1.2) and "only if" from the relation

$$\int dx\, \{f(x) - vG_{u+v} f(x)\}g(x) = E_u(f, G_{u+v}\, g).$$

Finally (v) and (vi) follow from (iv).

Before continuing we establish a simple but useful extension of Proposition 1.2 (iv).

COROLLARY 1.3. *Let g be a u-potential and let f on* $\mathbf{X}$ *satisfy*

$$0 \leq f \leq g \quad \text{[a.e. } dx]$$

$$vG_{u+v}f \leq f \quad \text{[a.e. } dx], \quad v > 0.$$

Then f belongs to $\mathbf{F}$ *and therefore f is a u-potential.*

Proof.

$$\int dx\, v\{f(x) - vG_{u+v} f(x)\}f(x) \leq \int dx\, v\{f(x) - vG_{u+v} f(x)\}g(x)$$
$$= \int dx\, v\{g(x) - vG_{u+v}\, g(x)\}f(x)$$
$$\leq \int dx\, v\{g(x) - vG_{u+v}\, g(x)\}g(x)$$

and the desired result follows from Proposition 1.1 and from Proposition 1.2 (iv).

1.7. DEFINITION. The Radon measure μ has finite energy if there exists a constant $c > 0$ such that

$$\int \mu(dx)f(x) \leq c\{E_1(f, f)\}^{1/2}$$

for f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. The collection of all such measures is denoted by $\mathfrak{M}$.

It is routine to show that if μ belongs to $\mathfrak{M}$, then for every $u > 0$ there exists a unique u-potential, written $G_u\mu$, such that

$$E_u(G_u\, \mu, g) = \int \mu(dx)g(x) \tag{1.3}$$

for g in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. The resolvent identity

$$G_u\mu = G_v\mu + (v - u)G_uG_v\mu$$

valid for μ in $\mathfrak{M}$ and for $u, v > 0$ is easily verified. It is convenient to introduce the special notation

$$\| \mu \|_u = \{E_u(G_u\mu, G_u\mu)\}^{1/2}$$

for $u > 0$ and μ in $\mathfrak{M}$. Important compactness properties of $\mathfrak{M}$ relative to the norms $\| \mu \|_u$ are established in

PROPOSITION 1.4. *Let $u > 0$ and let $\{\mu_n\}$ be a sequence in $\mathfrak{M}$.*

(i) *If $G_u\mu_n$ converges weakly to f in* **F**, *then f is a u-potential and indeed $f = G_u\mu$ where μ is the vague limit of the μ_n.*

(ii) *If $\| \mu_n \|_u$ is bounded and if $\mu_n \to \mu$ vaguely, then μ is in $\mathfrak{M}$ and $G_u\mu_n \to G_u\mu$ weakly in* **F**.

(iii) *$\mathfrak{M}$ is complete relative to $\| \; \|_u$.*

Proof. If $G_u\mu_n$ converges weakly to f, then clearly f is a u-potential and so $f = G_u\mu$ for some μ in $\mathfrak{M}$. But then $\int \mu_n(dx)g(x) \to \int \mu(dx)g(x)$ for g in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ and it follows directly that $\mu_n \to \mu$ vaguely, proving (i). Conclusions (ii) and (iii) follow routinely.

To make further progress we must validate (1.3) for general g in **F** which means in particular that we must represent g by a refinement which is well defined [a. e. μ] for every μ in $\mathfrak{M}$. The main tools for doing this are certain capacities associated with the Dirichlet norms E_u.

1.8. DEFINITION. For G an open subset of **X** let

$$\text{Cap}_u(G) = \inf E_u(f, f)$$

as f runs over the functions in F such that $f \geq 1$ a. e. on G. If no such f exist put $\text{Cap}_u(G) = +\infty$. For general Borel subsets A of X put

$$\text{Cap}_u(A) = \inf \text{Cap}_u(G)$$

as G runs over the open supersets of A.

1.9. DEFINITION. A Borel set A is a polar set if $\text{Cap}_u(A) = 0$.

Note that although in general the value of $\text{Cap}_u(A)$ depends on $u > 0$, still the condition $\text{Cap}_u(A) = 0$ is independent of $u > 0$. This follows from the easily verified estimates

$$E_u(f, f) \leq E_1(f, f) \leq (1/u)E_u(f, f), \qquad (1/u)E_u(f, f) \leq E_1(f, f) \leq E_u(f, f)$$

valid respectively for $0 < u < 1$ and $1 < u < +\infty$. Thus Definition 1.9 makes sense.

PROPOSITION 1.5. *Let $u > 0$ and let G be an open subset of* **X** *such that* $\text{Cap}_u(G) < \infty$.

(i) *There exists a unique function* p_u^G *in* F *such that* $E_u(p_u^G, p_u^G)$ *is minimal among* f *in* $\mathbf{F}$ satisfying $f \geq 1$ [a.e. dx] *on* G.

(ii) $0 \leq p_u^G \leq 1$ *and* $p_u^G = 1$ *on* G[a.e. dx].

(iii) p_u^G *is a u-potential and indeed* $p_u^G = G_u\mu$ *with* μ *concentrated on* $\mathrm{cl}(G)$, *the closure of* G.

Proof. Let L be the subset of f in $\mathbf{F}$ such that $f \geq 1$[a. e. dx] on G. Clearly L is a closed convex subset of the Hilbert space $\mathbf{F}$ with the inner product E_u and so (i) follows from the elementary theory of Hilbert spaces. Conclusion (ii) follows upon noting that if f belongs to L, then so does $\min(f, 1)$ and $\max(f, 0)$. That $p_u^G = G_u\mu$ for some μ in $\mathfrak{M}$ follows from Proposition 1.2(ii) since if g *in* $\mathbf{F}$ is ≥ 0 then $p_u^G + g$ belongs to L and so

$$E_u(p_u^G + g, p_u^G + g) \geq E(p_u^G, p_u^G).$$

To show that μ is actually concentrated on $\mathrm{cl}(G)$, it suffices to consider restrictions to $\mathrm{cl}(G)$ of functions in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ and to modify in an obvious way the proof of Proposition 1.2(i).

LEMMA 1.6. *For* ν *in* $\mathfrak{M}$, *for* $u > 0$ *and for* G *an open subset of* $\mathbf{X}$,

$$\nu(G) \leq \| \nu \|_u \{\mathrm{Cap}_u(G)\}^{1/2}.$$

Proof. If p_u^G were in $C_{\mathrm{com}}(\mathbf{X})$, the lemma would follow from (1.3). In general this is false and instead we apply an approximation argument of Fukushima. For $v > 0$ let $g_v = vG_{u+v}\nu$. Then $G_u g_v = vG_{u+v}G_u\nu$ which converges strongly (see 1.4.1) and therefore weakly to $G_u\nu$ as $v \uparrow \infty$. Thus by Proposition 1.4, $g_v(x)dx \to \nu$ vaguely and so

$$\begin{aligned}
\nu(G) &\leq \operatorname{Lim\,inf} \int_G g_v(x)\, dx \\
&\leq \operatorname{Lim\,inf} \int p_u^G(x) g_v(x)\, dx \\
&= \operatorname{Lim\,inf} E_u(p_u^G, G_u\, g_v) \\
&= E_u(p_u^G, G_u\, \nu) \\
&\leq \| \nu \|_u \{\mathrm{Cap}_u(G)\}^{1/2}.
\end{aligned}$$

COROLLARY 1.7. *If* μ *has finite energy, then* μ *charges no polar set.*

Next we establish some properties of Cap_u which permit the application of Choquet's general theory.

PROPOSITION 1.8. (i) $\mathrm{Cap}_u(G_1) \leq \mathrm{Cap}_u(G_2)$ *whenever* $G_1 \subset G_2$ *with both open.*

(ii) *If open* $G_n \uparrow G$, *then* $\mathrm{Cap}_u(G_n) \uparrow \mathrm{Cap}_u(G)$. *If* $\mathrm{Cap}_u(G)$ *is finite then also* $p_u^{G_n} \to p_u^G$ *strongly in* $\mathbf{F}$ *and* $p_u^{G_n}$ *increases to* p_u^G[a.e. dx].

(iii) (*strong sub-additivity*)

$$\mathrm{Cap}_u(G_1 \cup G_2) + \mathrm{Cap}_u(G_1 \cap G_2) \leq \mathrm{Cap}_u(G_1) + \mathrm{Cap}_u(G_2)$$

for G_1, G_2 open.

Proof. (i) is clear. To prove (ii) observe first that if $m < n$, then

$$g = \min(p_u^{G_m}.\ p_u^{G_n}) \geq 1 \quad \text{on } G_m$$

and therefore $E_u(g, g) \geq E_u(p_u^{G_m}, p_u^{G_m})$. On the other hand the argument used to prove Proposition 1.2 (iii) shows that

$$E_u(p_u^{G_m}, p_u^{G_m}) \geq E_u(g, g) + E_u(p_u^{G_m} - g, p_u^{G_m} - g)$$

and we conclude that $p_u^{G_m} = g$ or $p_u^{G_m} \leq p_u^{G_n}$ [a.e. dx) and so $p_u^{G_n}$ increases with n [a. e. dx]. Suppose now that sup $\mathrm{Cap}_u(G_n)$ is finite. Then $p_u^{G_n}$ converges weakly to some f in **F** and it is a simple matter to check that $f = p_u^G$. To complete the proof of (ii) it only remains to show that $p_u^{G_n} \to p_u^G$ strongly, which follows from the above argument with $p_u^{G_m}$ and $p_u^{G_n}$ playing the role of g and $p_u^{G_m}$. To prove (iii) we remark first that for G open with $\mathrm{Cap}_u(G) < +\infty$, we have $\mathrm{Cap}_u(G) = E_u(p_u^G, f)$ whenever $f = 1$ [a. e. dx] on G. This is because

$$E_u(p_u^G + t[f - p_u^G], p_u^G + t[f - p_u^G]) \geq E_u(p_u^G, p_u^G) \quad \text{for all real } t.$$

Therefore if $\mathrm{Cap}_u(G_1)$ and $\mathrm{Cap}_u(G_2)$ are finite, then

$$\mathrm{Cap}_u(G_1) + \mathrm{Cap}_u(G_2) - \mathrm{Cap}_u(G_1 \cup G_2) - \mathrm{Cap}_u(G_1 \cap G_2)$$
$$= E_u(p_u^{G_1 \cup G_2}, p_u^{G_1} + p_u^{G_2} - p_u^{G_1 \cup G_2} + p_u^{G_1 \cap G_2})$$

which is ≥ 0 since

$$p_u^{G_1} + p_u^{G_2} - p_u^{G_1 \cup G_2} - p_u^{G_1 \cap G_2} \geq 0 \text{ [a. e. } dx] \quad \text{on } G_1 \cup G_2.$$

(If $f \geq 0$ on $G_1 \cup G_2$, then $E_u(p_u^{G_1 \cup G_2} + tf, p_u^{G_1 \cup G_2} + tf) \geq 0$ for all $t > 0$ etc.) The case when either $\mathrm{Cap}_u(G_1)$ or $\mathrm{Cap}_u(G_2)$ is infinite is immediate and so the proof is complete.

We apply Choquet's theory of capacities (as presented for example in [22, Chapter III]) to get

THEOREM 1.9. *For any Borel set A and for any $u > 0$,*

$$\mathrm{Cap}_u(A) = \sup \mathrm{Cap}_u(K)$$

as K runs over the compact subsets of A.

COROLLARY 1.10. *A Borel set A is polar if and only if* $\mathrm{Cap}_u(K) = 0$ *for every compact subset K of A.*

Now we are ready to introduce our refinements.

1.10 DEFINITION. A property is valid quasi-everywhere (abbreviated q.e.)

if the exceptional set is polar. Two functions are quasi-equivalent if the set on which they differ is polar, that is, if they are equal quasi-everywhere.

1.11 DEFINITION A function f on $\mathbf{X}$ is quasi-continuous if there exists a decreasing sequence of open sets G_n with $\mathrm{Cap}_u(G_n) \downarrow 0$ such that f is continuous on $\mathbf{X} - G_n$ for every n.

THEOREM 1.11 *Each f in $\mathbf{F}$ has a representative uniquely defined up to quasi-equivalence such that*:

1.11.1. *f is quasi-continuous and Borel measurable.*

1.11.2. *Whenever $f_n \to f$ strongly in $\mathbf{F}$, then, for a subsequence, $f_n \to f$* q.e.

Proof. Uniqueness is clear from 1.11.2. To establish existence, fix f, g in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ and let $G = \{x : |f(x) - g(x)| > \varepsilon\}$. Then $(1/\varepsilon)|f - g| \geq 1$ on G and so

$$\mathrm{Cap}_u(G) \leq (1/\varepsilon^2)E_1(f - g, f - g). \tag{1.4}$$

This suffices to establish the existence for f in $\mathbf{F}$ of a quasi-continuous version characterized by the condition that whenever f_n in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ converges to f in $\mathbf{F}$ then a subsequence converges to f q.e. A routine argument serves to remove the restriction on f_n and the proof is complete.

The statement of uniqueness in Theorem 1.11 will be improved in Lemma 1.15 below.

1.12. Convention. From now on every f in $\mathbf{F}$ is represented by the version specified up to quasi-equivalence in Theorem 1.11.

We extend (1.3) and (1.4) in

THEOREM 1.12. *For f in $\mathbf{F}$, for μ in $\mathfrak{M}$ and for $u, \varepsilon > 0$,*

$$\int \mu(dx)f(x) = E_u(G_u\mu, f), \tag{1.5}$$

$$\mathrm{Cap}_u\{x : |f(x)| > \varepsilon\} \leq (1/\varepsilon^2)E_u(f, f).$$

Proof. The first relation has already been established for f in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ and follows in general by a passage to the limit using 1.11.2 and Corollary 1.7 since

$$\int \mu(dx)|f(x) - g(x)| = E_u(G_u\mu, |f - g|) \leq \|\mu\|_u\{E_u(f - g, f - g)\}^{1/2}$$

for f, g in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$. The second relation is established for general f in $\mathbf{F}$ by choosing f_n in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ such that $f_n \to f$ q.e. and in $\mathbf{F}$, by noting that

$$\{x : |f(x)| > \varepsilon\} \subset \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{x : |f_m(x)| > \varepsilon\}$$

except for a polar set, and by applying (1.4) to the right side in an obvious way.

Now we are ready to prove

THEOREM 1.13 (*Maximum Principle*). *Let* $f = G_u \mu$ *and* $g = G_u \nu$ *be u-potentials with* $g \geq f$ [a.e. μ]. *Then* $g \geq f$ *q.e.*

Proof. Let $h = \min(f, g)$. Then

$$E_u(h, f) = \int \mu(dx)h(x) = \int \mu(dx)f(x) = E_u(f, f)$$

and

$$E_u(h, h) = E_u(f, f) + E_u(h - f, h - f)$$

which implies $h = f$ by Proposition 1.2 (v) and Proposition 1.2 (ii).

Applying Theorem 1.13 with $g = \min(f, c)$ we get

COROLLARY 1.14. *Let* $f = G_u \mu$ *be a u-potential such that* $f \leq c$ [a.e. μ]. *Then* $f \leq c$ q.e.

We turn now to some technical results. The proof of the first is Fukushima's [13].

LEMMA 1.15. *Let* f, g *be quasi-continuous on an open subset* G *of* $\mathbf{X}$. *If* $f \geq g$ [a.e. dx] *then also* $f \geq g$ q.e.

Proof. We show that $A = \{x \text{ in } G : f(x) < g(x)\}$ satisfies $\text{Cap}_1(A) = 0$. Fix $\varepsilon > 0$ and choose an open subset ω of G such that $\text{Cap}_1(\omega) < \varepsilon$ and f, g are continuous on $G - \omega$. Assume first that ω has the following property: if x belongs to $G - \omega$, then every neighborhood U of x satisfies $\int_{U-\omega} dx > 0$. Then the condition $g \leq f$ [a.e. dx] guarantees that A is contained in ω and we are done. In general it suffices to replace ω by the open set ω' of x in G which have an open neighborhood U_x satisfying $\int_{U_x - \omega} dx = 0$.

LEMMA 1.16. *Let* ν *be a bounded Radon measure such that*

$$\nu(G) \leq c \, \text{Cap}_1(G)$$

for all open G *and for a fixed constant* c. *Then* ν *has finite energy.*

Proof. It suffices to show that

$$\int \nu(dx)f(x) \leq \text{constant}$$

for $f \geq 0$ in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ satisfying $E_1(f, f) = 1$. But this follows from

$$\int \nu(dx)f(x) \leq \nu\{x : f(x) < 1\} + \textstyle\sum_{k=0}^{\infty} 2^{k+1}\nu\{x : 2^k \leq f(x) < 2^{k+1}\}$$

$$\leq \int \nu(dx) + c \textstyle\sum_{k=0}^{\infty} 2^{k+1} \, \text{Cap}_1\{x : f(x) \geq 2^k\}$$

$$\leq \int \nu(dx) + c \textstyle\sum_{k=0}^{\infty} 2^{k+1} 2^{-2k}.$$

LEMMA 1.17. *Every Radon measure ν which charges no polar set is the vague limit of an increasing sequence of measures ν_n, each having finite energy.*

Proof. It suffices to consider the case when ν is bounded and because of Lemma 1.16 it suffices to establish the following two results.

1.13.1. *There exist real $\alpha_N \downarrow 0$ such that for any Borel set B the inequality $\nu(B) \geq N \operatorname{Cap}_1(B)$ implies that $\nu(B) \leq \alpha_N$.*

1.13.2. *For each N there exists a Borel subset X_N of $\mathbf{X}$ such that $\nu(X_N) \leq \alpha_N$ and such that $\nu(A) \leq N \operatorname{Cap}_1(A)$ for every Borel subset A of $\mathbf{X} - X_N$.*

If Statement 1.13.1 is false, then there exist Borel sets B_n such that

$$\nu(B_n) \geq \alpha > 0 \quad \text{and} \quad \operatorname{Cap}_1(B_n) \leq 2^{-n-1}.$$

But then $\nu(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B_n) \geq \alpha$ while

$$\operatorname{Cap}_1(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B_n) \leq \sum_{n=m}^{\infty} 2^{-n-1} = 2^{-m}$$

for all m which contradicts our hypothesis that ν charges no polar set. This proves 1.13.1. To prove 1.13.2 we employ one of the standard techniques for proving the Radon Nikodym theorem. First let

$$\alpha = \inf \{N \operatorname{Cap}_1(A) - \nu(A)\}$$

as A runs over the Borel subsets of $\mathbf{X}$. Clearly $-\nu(\mathbf{X}) \leq \alpha \leq 0$. Choose A_1 such that

$$N \operatorname{Cap}_1(A_1) - \nu(A_1) \leq \tfrac{1}{2}\alpha.$$

Then for $A \subset \mathbf{X} - A_1$, we have

$$\alpha \leq N \operatorname{Cap}_1(A \cup A_1) - \nu(A \cup A_1)$$
$$\leq N \operatorname{Cap}_1(A) + N \operatorname{Cap}_1(A_1) - \nu(A) - \nu(A_1)$$

which implies that $N \operatorname{Cap}_1(A) - \nu(A) \geq \tfrac{1}{2}\alpha$. Continuing in this way we find a sequence of disjoint Borel sets $A_1, A_2, \cdots$ such that $N \operatorname{Cap}(A_n) \leq \nu(A_n)$ for all n and such that

$$N \operatorname{Cap}_1(A) - \nu(A) \geq 2^{-n}\alpha$$

for all subsets A of $\mathbf{X} - (A_1 \cup \cdots \cup A_n)$. Finally we take $X_N = \bigcup_{n=1}^{\infty} A_n$ and apply 1.13.1.

Next we establish a simple but useful extension of Lemma 1.17.

COROLLARY 1.18. *Every Radon measure ν which charges no polar set is the vague limit of an increasing sequence of measures ν_n in $\mathfrak{M}$ each having a bounded 1-potential.*

Proof. By Lemma 1.17 and a standard diagonalization argument it suffices to consider ν in $\mathfrak{M}$ and then the corollary follows upon approximating ν by its restriction to the set where $G_1 \nu \leq n$ and then applying the maximum principle (Theorem 1.13).

We finish this section by generalizing the notion of a u-potential.

1.14. DEFINITION. A generalized u-potential $(u > 0)$ is a Borel function h satisfying:

1.14.1. $0 \leq h < +\infty$ q.e.

1.14.2. There exists a Radon measure μ charging no polar set such that

$$\int \nu\,(dx)h(x) = \int \mu\,(dx)G_u\,\nu(x) \tag{1.6}$$

for all ν in $\mathfrak{M}$. We write $h = G_u\,\mu$.

1.14.3. For each compact subset K of $\mathbf{X}$ there exists a u-potential g_K such that $g_K \geq h$ q.e. on K.

Condition 1.14.3 is technical in nature and seems rather artificial at this stage. However it seems to be just what is required to permit us to restrict our attention to Radon measures. (See the proof of Theorem 3.12 below.)

1.15. *Notation.* $\mathfrak{M}^e$ is the collection of Radon measures μ charging no polar set such that $G_u\,\mu$ is a generalized u-potential for all $u > 0$.

We will see that $\mathfrak{M}^e$ rather than $\mathfrak{M}$ is the natural setting for much of our work. An argument of Blumenthal and Getoor [2, p. 260] serves to establish the following uniqueness result.

LEMMA 1.19. *If μ, ν belong to $\mathfrak{M}^e$ and if $G_u\,\mu = G_u\,\nu$ q.e. for some $u > 0$, then $\mu = \nu$.*

2. Construction of processes

In this section we construct a family of strong Markov processes associated with the regular Dirichlet space of Section 1. As mentioned in the introduction, this has already been done by Fukushima [13] using different techniques.

We begin by applying the spectral theory to the resolvent operators G_u considered as bounded symmetric operators on the Hilbert space $\mathbf{F}$ relative to any of the inner products E_u. This yields a unique family of bounded linear operators P^t, $t > 0$ on $\mathbf{F}$ satisfying:

2.1.1. Each P^t is contractive and symmetric relative both to the Dirichlet norm E and to the standard inner product on $L^2(dx)$.

2.1.2. $P^tP^s = P^{t+s}$ for $s, t > 0$.

2.1.3. $G_u = \int_0^\infty dt\, e^{-ut}P^t$ for $u > 0$.

Now we apply Laplace inversion in the spirit of Feller [10, vol. 2, XIII. 4]. Tchebychev's inequality applied to the appropriate Poisson variables shows that for any bounded continuous function α on $[0, \infty)$

$$\alpha(t) = \mathrm{Lim} \sum_{k=0}^{\infty} e^{-nt}\alpha(k/n)(tn)^k/k! \quad (n \uparrow \infty)$$

boundedly and so for any $\varepsilon > 0$ and for any f in $\mathbf{F}$

$$\int_0^\infty dte^{-\varepsilon t}\alpha(t)P^t f = \mathrm{Lim} \int_0^\infty dte^{-\varepsilon t}P^t f \sum_{k=0}^{\infty} e^{-nt}\alpha(k/n)(tn)^k/k! \quad (n \uparrow \infty)$$

$$= \mathrm{Lim} \sum_{k=0}^{\infty} \alpha(k/n)(nG_{n+\varepsilon})^k G_{n+\varepsilon} f \quad (n \uparrow \infty)$$

where the integrals are to be interpreted as strong limits of Riemann sums. Now condition 1.1.1 and Proposition 1.2 (iv) lead to:

2.1.4. $f \geq 0$ q.e. implies that $P^t f \geq 0$ q.e. and $f \leq 1$ q.e. implies that $P^t f \leq 1$ q.e.

2.1.5. $e^{-ut}P^t f \leq f$ q.e. for $u, t > 0$ if f is a u-potential.

Note. Before continuing we remind the reader of our Convention 1.12 that functions in F are specified up to quasi-equivalence. This plays an important role in our construction.

To make further progress it is necessary to introduce transition probabilities $P^t(x, dy)$ satisfying in the appropriate sense

$$P^t f(x) = \int P^t(x, dy) f(y), \tag{2.1}$$

$$P^{t+s}(x, \Gamma) = \int P^t(x, dy) P^s(y, \Gamma). \tag{2.2}$$

For technical reasons we proceed indirectly, considering first the action of the P^t on measures.

LEMMA 2.1. *Let ν be a bounded Radon measure which charges no polar set. Then for each $t > 0$ there exists a unique Radon measure νP^t, also charging no polar set, such that*

$$\int \nu P^t(dx) \leq \int \nu(dx), \tag{2.3}$$

$$\int (\nu P^t)(dx) f(x) = \int \nu(dx) P^t f(x), \quad f \geq 0, f \text{ in } \mathbf{F}. \tag{2.4}$$

Moreover if ν has finite energy, then so does $P^t\nu$ and indeed for all $u > 0$,

$$e^{-ut}\| P^t\nu \|_u \leq \| \nu \|_u . \tag{2.5}$$

Proof. If $f_n \geq 0$ belong to $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$ and if $f_n \downarrow 0$ pointwise, then $f_n \downarrow 0$ uniformly and by 2.1.4 also $P^t f_n \downarrow 0$ uniformly q.e. Then $\int \nu(dx) P^t f_n(x) \downarrow 0$ and so by the Daniell approach to integration [20, p. 29] there exists a unique Radon measure νP^t satisfying (2.4) for f in $\mathbf{F} \cap C_{\mathrm{com}}(\mathbf{X})$. To show that νP^t charges no polar set it suffices to show that $\nu P^t(K) = 0$ whenever K is compact and $\text{Cap}_1(K) = 0$. To do this, consider such a K and let open $G \downarrow K$ with $\text{Cap}_1(G) \downarrow 0$. Then $p_1^G \to 0$ strongly in $\mathbf{F}$ and so also $P^t p_1^G \to 0$ strongly in $\mathbf{F}$. Since the p_1^G decrease monotonically q.e. for a sequence (see the proof of Proposition 1.8), it follows from 2.1.4 and from Theorem 1.11.2 that for a sequence $P^t p_1^G \downarrow 0$ q.e. and there $\int \nu(dx) P^t p_1^G(x) \downarrow 0$. Fix one such G and $\varepsilon > 0$ and choose f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ with $0 \leq f \leq 1$ such that $f \geq \frac{1}{2}$ on K, and $f \leq \varepsilon$ on $\mathbf{X} - G$. Then $f - p_G^1 \leq \varepsilon$ q.e. and so

$$\int (\nu P^t)(dx) f(x) - \int \nu(dx) P^t p_1^G(x)$$
$$= \int \nu(dx) P^t(f - p_1^G)(x) \leq \varepsilon \int \nu(dx)$$

and it follows readily that $\nu P^t(K) = 0$. The extension of (2.4) to general f in $\mathbf{F}$ and the verification of (2.3) and (2.5) are routine.

It follows directly from the semigroup property 2.1.2 and from (2.4) that

$$\nu P^t P^s = \nu P^{t+s}$$

for $s, t > 0$ and for ν a bounded measure which charges no polar set. Now we are ready for

LEMMA 2.2. *There exist subprobabilities $P^t(x, dy)$ on $\mathbf{X}$ defined for $t > 0$ and for x in $\mathbf{X}$ such that*

(i) *$P^t(\cdot, \Gamma)$ is Borel measurable for any $t > 0$ and for any Borel subset Γ of $\mathbf{X}$.*

(ii) *For fixed $t > 0$ and f in $\mathbf{F}$, equation (2.1) is valid for* q.e. *x.*

(iii) *For fixed $s, t > 0$ there exists a polar set N (depending on s and t) such that (2.2) is valid for all Borel subsets Γ of $\mathbf{X}$.*

Proof. Clearly there exist subprobabilities $P^t(x, dy)$ *satisfying* (i) and such that (2.1) is valid q.e. for fixed $t > 0$ and fixed f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. Therefore

$$\int \nu(dx) \int P^t(x, dy) f(y) = \int (\nu P^t)(dy) f(y) \tag{2.6}$$

for $t > 0$, for ν a bounded measure which charges no polar set, and for f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. The restriction on f can certainly be removed and then (ii) follows easily with the help of (2.4). Finally (iii) follows directly from (ii) and from 2.1.2.

Note. In the proof of Lemma 2.2 we used implicitly the fact that if A is a Borel set which is not polar, then there exists a bounded measure ν charging no polar set such that $\nu(A) > 0$. The existence of such a ν can be established by first using Corollary 1.10 to replace A by a compact set and then considering open G_n decreasing to A and applying Proposition 1.5 (iii) and Proposition 1.4 (ii).

At this stage of the argument we restrict our attention to rational times in order to pick out a single exceptional polar set. First choose N_1 polar such that (2.2) is valid for all rational $s, t > 0$ and for x in $\mathbf{X} - N_1$. From Lemma 2.1 and from (2.6) it follows that the set of x in $\mathbf{X}$ for which $P^t(x, N_1) > 0$ for any rational $t > 0$ is polar. Thus there exists polar N_2 containing N_1 such that $P^t(x, N_1) = 0$ for x in $\mathbf{X} - N_2$ and for rational $t > 0$. Continuing in this way and taking the union, we prove

LEMMA 2.3. *There exists a polar set N such that (2.2) is valid for all rational $s, t > 0$ and for x in $\mathbf{X} - N$ and such that*

$$P^t(x, N) = 0$$

for x in $\mathbf{X} - N$ and for all rational $t > 0$.

Next we introduce the *preliminary sample space*. This is the collection Ω_o of mappings ω from the nonnegative rationals into the *augmented state space* $\mathbf{X} \cup \{\partial\}$. Here ∂ is the usual "dead point" which we adjoin to $\mathbf{X}$ as an isolated point when $\mathbf{X}$ is compact and as the "point at infinity" otherwise. Functions f on $\mathbf{X}$ are automatically extended to $X \cup \{\partial\}$ so that $f(\partial) = 0$. For $t \geq 0$ the past $\mathfrak{F}_t$ is the sigma algebra generated by the coordinate variables $\omega(s)$ with $s \leq t$. The *Borel algebra* $\mathfrak{F}$ is the sigma algebra generated by all of the coordinate variables $\omega(s)$, $s \geq 0$. Standard arguments [6, p. 99] establish the following two theorems.

THEOREM 2.4. *For x outside the exceptional set N of Lemma 2.3, there is a unique probability $\mathcal{P}_x$ on the Borel algebra $\mathfrak{F}$ of Ω_o such that*

$$\mathcal{E}_x f_0[\omega(t_0)] f_1[\omega(t_1)] \cdots f_n[\omega(t_n)] = \int P^{t_1}(x, dy_1) \int P^{t_2 - t_1}(y_1, dy_2) \cdots \int P^{t_n - t_{n-1}}(y_{n-1}, dy_n) f_0(x) f_1(y_1) \cdots f_n(y_n)$$

for $0 = t_0 < t_1 < \cdots < t_n$ all rational and for Borel $f_0, \cdots, f_n \geq 0$.

Of course $\mathcal{E}_x$ denotes the usual expectation functional corresponding to the probability $\mathcal{P}_x$.

For $t \geq 0$ rational let $\theta(t)$ be the shift transformation defined on Ω_o by

$$\theta(t)\omega(s) = \omega(t + s).$$

Then

THEOREM 2.5 (*Simple Markov property*). *For x outside the exceptional set N of Lemma 2.3, for $t \geq 0$ rational and for $\xi \geq 0$ a Borel function on Ω_o,*

$$\mathcal{E}_x(\theta(t)\xi \mid \mathfrak{F}_t) = \mathcal{E}_{\omega(t)} \xi \quad \text{[a.e. } \mathcal{P}_x]. \tag{2.7}$$

Of course $\mathcal{E}(\ \mid\ )$ denotes the usual conditional expectation and $\theta(t)\xi$ is defined by

$$\theta(t)\xi(\omega) = \xi(\theta(t)\omega).$$

Note that $\omega(t)$ avoids N of Lemma 2.3 with $\mathcal{P}_x$ probability one so that $\mathcal{E}_{\omega(t)}\,\xi$ is well defined [a.e. $\mathcal{P}_x$].

Our *final sample space* Ω is the subset of ω in Ω_o having one sided limits $\omega(t \pm 0)$ for all real $t \geq 0$. Standard arguments [22, p. 60] show that Ω is a Borel subset of Ω_o. The starting point for establishing regularity is

THEOREM 2.6. *There exists a polar set N satisfying the conclusion of Lemma 2.3 such that for x in $\mathbf{X} - N$ the probability $\mathcal{P}_x$ is concentrated on the final sample space Ω of trajectories ω having one sided limits $\omega(t \pm 0)$ for all real $t \geq 0$.*

Proof. Let R_1 be a nonnegative random variable which is exponentially distributed with rate 1 (that is, $\mathcal{P}_x(R_1 > t) = e^{-t}$) and which is independent of the Borel algebra $\mathfrak{F}$. The theorem follows by routine arguments once we show that for each f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ there exists a polar set $N(f)$ such that for x in $\mathbf{X} - N(f)$ the set of ω in Ω_o for which $f[\omega(t)]$ has one sided limits for all real $t < R_1$ has $\mathcal{P}_x$ probability one. (We take the usual liberty of assuming that all structure on Ω_o has been transferred to an appropriate augmented sample space on which R_1 is also defined.) Fix one such f and choose f_n in $\mathbf{F}$ having the form $f_n = G_1 \varphi_n$ with φ_n bounded and square integrable such that $f_n \to f$ strongly in $\mathbf{F}$ and also quasi-everywhere. Choose decreasing open G_m with $\text{Cap}_1(G_m) \downarrow 0$ such that $f_n \to f$ uniformly on the complement of each G_m. With the help of the simple Markov property Theorem 2.5, it is easy to check that for a fixed nonnegative φ which is bounded and square integrable and for q.e. x the process

$$\{e^{-t} G_1 \varphi[\omega(t)],\ t \geq 0 \text{ and rational}\}$$

is an L^1 bounded supermartingale relative to $\mathcal{P}_x$ and so by standard supermartingale estimates [22, Part B] has one sided limits everywhere for real $t \geq 0$ and in particular for real $t < R_1$. The same is true with $G_1 \varphi$ replaced by any f_n and it only remains to show that for q.e. x,

(2.8) $\mathcal{P}_x\{\omega : \omega(t)$ is in G_m for some nonnegative rational $t < R_1\}$ decreases to 0 as $m \uparrow \infty$.

Fix a finite subset S of nonnegative rationals and put

$$\sigma_m = \text{minimum}\ \{t \text{ in } S : \omega(t) \text{ is in } G_m\}$$

with the understanding that $\sigma_m = +\infty$ when not otherwise defined. Since the process

$$\{e^{-t} p_1^{G_m}[\omega(t)],\ t \text{ in } S\}$$

is a supermartingale by 2.1.5 and since σ_m is a stopping time in the sense of [22, T28], we have

$$\begin{aligned} \mathcal{P}_x[\sigma_m < R_1] &= \mathcal{E}_x I(\sigma_m < R_1) p_1^{G_m}[\omega(\sigma_m)] \\ &= \mathcal{E}_x\, e^{-\sigma_m} p_1^{G_m}[\omega(\sigma_m)] \qquad (2.9) \\ &\leq p_1^{G_m}(x) \end{aligned}$$

which decreases to 0 as $m \uparrow \infty$ for q.e. x. The desired estimate for (2.8) follows and the proof of the theorem is complete since the estimate (2.9) is independent of S.

Note. We are using the symbol $I(A)$ to denote the indicator of the set A.

For convenient future reference we record here a result which was established in the course of proving Theorem 2.6.

LEMMA 2.7. *For any particular version of a function f in* $\mathbf{F}$ *there exists a polar set N such that for x in* $\mathbf{X} - N$ *and for* [a.e. $\mathcal{P}_x$] *trajectory ω in Ω the range of $X(t, \omega)$ for $0 \leq t < +\infty$ belongs to a subset of the augmented space* $\mathbf{X} \cup \{\partial\}$ *on which f is defined and continuous.*

2.2. *Extension.* For ν a Radon measure on $\mathbf{X}$ which charges no polar set let $\mathcal{P}_\nu$ be the unique measure on the Borel algebra $\mathfrak{F}$ of Ω satisfying

$$\mathcal{E}_\nu \xi = \int \nu(dx) \mathcal{E}_x \xi \tag{2.10}$$

for $\xi \geq 0$ and Borel measurable on Ω. (This makes sense since $\mathcal{E}_x \xi$ is Borel measurable on $\mathbf{X} - N$.)

2.3. *Conventions.* From now on we work with Ω instead of Ω_o and we assume that all structure on Ω_o is transferred in the obvious way to Ω. For $t \geq 0$ and *real* we define $X(t, \omega)$ on Ω by

$$X(t, \omega) = \text{Lim } \omega(s)$$

with the limit taken as rational s *decrease* to t. Then for all ω in Ω the trajectory $X(t, \omega)$ is right continuous and has one sided limits everywhere for real $t \geq 0$. We will generally suppress ω in the notation. For typographical convenience we introduce the nonstandard notation

$$f(t) = f[X(t)]$$

for functions f on $\mathbf{X}$. We also adopt the convention

$$X(\infty) = \partial. \tag{2.11}$$

For real $t \geq 0$, and for $f \geq 0$ on $\mathbf{X}$ we define

$$P^t f(x) = \mathcal{E}_x f(t), \qquad G_u f(x) = \mathcal{E}_x \int_0^\infty dt e^{-ut} f(t)$$

for x in $\mathbf{X} - N$ where N is the exceptional polar set of Theorem 2.6. This is consistent with all previous definitions.

A simple argument using the fact that $G_u f$ belongs to $\mathbf{F}$ whenever f does suffices to establish a useful extension of Corollary 1.3.

LEMMA 2.8. *Let $f \geq 0$ and defined* q.e. *satisfy the hypotheses of Corollary* 1.3 *and in addition*

$$\text{Lim } vG_{u+v} f = f \text{ q.e.} \quad (v \to \infty).$$

Then f is a u-potential and is the unique quasi-continuous version of Theorem 1.11.

We turn now to the strong Markov property.

2.4. Definition. A stopping time is a nonnegative Borel measurable function T on Ω (possibly taking the value $+\infty$) such that for each $t > 0$ the subset $[T < t]$ of Ω belongs to the past $\mathfrak{F}_t$. The corresponding past $\mathfrak{F}_T$ is the sigma algebra of Borel subsets Γ of Ω such that for all $t > 0$ the intersection $\Gamma \cap [T < t]$ belongs to $\mathfrak{F}_t$. The shift transformation $\theta(T)$ is defined on the set $[T < +\infty]$ by

$$\theta(T)\omega(s) = \omega(T(\omega) + s).$$

Theorem 2.9. *Let T be a stopping time. Then*

(i) *The coordinate $X(T, \omega) = X(T(\omega), \omega)$ is $\mathfrak{F}_T$ measurable. (Recall our convention* (2.9).)

(ii) *Let ν be a bounded measure which charges no polar set and for $u \geq 0$ let ν' be the measure on $\mathbf{X}$ defined by*

$$\int \nu'(dx)f(x) = E_\nu e^{-uT}f(T).$$

Then ν' charges no polar set. Indeed if ν has finite energy then so does ν' and for $u > 0$,

$$\| \nu' \|_u \leq \| \nu \|_u .$$

(iii) (*Strong Markov property*) *There exists a polar set N independent of T which satisfies the conclusion of Theorem* 2.6 *and such that for x in $\mathbf{X} - N$ and for $\xi \geq 0$ and Borel on Ω,*

$$\mathcal{E}_x(\theta(T)\xi \mid \mathfrak{F}_T) = \mathcal{E}_{x(T)}\, \xi \quad [\text{a.e. } \mathcal{P}_x].$$

Proof. Fix a sequence of rational valued stopping times T_n which decrease to T as $n \uparrow \infty$. (For example, take T_n to be the smallest positive number of the form $k2^{-n}$ which is $\geq T$.) Then for K compact and for open $G_m \downarrow K$,

$$[T < t] \cap [X_T \in K] = \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{[T_n < t] \cap [X(T_n) \in G_m]\}$$

and the $\mathfrak{F}_T$ measurability of X_T follows. In proving (ii) it suffices becauses of Lemma 1.17 to consider ν having finite energy and because of Proposition 1.4 (ii) to take $T = T_m$ for some m. But then (ii) follows since for any μ in $\mathfrak{M}$, $\{e^{-t}G_1\mu(t), t \geq 0\}$ is a supermartingale relative to $\mathcal{P}_\nu$. To prove (iii), choose a polar set N satisfying the conclusion of Theorem 2.6 and such that for x in $\mathbf{X} - N$ and for [a.e. $\mathcal{P}_x$] trajectory ω the range of $X(t, \omega)$ for $0 \leq t < +\infty$ belongs to a subset of the augmented space $\mathbf{X} \cup \{\partial\}$ on which the functions

$$f_0(x), P^{t_1}f_1 P^{t_2-t_1}f_2 \cdots P^{t_m-t_{m-1}}f_m(x)$$

are defined, bounded and continuous for all choices of $f_0, \cdots, f_m$ belonging to a fixed countable dense set in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$ and of $0 < t_1 < \cdots < t_m$ rational.

The possibility of choosing such an N follows from Lemma 2.7. Finally put

$$\xi = f_0(0)f_1(t_1) \cdots f_m(t_m),$$

note that

$$\mathcal{E}_x I(A)I(T < +\infty)\theta(T_n)\xi$$
$$= \mathcal{E}_x I(A)I(T < +\infty)f_0(T_n)(P^{t_1}f_1 \cdots P^{t_m - t_{m-1}}f_m)(T_n)$$

is valid for A in $\mathfrak{F}_T$ by the simple Markov property Theorem 2.5, and pass to the limit $n \uparrow \infty$.

THEOREM 2.10 (*Quasi-left-continuity*). *There exists a polar set N satisfying Theorem* 2.9 (iii) *and such that the following is true for x in $\mathbf{X} - N$. If T_n, T are stopping times such that $T_n \uparrow T$* [a.e. $\mathcal{P}_x$], *then*

$$X(T) = \operatorname{Lim} X(T_n) \quad (n \uparrow \infty)$$

[a.e. $\mathcal{P}_x$] *on the set* $[T < +\infty]$.

Proof. Let R_1 be as in the proof of Theorem 2.6. Fix $m < n$ and f in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. Then by Fubini's theorem and the strong Markov property Theorem 2.9-(iii)

$$\mathcal{E}_x\left(I(T_n < R_1)\int_{T_n}^{R_1} dtf(t) \mid \mathfrak{F}_{T_m}\right) = \mathcal{E}_x\left(\left(\int_{T_n}^{\infty} dte^{-t}f(t) \mid \mathfrak{F}_{T_m}\right)\right)$$
$$= \mathcal{E}_x(e^{-T_n}G_1 f(T_n) \mid \mathfrak{F}_{T_m})$$
$$= \mathcal{E}_x(I(T_n < R_1)G_1 f(T_n) \mid \mathfrak{F}_{T_m})$$

for x in $\mathbf{X} - N$ with N as in Theorem 2.9 (iii). The analagous relation is valid with T_n replaced by T. For q.e. x we have

$$\mathcal{E}_x \int_0^{R_1} dtf(t) < +\infty$$

so that the dominated convergence theorem for conditional expectations [19, p. 348] is applicable. Letting $n \to \infty$, noting that $\mathcal{P}_x(T = R_1) = 0$ and applying Lemma 2.7 we obtain

$$\mathcal{E}_x(I(T < R_1)G_1 f(\operatorname{Lim} X(T_n)) \mid \mathfrak{F}_{T_m}) = \mathcal{E}_x(I(T < R_1)G_1 f(X(T)) \mid \mathfrak{F}_{T_m})$$

for x in $\mathbf{X} - N$ where N is a polar set depending only on f. Since this is true for all m and since for bounded Borel g the function $g(\operatorname{Lim} X(T_n))$ is measurable with respect to the sigma algebra generated by the union of the $\mathfrak{F}_{T_m}$ (see Theorem 2.9 (i)), we have

$$\mathcal{E}_x I(T < R_1)g(\operatorname{Lim} X(T_n))G_1 f(\operatorname{Lim} X(T_n))$$
$$= \mathcal{E}_x I(T < R_1)g(\operatorname{Lim} X(T_n))G_1 f(X(T))$$

and the theorem follows since we can replace $G_1 f$ by f after a passage to the limit as in the proof of Theorem 2.6.

Note. The above proof of Theorem 2.10 is essentially the one given by Kunita and Watanabe [26].

We turn now to the hitting times

$$\sigma(A) = \inf\{t \geq 0 : X(t) \text{ is in } A\}$$

defined for Borel subsets A of the augmented space $\mathbf{X} \cup \{\partial\}$ with the understanding that $\sigma(A) = +\infty$ if not otherwise defined. We first consider open sets.

THEOREM 2.11. *Let G be an open subset of the augmented state space $\mathbf{X} \cup \{\partial\}$.*

(i) *$\sigma(G)$ is a stopping time.*

(ii) *If G has finite capacity, then for any $u > 0$,*

$$\mathcal{E}_x \exp\{-u\sigma(G)\} = p_u^G(x) \quad \text{q.e.} \tag{2.12}$$

(iii) *The left side of (2.12) is always quasi-continuous.*

Proof. (i) is standard. To prove (ii) let $h(x)$ be the minimum of the two sides of (2.12) and note that h satisfies the hypotheses of Lemma 2.8 as can be verified by routine arguments. Then from Proposition 1.5 (i) and Proposition 1.2 (ii) we conclude that $h = p_u^G$ q.e. and so the left side of (2.12) dominates the right side q.e. The opposite inequality follows in the same way as (2.9). Finally (iii) follows upon applying Lemma 2.8 to the minimum of the left side of (2.12) and $p_u^{G'}$ where G' is a general open set with finite capacity.

THEOREM 2.12 (*Measurability of hitting times*). *Let A be a Borel subset of the augmented state space $\mathbf{X} \cup \{\partial\}$.*

(i) *There exist stopping times $\sigma_*(A)$, $\sigma^*(A)$ and a polar set N depending on A such that $\sigma_*(A) \leq \sigma(A) \leq \sigma^*(A)$ everywhere on Ω and such that*

$$\sigma_*(A) = \sigma^*(A) \text{ [a.e. } P_x]$$

for x in $\mathbf{X} - N$.

(ii) *Possible choices for $\sigma_*(A)$ and $\sigma^*(A)$ are*

$$\sigma^*(A) = \operatorname{Lim} \sigma^*(K_n) \quad (n \uparrow \infty)$$

$$\sigma_*(A) = \operatorname{Lim} \sigma(G_n) \quad (n \uparrow \infty)$$

where the K_n form a particular increasing sequence of compact subsets of A and the G_n form a particular decreasing sequence of open supersets of A.

(iii) *For $u > 0$ the function*

$$h(x) = \mathcal{E}_x \exp\{-u\sigma(A)\}$$

is quasi-continuous.

Proof. First consider A compact, let G_n be any decreasing sequence of open sets whose intersection is A and define $\sigma_*(A) = \text{Lim}\ \sigma(G_n)\ (n \uparrow \infty)$. The set $[\sigma_*(A) = \sigma(A)]$ is clearly Borel and by Theorem 2.10 there exists a polar set N such that $P_x(\sigma_*(A) = \sigma(A)) = 1$ for x in $\mathbf{X} - N$. Therefore (i) and (ii) are satisfied if we define $\sigma^*(A) = \sigma_*(A)$ whenever $\sigma_*(A) = \sigma(A)$ and $\sigma^*(A) = +\infty$ otherwise. To handle general Borel A we follow Hunt [16] and apply the Choquet extension theorem. Fix a bounded measure μ charging no polar set and define $\lambda(A)$ for A open or compact by

$$\lambda(A) = \int \mu\,(dx)\mathcal{E}_x \exp\{-\sigma(A)\}. \tag{2.13}$$

We want to conclude that for any Borel set A

$$\text{supremum}\ \lambda(K) = \text{infimum}\ \lambda(G) \tag{2.14}$$

as K runs over the compact subsets of A and as G runs over the open supersets of A. To do this we must check that λ satisfies the following conditions. (See [2, p. 53].)

2.4.1 $\lambda(G_1) \leq \lambda(G_2)$ whenever $G_1 < G_2$ with G_1, G_2 open.

2.4.2. For K compact, $\lambda(K) = \text{infimum}\ \lambda(G)$ as G runs over the open supersets of K.

2.4.3. $\gamma(G_1 \cup G_2) + \lambda(G_1 \cap G_2) \leq \lambda(G_1) + \lambda(G_2)$ for G_1, G_2 open.

Property 2.4.1 is clear, 2.4.2 follows from (ii) for A compact, and 2.4.3 follows from

$$\mathcal{E}_x \exp\{-\sigma(G)\} = \mathcal{P}_x(\sigma(G) < R_1)$$

and from

$$I(\sigma(G_1 \cup G_2) < R_1) + I(\sigma(G_1 \cap G_2) < R_1) \leq I(\sigma(G_1) < R_1) + I(\sigma(G_2) < R_1).$$

(Of course R_1 is defined as in the proof of Theorem 2.6.) Thus (2.14) is established. To complete the proof fix a Borel set A and take μ in (2.13) equivalent to dx. Then there exist increasing compact subsets K_n of A and decreasing open supersets G_n of A such that $\sigma_*(A) = \sigma^*(A)$ [a.e. P_x] for [a.e. dx] point x in $\mathbf{X}$ with $\sigma_*(A)$ and $\sigma^*(A)$ as in conclusion (ii) of the theorem. But the proof of Theorem 2.11 shows that the functions

$$h_*(x) = \mathcal{E}_x \exp\{-\sigma_*(A)\}, \qquad h^*(x) = \mathcal{E}_x \exp\{-\sigma^*(A)\}$$

are quasi-continuous. Thus by Lemma 1.15 $h_* = h^*$ quasi-everywhere and we are done.

It follows directly from (2.12) and from Theorem 2.12 (ii) that if N is polar, then $\mathcal{P}_x(\sigma(N) = +\infty) = 1$ for q.e. x. Thus reasoning as in the paragraph preceding Lemma 2.3 we prove

THEOREM 2.13. *There exists a polar set N satisfying the conclusion of Theorem* 2.10 *such that $\mathcal{P}_x(\sigma(N) = +\infty) = 1$ for x in $\mathbf{X} - N$.*

From the strong Markov property Theorem 2.9 (iii) and the obvious identity

$$\mathcal{P}_\partial(X(t) = \partial \text{ for all } t \geq 0) = 1$$

follows

THEOREM 2.14. *There exists a stopping time ζ, called the life time, and a polar set N such that for x in $\mathbf{X} - N$ and for* [a.e. $\mathcal{P}_x$] *trajectory ω in Ω,*

$$X(t) \text{ belongs to } \mathbf{X} \text{ for } t < \zeta,$$

$$X(t) = \partial \quad \text{for } t \geq \zeta.$$

3. Additive functional and balayage

For $u > 0$ and ν in $\mathfrak{M}$ and for q.e. x the process

$$\{e^{-ut}G_u\nu(t),\ t \geq 0\}$$

is a supermartingale relative to $\mathcal{P}_x$. In this section we use techniques similar to those of Blumenthal and Getoor [2, IV.3] to construct additive functionals which generate such supermartingales in the sense of [22, VII] and which satisfy certain regularity conditions. We begin with a special case. For $u, t \geq 0$ and for nonnegative g bounded and square integrable on $\mathbf{X}$ let

$$A_u(g; t) = \int_0^t ds e^{-us} g(s).$$

These functionals are certainly everywhere defined on Ω and with the help of Theorem 2.9 it is easy to verify

3.1. There exists a polar set N satisfying the hypotheses of Theorem 2.13 such that for every stopping time T the functional $A_u(g; T)$ is measurable with respect to the past $\mathfrak{F}_T$ and such that for $u > 0$, for x in $\mathbf{X} - N$, and for [a.e. $\mathcal{P}_x$] trajectory ω,

$$\mathfrak{F}_T A_u(g; \infty) = A_u(g; T) + e^{-uT} G_u g(T), \tag{3.1}$$

$$e^{-uT}\theta(T) A_u(g; t) = A_u(g; T + t) - A_u(g; T) \tag{3.2}$$

with the second equation making sense and being valid for all $t \geq 0$ whenever $T(\omega) < +\infty$.

3.2. *Notation.* In paragraph 3.1 and at various points in the sequel we use Hunt's notation $\mathfrak{F}_T \xi$ instead of $\mathcal{E}(\xi \mid \mathfrak{F}_T)$ to denote the conditional expectation.

The necessary preliminaries are established in

LEMMA 3.1. *Let ν be a bounded measure in $\mathfrak{M}$ such that $G_u \nu$ is bounded* q.e. *for one (and therefore all) $u > 0$. Then there exists a polar set N and a sequence $v(k) \uparrow \infty$ such that for x in $\mathbf{X} - N$ and for* [a.e. $\mathcal{P}_x$] *trajectory ω,*

$$\operatorname{Lim} A_u(v(k) G_{v(k)} \nu; t) \quad (k \uparrow \infty)$$

exists uniformly in $0 \leq t \leq +\infty$ for all $u > 0$.

Proof. We establish the lemma first for a fixed $u > 0$ when there is no harm in using $g_k = [u + v(k)]G_{v(k)}\nu$ in place of $v(k)G_{v(k)}\nu$. Following Blumenthal and Getoor, we fix $l > k$ and estimate

$$\begin{aligned}
&\frac{1}{2}\,\mathcal{E}_x\{A_u(g_l\,;\,\infty) - A_u(g_k\,;\,\infty)\}^2 \\
&\quad = \mathcal{E}_x \int_0^\infty dt e^{-ut}\{g_l(t) - g_k(t)\} \int_t^\infty ds e^{-us}\{g_l(s) - g_k(s)\} \\
&\quad = \mathcal{E}_x \int_0^\infty dt e^{-2ut}\{g_l(t) - g_k(t)\}\{G_u g_l(t) - G_u g_k(t)\} \\
(3.3)\qquad &\quad \le \mathcal{E}_x \int_0^\infty dt e^{-ut} g_l(t)\{G_u \nu(t) - G_u g_k(t)\} \\
&\quad \le 2^{-3k}\mathcal{E}_x \int_0^\infty dt e^{-ut} g_l(t) \\
&\qquad + c[\mathcal{P}_x(\sup\{G_u\nu(t) - G_u g_k(t)\} \ge 2^{-3k})]^{1/2}\left[\mathcal{E}_x\left\{\int_0^{R_u} dt g_l(t)\right\}^2\right]^{1/2} \\
&\quad \le 2^{-3k}c + (\sqrt{2})c^2[\mathcal{P}_x(\sup\{G_u\nu(t) - G_u g_k(t)\} \ge 2^{-3k})]^{1/2}
\end{aligned}$$

for q.e. x with the supremum taken over $0 \le t < R_u$ (see the proof of Theorem 2.6) and with $c > 0$ a constant which dominates $G_u\nu$ q.e. Choose $v(k) \uparrow \infty$ such that

$$E_u(G_u\nu - G_u g_k\,,\,G_u\nu - G_u g_k) < (512)^{-3k}.$$

The estimate (1.5) guarantees that the subset of $\mathbf{X}$ where

$$G_u\nu - G_u g_k \ge 2^{-3k}$$

has u-capacity $< (128)^{-3k}$. Choose open G_k such that $G_u\nu - G_u g_k < 2^{-3k}$ on $\mathbf{X} - G_k$ and such that $\mathrm{Cap}_u(G_k) \le (128)^{-3k}$. Then by the very definition of Cap_u and by Proposition 1.5 (i) the set where $p_u^{G_k} \ge 4^{-3k}$ has u-capacity $\le 2^{-3k}$ and from (2.12) it follows directly that for q.e. x there exists a finite integer $m(x)$ such that

$$\mathcal{P}_x(\sigma(G_k) < R_u) \le 4^{-3k}$$

for $k \ge m(x)$. Combining this with (3.3) we conclude that for q.e. x and for $l > k \ge m(x)$

$$\frac{1}{2}\,\mathcal{E}_x\{A_u(g_l\,;\,\infty) - A_u(g_k\,;\,\infty)\}^2 \le 2^{-3k}c + (\sqrt{2})c^2 2^{-3k}.$$

and therefore by the maximal inequality for L^2 bounded martingales [22, p. 81]

$$\mathcal{P}_x(\sup|\mathfrak{F}_t A_u(g_l\,;\,\infty) - \mathfrak{F}_t A_u(g_k\,;\,\infty)| \ge 2^{-k}) \le c'\,2^{-k}$$

with the supremum taken over $0 \le t < +\infty$ and with c' a second constant. The desired uniform convergence of $A_u(g_k\,;\,t)$ now follows with the help of (3.1). Finally, by a standard diagonalization procedure we can extend the result to rational $u > 0$ and then to real $u > 0$ after a straightforward estimate along individual sample paths.

3.3. Definition. For ν and $v(k)$ as in Lemma 3.1 let

$$A_u(\nu; t) = \text{Lim sup } A_u(v(k)G_{v(k)}\,\nu; t) \quad (k \uparrow \infty)$$

for $u > 0$ and for $0 \leq t \leq +\infty$ and let

$$A(\nu; t) = \text{Lim sup } A_{(1/n)}(\nu; t) \quad (n \uparrow \infty)$$

for $0 \leq t \leq +\infty$.

Elementary arguments establish

Theorem 3.2. *For ν as in Lemma 3.1 there exists a polar set N such that the following is true for x in $\mathbf{X} - N$.*

(i) *For every stopping time T and for $u \geq 0$ the functional $A_u(\nu; T)$ is $\mathfrak{F}_T$ measurable and for $u > 0$*

$$\mathfrak{F}_T A_u(\nu; \infty) = A_u(\nu; T) + e^{-uT} G_u \nu(T) \quad [\text{a.e. } P_x].$$

(ii) *For* [a.e. $\mathcal{P}_x$] *trajectory ω,*

$$A(\nu; 0) = 0,$$

$$A(\nu; \cdot) \textit{ is continuous and increasing},$$

$$A_u(\nu; t) = \int_0^t A(\nu; ds)e^{-us},$$

with the last identity valid for $u > 0$ and $0 \leq t \leq +\infty$. Moreover if T is a stopping time such that $T(\omega) < +\infty$, then for $u, t \geq 0$

$$e^{-uT}\theta(T)A_u(\nu; t) = A_u(\nu; T + t) - A_u(\nu; T).$$

3.4. *Extension.* For ν in $\mathfrak{M}^e$ put

$$A_u(\nu; t) = \textstyle\sum_{n=1}^{\infty} A_u(\nu_n; t), \qquad A(\nu; t) = \textstyle\sum_{n=1}^{\infty} A(\nu_n; t)$$

where $\{\nu_n\}$ is any sequence of measures each satisfying the hypotheses of Lemma 3.1 and such that $\nu = \sum_{n=1}^{\infty} \nu_n$. The existence of such a sequence is guaranteed by Corollary 1.18. Elementary arguments extend Theorem 3.2 to ν in $\mathfrak{M}^e$.

Remark. Meyer's uniqueness results [22, VII. 3] guarantee that if $A_u^*(\nu; t)$ is any other family of functionals satisfying the conclusion of Theorem 3.2 with a corresponding polar set N^*, then for x in $\mathbf{X} - (N \cup N^*)$ and for [a.e. $\mathcal{P}_x$] trajectory ω,

$$A_u(\nu; t) = A_u^*(\nu; t)$$

for all $u, t \geq 0$.

3.5. *Notation.* The symbol $\varphi \cdot \nu$ denotes the measure which is absolutely continuous with respect to ν and has Radon-Nikodym derivative φ.

To avoid the need for continual reference to exceptional polar sets as in Theorem 3.2 and Theorem 2.13, we introduce at this point some nonstandard

3.6. *Terminology.* A property is valid *quasi-surely* on Ω if the exceptional set has outer $\mathcal{P}_x$ measure 0 for q.e. x in $\mathbf{X}$.

THEOREM 3.3. *Let $\varphi \geq 0$ be Borel on* $\mathbf{X}$ *and let ν be a Radon measure on* $\mathbf{X}$ *such that both ν and $\varphi \cdot \nu$ belong to $\mathfrak{M}^e$. Then quasi-surely on Ω*

$$A(t, \varphi \cdot \nu) = \int_0^t A\ (ds; \nu)\varphi(s)$$

for $0 \leq t \leq +\infty$.

Proof. Evidently it suffices to consider $A_1(t, \varphi \cdot \nu)$ and $A_1(t, \nu)$ in place of $A\ (t, \varphi \cdot \nu)$ and $A\ (t, \nu)$ and to assume that φ is bounded and that ν satisfies the hypotheses of Lemma 3.1. The proof of Lemma 3.1 and the remark following 3.4 show that for an appropriate sequence of $v(k) \uparrow \infty$, and quasi-surely on Ω

$$A_1(t, [1 + v(k)]G_{v(k)}\, \nu) \to A_1(t, \nu)$$

$$A_1(t, \varphi[1 + v(k)]G_{v(k)}\, \nu) \to A_1(t, \varphi \cdot \nu)$$

uniformly in $t \geq 0$ and the theorem follows.

We introduce next the inverse process

$$B(\nu; s) = \inf \{t > 0 : A\ (\nu; t) > s\}$$

with the understanding that $B(\nu; s) = +\infty$ when the t-set on the right is empty or when $A\ (\nu; \cdot\ )$ fails to satisfy the first two conditions in Theorem 3.2 (ii). Clearly each $B(\nu; s)$ is a stopping time and consequently for $u > 0$,

$$h_u^\nu(x) = \mathcal{E}_x\, e^{-uB(\nu;0)}$$

is Borel on the complement of an appropriate polar set and the set

$$M^\nu = \{x : h_1^\nu(x) = 1\} \tag{3.4}$$

is a Borel set. Evidently 1 can be replaced by $u > 0$ in (3.4). We will generally suppress the superscript ν when no confusion is possible. A point x in $\mathbf{X}$ is *ν-regular* if it belongs to M. A Radon measure μ is ν-regular if it is concentrated on M.

Now we are ready to introduce a form of balayage which is convenient for our purposes. First we denote by $\mathfrak{M}(\nu)$ the collection of ν-regular measures μ having finite energy and we prove

LEMMA 3.4. *For ν in $\mathfrak{M}^e$, the collection $\mathfrak{M}(\nu)$ is convex and closed relative to $\|\ \ \|_1$ (and therefore to any of the norms $\|\ \ \|_u$).*

Proof. Convexity is obvious. To show that $\mathfrak{M}(\nu)$ is closed, consider μ_n in $\mathfrak{M}(\nu)$ and μ in $\mathfrak{M}$ such that $\| \mu - \mu_n \|_1 \to 0$. It suffices to show that $\varphi \cdot \mu$ belongs to $\mathfrak{M}(\nu)$ for bounded nonnegative Borel φ with compact support K. Evidently $\varphi \cdot \mu$ is concentrated on M if and only if $\int \varphi \cdot \mu(dx) h_u(x)$ is independent of $u > 0$ and it suffices to show that this property is preserved by $\| \quad \|_1$ convergence in $\mathfrak{M}$. But it follows from Theorems 2.11 and 2.12 that h_u agrees q.e. on K with a u-potential and so the desired result follows from (1.5).

We denote by $\mathbf{F}_u(\nu)$, or simply $\mathbf{F}_u$, the closed linear subspace of $\mathbf{F}$ spanned by $G_u \mathfrak{M}(\nu)$. Since $\mathfrak{M}$ and therefore $\mathfrak{M}(\nu)$ is complete, standard Hilbert space arguments establish for each μ in $\mathfrak{M}$ the existence of a unique measure μ' in $\mathfrak{M}(\nu)$ such that $\| \mu - \mu' \|_u$ is minimal. We denote this balayaged measure by $\pi_u \mu$ and we characterize it in the following lemma. The proof given is essentially that of Cartan [3].

LEMMA 3.5. (i) *$G_u \pi_u \mu$ is the E_u-orthogonal projection of $G_u \mu$ onto $\mathbf{F}_u(\nu)$*
(*ii*) $G_u \mu \geq G_u \pi_u \mu$ q.e. *and* $G_u \mu = G_u \pi_u \mu$ q.e. *on* M.
(iii) $G_u \pi_u \mu =$ infimum $\{f$ *in* $G_u \mathfrak{M} : f \geq G_u \mu$ q.e. *on* $M\}$ q.e.
(iv) *$G_u \pi_u \mu$ is the unique element of minimal E_u norm among the set of f in $\mathbf{F}$ such that $f \geq G_u \mu$* q.e. *on* M.

Proof. From the relation $\| \mu - \pi_u \nu \|_u \leq \| \mu - \lambda \|_u$ for all λ in $\mathfrak{M}(\nu)$ follows

$$E_u(G_u \mu - G_u \pi_u \mu, G_u \lambda - G_u \pi_u \mu) \leq 0$$

for all λ in $\mathfrak{M}(\nu)$. Taking $\lambda = 0$ and $\lambda = 2\pi_u \mu$, we conclude that

$$E_u(G_u \mu - G_u \pi_u \mu, G_u \pi_u \mu) = 0$$

and therefore

$$E_u(G_u \mu - G_u \pi_u \mu, G_u \lambda) \leq 0 \tag{3.5}$$

for all λ in $\mathfrak{M}(\nu)$. The inequality (3.5) implies that $G_u \mu \leq G_u \pi_u \mu$ q.e. on M and then the previous relation implies that $G_u \mu = G_u \pi_u \mu$ [a.e. $\pi_u \mu$] and therefore by the maximum principle (Theorem 1.13), $G_u \pi_u \mu \leq G_u \mu$ q.e., proving (ii). This together with the inequality (3.5) yields the equality

$$E_u(G_u \mu - G_u \pi_u \mu, G_u \lambda) = 0 \tag{3.6}$$

for λ in $\mathfrak{M}(\nu)$ which is equivalent to (1). Conclusion (iii) follows from the maximum principle. Finally (iv) is easily established for $f \geq 0$ using (1.5), which suffices because of the contractivity of E_u.

At this point it is convenient to introduce the complement $D = \mathbf{X} - M$ and the corresponding "killed" process

$$X^D(t) = \begin{cases} X(t) & \text{for } t < B(\nu; 0) \\ \partial & \text{for } t \geq B(\nu; 0). \end{cases}$$

We will be interested primarily in the corresponding resolvent operators

$$G_u^D f(x) = \mathcal{E}_x \int_0^{B(\nu;0)} dt e^{-ut} f(t) \tag{3.7}$$

and in particular, their connection with the operators

$$(3.8) \qquad H_u f(x) = \mathcal{E}_x \exp\{-uB(\nu; 0)\} f[B(\nu; 0)].$$

We note in particular the familiar and easily established identities

$$(3.9) \qquad G_u = G_u^D + H_u G_u$$

$$(3.10) \qquad H_u = H_v + (v - u) G_u^D H_v .$$

Basic results for these operators are collected in

THEOREM 3.6. (i) *If f belongs to* $\mathbf{F}$, *then $H_u f$ is the quasi-continuous version of the E_u-orthogonal projection of f onto the linear subspace* $\mathbf{F}_u$.

(ii) *For μ in $\mathfrak{M}$ and for $f \geq 0$ on M,*

$$\int \pi_u \mu\,(dy) f(y) = \int \mu\,(dx) H_u f(x).$$

(iii) *The operators G_u^D, $u > 0$ form a symmetric resolvent on $L^2(D, dx)$. The corresponding Dirichlet space $(\mathbf{F}^D, E^D)$ is given by*

$$\mathbf{F}^D = \{f \textit{ in } \mathbf{F} : f = 0 \text{ q.e. } \textit{on } M\}$$

$$E^D(f, g) = E(f, g), \quad f, g \textit{ in } \mathbf{F}^D.$$

Moreover $\mathbf{F}^D$ is precisely the E_u-orthogonal complement of $\mathbf{F}_u$ in $\mathbf{F}$.

(iv) *Let $\mu = r(x)\,dx$ with $r(x) > 0$* [a.e. *dx*] *on D and with $r(x)$ square integrable on D. If $\Gamma \subset M$ is null for $\pi_1 \mu$, then $H_u(x, \Gamma) = 0$ for* q.e. *x in D. Also $\pi_u \lambda$ is absolutely continuous relative to $\pi_1 \mu$ for all λ in $\mathfrak{M}$.*

Proof. We begin by proving (i), after first noting that by elementary approximation arguments it suffices to consider the special case $f = G_u \mu$ with μ in $\mathfrak{M}$. The estimate

$$(3.11) \qquad H_u f \leq f \quad \text{q.e.}$$

can be established by a straight forward calculation when $\mu = g(x)\,dx$ and follows in general by the usual approximation. A similar argument, using Lemma 2.8, shows that $H_u f$ is a u-potential. Since $f = H_u f$ q.e. on M, clearly $f - H_u f$ belongs to the E_u-orthogonal complement of $\mathbf{F}_u$. (Apply (1.5).) Now $f = G_u \pi_u \mu$ q.e. on M and so $H_u f = H_u G_u \pi_u \mu$. Therefore the estimate (3.11) for $G_u \pi_u \mu$ in place of f implies that $G_u \pi_u \mu \geq H_u f$ q.e. Then

$$E_u(G_u \pi_u \mu, G_u \pi_u \mu) \geq E_u(H_u f, H_u f)$$

(using (1.5)) and since $G_u \pi_u \mu$ is already known to be the E_u-orthogonal projection of f onto F_u it follows that actually $H_u f = G_u \pi_u \mu$ and (i) is proved. Conclusion (ii) is an immediate consequence of (i). We turn now to (iii). The resolvent identities can be established by a straight forward computation which we omit. Symmetry follows from symmetry for G_u, from (3.9) and from the following version of Hunt's duality relation [16, p. 168],

$$\int \lambda\,(dx)H_u G_u \mu(x) = \int \mu\,(dx)H_u G_u \lambda(x), \tag{3.12}$$

which is an immediate consequence of (i). Finally, the statements about the corresponding Dirichlet space are easily established with the help of (3.9). Before going on to prove (iv) we note that (ii) applied to (3.10) yields the dual identity

$$\pi_u = \pi_v + (v - u)\pi_v G_u^D \tag{3.13}$$

where G_u^D is defined on measures in the usual way. It follows in particular that $\pi_1 \mu$ and $\pi_u \mu$ are equivalent measures. Now let $\Gamma \subset M$ be null for $\pi_1 \mu$ and therefore for $\pi_u \mu$. From (ii) it follows that $H_u(x, \Gamma) = 0$ for [a.e. dx] x in D. Then $vG_{u+n} H_u 1_\Gamma = 0$ q.e. and therefore $vG_{u+v}^D H_u 1_\Gamma = 0$ q.e. But from (3.9) it follows that $vG_{u+v}^D H_u 1_\Gamma$ increases to $H_u 1_\Gamma$ q.e. on D as $v \uparrow \infty$ and we conclude that $H_u(x, \Gamma) = 0$ for q.e. x in D as required. The remainder of (iv) follows easily with the help of (ii).

Next we establish an important alternate description for $\mathfrak{M}(\nu)$.

LEMMA 3.7. *Let ν belong to $\mathfrak{M}^e$. Then $\mathfrak{M}(\nu)$ is the $\| \ \|_1$ closure in $\mathfrak{M}$ of the set of measures μ in $\mathfrak{M}$ which are absolutely continuous with respect to ν.*

Proof. We show first that ν is concentrated on M. We have seen in the proof of Lemma 3.4 that h_1 agrees q.e. on every compact set with a 1-potential and so Lemma 2.7 implies in particular that quasi-surely $h_1(t)$ is everywhere right continuous. It follows easily upon approximating.

$$\int_0^\infty A(\nu; dt)e^{-t}h_1(t)$$

by sums

$$\sum_{k=0}^\infty \int_{k/n}^{(k+1)/n} A(\nu; dt)e^{-t}h_1[(k + 1)/n]$$

and applying the simple Markov property that for q.e. x

$$\mathcal{E}_x \int_0^\infty A(\nu; dt)e^{-t}h_1(t) = \mathcal{E}_x \int_0^\infty A(\nu; dt)e^{-t} \exp\left(- \{\theta(t)B(\nu; 0) + t\}\right)$$

But it is easy to see that quasi-surely the subset of t where $\theta(t)B(\nu; 0) > t$ is null relative to the t-measure $A(\nu; dt)$. Thus the last integral equals $G_1\nu(x)$ and it follows from Theorem 3.3 and from Lemma 1.19 that ν is concentrated on M. To complete the proof it suffices to show that if f in $\mathbf{F}$ is not E_1-orthogonal to $\mathbf{F}_1$, then it is false that $f = 0$ [a.e. ν]. But the hypothesis on f implies that $\int \mu(dx)H_1 |f| (x) > 0$ for some μ in $\mathfrak{M}$ and the right continuity of $f(t)$ (Lemma 2.7) implies that

$$\mathcal{E}_\mu \int_0^\infty A(\nu; dt) e^{-t} |f|(t) > 0$$

which by Theorem 3.3 implies in particular that $f \cdot \nu \neq 0$.

Similar arguments establish

LEMMA 3.8. *If ν belongs to $\mathfrak{M}^e$, then*

$$\sigma(M) = B(\nu; 0) \quad \text{q.s. } on\ \Omega.$$

In the remainder of this section we establish a characterization of generalized potential which will permit us to extend the domain of the balayage operators π_u to $\mathfrak{M}^e$. As a preliminary step we introduce for A closed in $\mathbf{X}$ the subset $\mathfrak{M}(A)$ of measure μ in $\mathfrak{M}$ which are concentrated on A. Clearly $\mathfrak{M}(A)$ is convex and by Proposition 1.4 it is closed relative to any of the norms $\| \ \|_u$. For μ in $\mathfrak{M}$ we denote by $\pi_u^A \mu$ the unique member of $\mathfrak{M}(A)$ such that $\| \mu - \pi_u^A u \|_u$ is minimal and we denote by $\mathbf{F}_u(A)$ the linear subspace of $\mathbf{F}$ spanned by $G_u \mathfrak{M}(A)$. The analogues of Lemma 3.5 and Theorem 3.6 can be established by exactly the same arguments—the role of M being played of course by A itself and the role of $B(\nu; 0)$ by $\sigma(A)$. (It can be shown that actually we are dealing only with a special case.) We also introduce the special notation

$$H_u^A f(x) = \mathcal{E}_x \operatorname{esp}\{-u\sigma(A)\} f[\sigma(A)]. \tag{3.14}$$

Now fix an increasing sequence of open X_n each having compact support and such that $\mathbf{X} = \bigcup_{n=1}^\infty X_n$ and put $A_n = \mathbf{X} - X_n$. We begin with

LEMMA 3.9. *Fix $u > 0$ and f in $\mathbf{F}$. Then as $n \uparrow \infty$,*

(i) $\exp\{-u\sigma(A_n)\} f[\sigma(A_n)] \to 0$ *quasi-surely on* Ω,

(ii) $H_u^{A_n} f \to 0$ *strongly in* $\mathbf{F}$,

(iii) $H_u^{A_n} f \to 0$ q.e. *whenever f is a u-potential.*

Proof. By quasi-left-continuity, either $\sigma[A_n] \uparrow +\infty$ or $X[\sigma(A_n)] \to \partial$ quasi-surely on Ω and (i) follows from Lemma 2.7. The operators $H_u^{A_n}$ form (Theorem 3.6) a decreasing family of E_u-orthogonal projections on $\mathbf{F}$. If f is a u-potential, then

$$\operatorname{Lim} H_u^{A_n} f = G_u \lambda$$

exists strongly in $\mathbf{F}$ with λ concentrated on $\bigcap A_n$ which implies that λ and therefore $f = 0$. The proof of (ii) and (iii) is now routine.

A simple application of the dominated convergence theorem for infinite series extends Lemma 3.9 (iii) to

LEMMA 3.10. *If h is a generalized u-potential, then $H_u^{A_n} h \downarrow 0$ q.e. as $n \to \infty$.*

Now we are ready to characterize generalized u-potentials.

THEOREM 3.11. *Let Borel h on $\mathbf{X}$ be nonnegative and finite* q.e. *The following conditions are sufficient for h to be a generalized u-potential.*

3.11.1. *For each compact subset K of $\mathbf{X}$ exists a u-potential h_K such that $h_K = h$* q.e. *on K.*
3.11.2. $H_u^{A_n} h \to 0$ q.e. *as $n \uparrow \infty$.*

Proof. Let h_n be the u-potential of 3.11.1 with $K = \text{cl}\,(X_n)$. Put $h_n = G_u\,\mu_n$ and denote by $\mu_n^{(m)}$ the restriction of μ_n to cl (X_m). The estimate

$$\begin{aligned}\| \mu_n^{(m)} \|_u^2 &\leq \int_{X_m} \mu_n(dx) G_u\,\mu_n(x) \\ &\leq \int_{X_m} \mu_n(dx) h_m(x) \\ &\leq \int \mu_n(dx) H_u^{\text{cl}(X_m)} h_m\,(x) \\ &= E_u(H_u^{\text{cl}(X_m)} G_u\,\mu_n\,, h_m) \\ &\leq E_u(h_m\,, h_m)\end{aligned}$$

shows that for each m the norm $\| \mu_n^{(m)} \|_u$ is bounded independent of n. Applying Proposition 1.4 and replacing μ_n by an appropriate subsequence, we can assume on the one hand that $\mu_n \to \mu$ vaguely where μ is a Radon measure on $\mathbf{X}$ and on the other hand that for each m, the potentials $G_u\,\mu_h^{(m)} \to G_u\,\mu^{(m)}$ weakly in $\mathbf{F}$. For $f \geq 0$ in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$,

$$\begin{aligned}\int dx\ h(x)\ f(x) &= \text{Lim} \int dx\, f(x) G_u\,\mu_n(x) \\ &= \text{Lim}\left\{\int_{\text{cl}(X_m)} \mu_n(dx) G_u f(x) + \int_{X - \text{cl}(X_m)} \mu_n(dx) G_u f(x)\right\}\end{aligned}$$

with the limit taken as $n \uparrow \infty$. For fixed m, the first term inside the bracket converges to

$$\int_{\text{cl}(X_m)} \mu(dx) G_u f(x)$$

as $n \uparrow \infty$. The second term is

$$\leq \int \mu_n(dx) H_u^{A_m} G_u f(x) = E_u(G_u f, H_u^{A_m} G_u\,\mu_n) \leq \int dx\, f(x) H_u^{A_m} h(x)$$

which $\to 0$ independent of n as $m \uparrow \infty$. It follows easily that h is a generalized u-potential and that $h = G_u\,\mu$.

We remark that our proof of Theorem 3.11 is a simple variant of the one given in [2, p. 268].

Now it is a simple matter to extend the domain of definition of the balayage operators π_u and establish

PROPOSITION 3.12. *Let μ belong to $\mathfrak{M}^e$ and let $u > 0$. Then there exists a unique measure $\pi_u \mu$ in $\mathfrak{M}^e$ satisfying*

$$H_u G_u \mu = G_u \pi_u \mu.$$

Moreover $\pi_u \mu$ is concentrated on M and

$$\int \mu(dx) H_u G_u \lambda(x) = \int \pi_u \mu(dx) H_u G_u \lambda(x)$$

$$= \int \pi_u \mu(dx) G_u \lambda(x)$$

for λ in $\mathfrak{M}^e$ (with the understanding that these quantities may be infinite).

4. Invariance under time reversal

We begin by extablishing a kind of universality property for the u-equilibrium distributions ν_u^K. These are defined for $u > 0$ and for K compact by

$$H_u^K 1 = G_u \nu_u^K. \tag{4.1}$$

We will make continual use in this section of the exponential holding times R_u, defined in the same way as R_1 except that R_u has density ue^{-ul}.

THEOREM 4.1. *Let ξ be a nonnegative Borel function on Ω and let μ in $\mathfrak{M}^e$ be supported by K compact. Then*

$$\mathcal{E}_\mu \xi = \mathcal{E}_{\nu_u K} \int_0^{R_u} A(\mu; dt)\theta(t)\xi.$$

Proof. It suffices to observe that by Theorem 3.3 the right side

$$= \int \nu_u^K(dx) G_u(\mathcal{E}_\cdot(\xi)\cdot\mu)(x) = \int \mu(dx)\mathcal{E}_x(\xi) G_u \nu_u^K(x)$$

and to apply (4.1).

Next define the last exit time

$$r_u^K = \sup\{t > 0 : t < R_u \text{ and } X(t-0) \text{ is in } K\}$$

with the understanding that $r_u^K = 0$ when not otherwise defined. It is easy to check that r_u^K is Borel measurable (relative to the obvious augmented sigma algebra) but not a stopping time. We will be concerned with the truncated trajectory

$$\omega_u^K(t) = \omega(t) \quad \text{for } 0 \le t < r_u^K$$

$$= \partial \quad \text{for } t \ge r_u^K$$

and the time reversed trajectory

$$J\omega_u^K(t) = \omega(r_u^K - t) \quad \text{for } 0 \le t < r_u^K$$
$$= \partial \quad \text{for } t \ge r_u^K$$

Our result on invariance under time reversal is

THEOREM 4.2. *Let ξ be a nonnegative Borel function on Ω and let ν_u^K be the u-equillibrium distribution for K compact in* **X**. *Then*

$$\int \mathcal{P}_{\nu_u K}(d\omega)\xi(\omega_u^K) = \int \mathcal{P}_{\nu_u K}(d\omega)\xi(J\omega_u^K). \tag{4.2}$$

Proof. By the usual passage to the limit, it suffices to consider

$$\xi = f_0(0)f_1(t_1) \cdots f_n(t_n)$$

with $f_i \ge 0$ bounded and belonging to **F**. In the following computation we use -0 to denote left hand limits and we apply Lemma 2.7 when appropriate without comment. The right side of (4.2)

$$= \mathcal{E}_{\nu_u K} I[r_u^K > t_n] f_n(r_u^K - t_n - 0) \cdots f_0(r_u^K - 0)$$

$$= \operatorname{Lim} \sum_{k=0}^{\infty} \mathcal{E}_{\nu_n K} I[t_n + (k/m) < r_u^K \le t_n + (k+1)m)]$$
$$\times f_n(k/m) \cdots f_0(t_n + (k/m))$$

$$= \operatorname{Lim} \sum_{k=0}^{\infty} \mathcal{E}_{\nu_u K} f_n(k/m) \cdots f_0(t_n + (k/m))$$
$$\times \exp\{-ut_n - (uk/m)\}\{H_u^K 1[t_n + (k/m)]$$
$$- e^{-u/m} H_u^K 1[t_n + (k+1)/m]\}$$

$$= \operatorname{Lim} \sum_{k=0}^{\infty} \int \nu_u^K(dx) e^{-ut_n} e^{-uk/m} P^{k/m} f_n P^{t_n - t_{n-1}} f_{n-1} \cdots$$
$$\times P^{t_1} f_0 \{H_u^K 1 - e^{-u/m} P^{1/m} H_u^K 1\}(x)$$

$$= \operatorname{Lim} \sum_{k=0}^{\infty} \int \nu_u^K(dx) G_u\{1 - e^{-u/m} P^{1/m}\} f_0 P^{t_1} f_1 \cdots$$
$$\times P^{t_n - t_{n-1}} f_n P^{k/m} \nu_u^K(x) e^{-ut_n} e^{-uk/m}$$

$$= \operatorname{Lim} \int \nu_u^K(dx) m \int_0^{1/m} dt\, e^{-ut} P^t f_0 \cdots P^{t_n - t_{n-1}} f_n e^{-ut_n}$$
$$\times \left\{ \sum_{k=0}^{\infty} (1/m) e^{-uk/m} P^{k/m} \nu_u^K \right\}(x)$$

$$= \int \nu_u^K(dx) f_0(x) P^{t_1} f_1 \cdots P^{t_n - t_{n-1}} f_n H_u^K 1(x) e^{-ut_n}$$

which is equal to the left side of (4.2).

5. The time changed process

Fix ν in $\mathfrak{M}^e$ and let $M = M^\nu$ as in Section 3. The time changed process itself is defined by

$$X^\nu(t) = X[B(\nu; t)].$$

However we will focus our attention on the corresponding resolvent operators

$$R_a \varphi(x) = \mathcal{E}_x \int_0^\infty dt\, e^{-at} \varphi[X^\nu(t)]$$

and the associated Dirichlet space. A simple change of variable establishes the formula

$$R_a \varphi(x) = \mathcal{E}_x \int_0^\infty A(\nu; dt) \exp\{-aA(\nu; t)\} \varphi(t).$$

Important tools for us will be the modified resolvent operators

$$R_{(u)a} \varphi(x) = \mathcal{E}_x \int_0^{R_u} A(\nu; dt) \exp\{-aA(\nu; t)\} \varphi(t).$$

We will generally regard the operators R_a and $R_{(u)a}$ as acting on functions defined q.e. on M. We begin with

LEMMA 5.1. *The family* $\{R_a, a > 0\}$ *and also for* $u > 0$ *the family* $\{R_{(u)a}, a > 0\}$ *is a symmetric resolvent relative to* $L^2(\nu)$.

Proof. It suffices to consider $R_{(u)a}$ since the analagous results for R_a can be established by passing to the limit $u \downarrow 0$. A simple computation gives

$$aR_{(u)a} 1 \leq 1. \tag{5.1}$$

To establish symmetry, assume first that ν is supported by a fixed compact set K and let ν_u^K be the u-equilibrium distribution for K. Then by Theorem 4.1, for φ, ψ nonnegative and defined quasi-everywhere on M,

$$\int \nu(dx) \psi(x) R_{(u)a} \varphi(x)$$

$$= \mathcal{E}_{\nu_u^K} \int_0^{R_u} A(\nu; ds) \int_0^{R_u} A(\nu; dt) \exp\{-a[A(\nu; t) - A(\nu; s)]\} \psi(s) \varphi(t)$$

and careful application of Theorem 4.2 yields the desired symmetry for this case. In general let ν_n be the restriction of ν to compact sets K_n which increase to $\mathbf{X}$ and pass to the limit for φ, ψ bounded and integrable relative to ν which is sufficient. Of course symmetry together with (5.1) implies that

$aR_{(u)a}$ is a contraction on $L^2(\nu)$ and it only remains to establish the resolvent identity. But this follows from the relation

$$\int_0^{R_u} A(\nu; dt)[e^{-aA(\nu;t)} - e^{-bA(\nu;t)}]\varphi(t)$$

$$= (b - a) \int_0^{R_u} A(\nu; ds)e^{-aA(\nu;s)} \int_s^{R_u} A(\nu; dt)e^{-bA(\nu;t)+bA(\nu;s)}\varphi(t)$$

upon applying Theorem 3.2 and approximating integrals by sums in an obvious way.

Remark. Up to this point we have discussed resolvents and Dirichlet spaces only when the underlying set is a separable locally compact Hausdorff space. This need not be the case with M, but the extension of relevant definitions and results is straight forward, and we take it for granted.

The corresponding Dirichlet spaces relative to $L^2(\nu)$ will be denoted by $(\mathbf{H}_{(0)}, Q)$ and by $(\mathbf{H}_{(u)}, Q_{(u)})$. The generators will be denoted by B and $B_{(u)}$. We will see eventually that $\mathbf{H}_{(u)}$ is actually independent of $u > 0$ at which point we will use the more suggestive notation $\mathbf{H}_{(+)}$. We impose the following restrictions which will be removed in Section 8.

(5.2.1) $$G_1\nu \leq c$$

(5.2.2) $$\int \nu(dx) \leq c$$

(5.2.3) $$\|\nu\|_1 \leq c$$

for some fixed constant c.

For $\varphi \geq 0$ on M define

$$R_{(u)}\varphi = \text{Lim}\, R_{(u)a}\varphi \quad (a \downarrow 0)$$

and note the obvious relation

(5.3) $$R_{(u)}\varphi = \gamma G_u(\varphi\cdot\nu) \quad [\text{a.e. } \nu].$$

In (5.3) *we use* γ *to denote restriction to* M. From now on we define $R_{(u)}\varphi$ up to quasi-equivalence on M by means of (5.3). From (5.2.1) it follows that $R_{(u)}$ is bounded in the uniform norm, and, since it is symmetric relative to ν, also in the $L^1(\nu)$ and $L^2(\nu)$ norms. A routine argument shows that $R_{(u)}$ is exactly the inverse to the generator $-B_{(u)}$ corresponding to the resolvent operators $R_{(u)a}$. Therefore if φ in $L^2(\nu)$ belongs to the domain of $B_{(u)}$, then there exists ψ in $L^2(\nu)$ such that $\varphi = R_{(u)}\psi$ and from (5.3) it follows that $H_u\varphi = H_u G_u(\psi\cdot\nu)$. Since both φ and ψ belong to $L^2(\nu)$, we have

$$\int \nu(dx)\varphi(x)\psi(x) < +\infty$$

and it follows easily that $H_u\varphi$ is in $\mathbf{F}$ and that

(5.4) $$Q_{(u)}(\varphi, \varphi) = E_u(H_u\varphi, H_u\varphi), \quad u > 0.$$

Approximating general φ in $\mathbf{H}_{(u)}$ by φ_n in the domain of $B_{(u)}$, we conclude that $\mathbf{H}_{(u)}$ is naturally contained in $\gamma\mathbf{F} \cap L^2(\nu)$ and that (5.4) is valid for φ in $\mathbf{H}_{(u)}$. To go the other way consider $\varphi \geq 0$ belonging to $L^2(\nu) \cap \gamma\mathbf{F}$. To show that φ belongs to $\mathbf{H}_{(u)}$ it suffices to find for any specified $\varepsilon > 0$, a function ψ in $L^2(\nu)$ satisfying

$$(5.5)\qquad \begin{aligned} &\int \nu(dx) \mid \varphi(x) - R_{(u)}\psi(x) \mid^2 < \varepsilon \\ &E_u(H_u\varphi - G_u(\psi\cdot\nu), H_u\varphi - G_u(\psi\cdot\nu)) < \varepsilon. \end{aligned}$$

Since $\varphi_n = \min(\varphi, n)$ belongs to $L^2(\nu) \cap \gamma\mathbf{F}$ and since $\varphi_n \to \varphi$ in $L^2(\nu)$ and $H_u\varphi_n \to H_u\varphi$ strongly in $\mathbf{F}$, we may assume that φ is bounded. For an appropriate sequence of $v \uparrow \infty$ the functions

$$vG_vH_u\varphi \to H_u\varphi \quad \text{strongly in } \mathbf{F}$$

and quasi-everywhere and therefore $\gamma vG_vH_u\varphi \to \varphi$ in $L^2(\nu)$. Therefore we may assume $H_u\varphi = G_u\lambda$ with λ in $\mathfrak{M}(\nu)$. From Lemma 3.7 it follows easily that there exist $\psi_n \geq 0$ in $L^2(\nu)$ such that $\psi_n\cdot\nu \to \lambda$ relative to $\| \quad \|_u$. Since $\theta\cdot\nu$ belongs to $\mathfrak{M}$ whenever $\theta \geq 0$ belongs to $L^2(\nu)$, it follows that

$$\gamma G_u(\psi_n\cdot\nu) \to \gamma G_u\lambda \text{ weakly in } L^2(\nu)$$

and so after replacing the ψ_n by a subsequence and then by Cesáro means [24, p. 80], we can conclude that $R_{(u)}\psi_n \to \varphi$ in $L^2(\nu)$. This establishes (5.5) and we have proved

LEMMA 5.2. *Assume the restrictions* (5.2). *Then*

$$\mathbf{H}_{(u)} = \gamma\mathbf{F} \cap L^2(\nu)$$

and (5.4) *is valid for* φ *in* $\mathbf{H}_{(u)}$.

Lemma 5.2 justifies writing $\mathbf{H}_{(+)}$ in place of $\mathbf{H}_{(u)}$ for $u > 0$ and we do so from now on. Next we show that it is possible to identify $\mathfrak{M}(\nu)$ with the measures having finite 0-energy relative to $(\mathbf{H}_{(+)}, Q_{(u)})$. Consider φ in $\mathbf{H}_{(+)}$ such that $Q_{(u)}(\varphi, \psi) \geq 0$ whenever ψ in $\mathbf{H}_{(+)}$ is ≥ 0 q.e. on M. Then

$$E_u(H_u\varphi, f) = E_u(H_u\varphi, H_uf) = Q_{(u)}(\varphi, \gamma f) \geq 0$$

whenever f in $\mathbf{F} \cap L^2(\nu)$ is ≥ 0 and if follows that there exists a unique λ in $\mathfrak{M}(\nu)$ such that $H_u\varphi = G_u\lambda$. But then

$$(5.6)\qquad Q_{(u)}(\varphi, \psi) = \int \lambda(dx)\psi(x)$$

for ψ in $\mathbf{H}_{(+)}$. Conversely fix λ in $\mathfrak{M}(\nu)$ and note that since for $\psi \geq 0$ in $\mathbf{H}_{(+)}$,

$$\int \lambda(dx)\psi(x) = E_u(G_u\lambda, H_u\psi) \leq c\{Q_{(u)}(\psi, \psi)\}^{1/2}$$

and since $\mathbf{H}_{(+)}$ is a Hilbert space relative to $Q_{(u)}$ itself (since $R_{(u)}$ is bounded), there exists a unique φ in $\mathbf{H}_{(+)}$ such that (5.6) is valid for general ψ in $\mathbf{H}_{(+)}$

and then $H_u \varphi = G_u \lambda$ (because of Lemma 3.7). We have proved

LEMMA 5.3. *Assume the restrictions* (5.2).

(i) *If φ in $\mathbf{H}_{(+)}$ satisfies $Q_{(u)}(\varphi, \psi) \geq 0$ whenever ψ in $\mathbf{H}_{(+)}$ is ≥ 0 q.e. on M, then there exists a unique λ in $\mathfrak{M}(\nu)$ such that*

$$\varphi = \gamma G_u \lambda, \qquad \int \lambda(dx)\psi(x) = Q_{(u)}(\varphi, \psi)$$

for ψ in $\mathbf{H}_{(+)}$. In this case we write $\varphi = R_{(u)} \lambda$.

(ii) *Conversely to each λ in $\mathfrak{M}(\nu)$ there corresponds a unique φ in $\mathbf{H}_{(+)}$ such that $\varphi = R_{(u)} \lambda$.*

LEMMA 5.3 extends (5.3) to

$$R_{(u)} \lambda = \gamma G_u \lambda \tag{5.7}$$

valid for λ in $\mathfrak{M}(\nu)$. More generally, we define $R_{(u)} \lambda$ by (5.7) for λ in $\mathfrak{M}^e$ concentrated on M. For $a > 0$ define $R_{(u)a} \lambda$ by

$$Q_{(u)a}(R_{(u)a} \lambda, \psi) = \int \lambda(dx)\psi(x)$$

valid for ψ in $\mathbf{H}_{(+)}$ when λ belongs to $\mathfrak{M}(\nu)$ and extend this definition by passage to the limit for λ in $\mathfrak{M}^e$ concentrated on M.

Next we note that for $0 < v < u$ and for φ, ψ in $\mathbf{H}_{(+)}$,

$$\begin{aligned} Q_{(u)}(\varphi, \psi) &= E_u(H_u \varphi, H_u \psi) \\ &= E_v(H_u \varphi, H_v \psi) + (u - v) \int dx H_u \varphi(x) H_v \psi(x) \\ &= Q_{(v)}(\varphi, \psi) + (u - v) \int dx H_u \varphi(x) H_v \psi(x) \end{aligned} \tag{5.8}$$

where we have used the fact that $H_u \varphi - H_v \varphi$, since it vanishes quasi-everywhere on M, is E_u-orthogonal to $\mathbf{F}_u$ and E_v-orthogonal to $\mathbf{F}_v$. This motivates our studying the symmetric bilinear form

$$U_{v,u}(\varphi, \psi) = (u - v) \int dx H_u \varphi(x) H_v \psi(x)$$

defined for φ, ψ in $\mathbf{H}_{(+)}$ and for $0 < v < u$. We can also write

$$U_{v,u}(\varphi, \psi) = (u - v) \int \pi_v H_u \varphi\,(dy)\psi(y)$$

where π_v is the balayage operator of Section 3. For typographical convenience we have introduced the special notation

$$\pi_v \varphi = \pi_v(\varphi \cdot dx).$$

From the computation (5.8) and from Lemma 5.3 it follows (since $\mathbf{H}_{(+)}$ is a Hilbert space relative to both $Q_{(u)}$ and $Q_{(v)}$) that the measure $\pi_v H_u \varphi$ belongs to $M(\nu)$ for φ in $\mathbf{H}_{(+)}$. From now on we write simply

$$U_{v,u}\varphi = (u - v)\pi_v H_u \varphi \tag{5.9}$$

for φ in $\mathbf{H}_{(+)}$. Now for φ, ψ in $L^2(\nu)$ and for $a > 0$,

$$\begin{aligned}\int \nu\,(dx)\psi(x)R_{(v)a}\varphi(x) &= \int \nu\,(dx)\varphi(x)R_{(v)a}\psi(x)\\ &= Q_{(u)a}(R_{(u)a}\varphi, R_{(v)a}\psi)\\ &= Q_{(v)a}(R_{(u)a}\varphi, R_{(v)a}\psi) + U_{v,u}(R_{(u)a}\varphi, R_{(v)a}\psi)\\ &= \int \nu\,(dx)\psi(x)R_{(u)a}\varphi(x) + U_{v,u}(R_{(u)a}\varphi, R_{(v)a}\psi)\end{aligned} \tag{5.10}$$

and it follows that

$$R_{(v)a}\varphi = R_{(u)a}\varphi + R_{(u)a}U_{v,u}R_{(v)a}\varphi \tag{5.11}$$

in the notation of (5.9) and of Lemma 5.3.

Next we apply (3.10) to conclude that

$$U_{v,u}(\varphi, \varphi) = (u - v)\int dx H_v\varphi(x)\{H_v\varphi(x) - (u - v)G_u^D H_v\varphi(x)\}$$

and therefore $U_{v,u}$ is nonnegative definite. This leads in turn to the familiar Cauchy-Schwarz inequality

$$|U_{v,u}(\varphi, \psi)| \leq \{U_{v,u}(\varphi, \varphi)U_{v,u}(\psi, \psi)\}^{1/2} \tag{5.12}$$

We also note the relationship

$$U_{v,u} = U_{v,w} + U_{w,u}, \quad 0 < v < w < u, \tag{5.13}$$

which is a direct consequence of (5.8). Now we are ready to pass to the limit $v \downarrow 0$. From (5.8) it follows that

$$U_{v,u}(\varphi, \varphi) \leq Q_{(u)}(\varphi, \varphi)$$

for φ in $\mathbf{H}_{(+)}$ independent of $0 < v < u$. Now (5.12), (5.13) and Lemma 5.3 imply that for $\varphi \geq 0$ in $\mathbf{H}(+)$,

$$U_{0,u}\varphi = \lim U_{v,u}\varphi \quad (v \downarrow 0) \tag{5.14}$$

exists and belongs to $\mathfrak{M}(\nu)$. The extension of (5.14) to general φ in $\mathbf{H}_{(+)}$ is immediate. Now passing to the limit $v \downarrow 0$ in (5.11) yields for $a > 0$ and φ in $L^2(\nu)$,

$$R_a\varphi = R_{(u)a}\varphi + R_{(u)a}U_{0,u}R_a\varphi. \tag{5.15}$$

Next consider $\varphi = R_a \psi$ with ψ in $L^2(\nu)$. Then also

$$\varphi = R_{(u)a} \psi + R_{(u)a} U_{0,u} \varphi$$

which means that φ belongs to $\mathbf{H}_{(+)}$ and that for θ in $\mathbf{H}_{(+)}$,

$$Q_{(u)a}(\varphi, \theta) = \int \nu\,(dx)\psi(x)\theta(x) + U_{0,u}(\varphi, \theta).$$

In particular

$$Q_{(u)a}(\varphi, \varphi) = Q_a(\varphi, \varphi) + U_{0,u}(\varphi, \varphi).$$

On the other hand if $\varphi \geq 0$ belongs to $\mathbf{H}_{(+)}$ then

$$a \int \nu\,(dx)\{\varphi(x) - aR_a \varphi(x)\}\varphi(x) \leq a \int \nu\,(dx)\{\varphi(x) - aR_{(u)a}\varphi(x)\}\varphi(x)$$

which implies that φ belongs to $\mathbf{H}_{(0)}$. (See Proposition 1.1.). These results plus an approximation of φ in $\mathbf{H}_{(0)} \cap \gamma\mathbf{F}$ by φ_n in the domain of B prove

PROPOSITION 5.4. *Assume the restriction* (5.2).

(i) *The domain of* B *is contained in* $\mathbf{H}_{(+)}$.

(ii) $\mathbf{H}_{(+)} = \mathbf{H}_{(0)} \cap \gamma\mathbf{F}$.

(iii) *For* $u > 0$ *and* φ *in* $\mathbf{H}_{(+)}$,

$$Q_{(u)}(\varphi, \varphi) = Q(\varphi, \varphi) + U_{0,u}(\varphi, \varphi).$$

Before continuing, we note a simple but important consequence of Lemma 5.2 and of Proposition 5.4 (iii).

LEMMA 5.5. *If* f *belongs to* $\mathbf{F}$, *then*

$$u \int dx H_0 f(x) H_u f(x)$$

converges absolutely for all $u > 0$ *and converges to* 0 *as* $u \downarrow 0$.

Proof. It suffices to observe that it is always possible to choose ν such that f belongs to $L^2(\nu)$.

We consider now the functions

$$\text{(5.16)} \qquad \begin{aligned} p(x) &= \mathcal{P}_x(B(\nu; 0) = \zeta = +\infty) \\ q(x) &= \mathcal{P}_x(B(\nu; 0) = +\infty \text{ and } \zeta < +\infty). \end{aligned}$$

Recall that ζ is the lifetime of the process. The function p is the "passive part" of 1 in the sense of Feller [9]. The sum $q + H_0 1$ is the "active part" and of course

$$\text{(5.17)} \qquad 1 = H_0 1 + p + q.$$

We note first that for $0 < v < u$ and for $f \geq 0$ in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$,

$$
\begin{aligned}
&\int u\pi_u q\,(dy)f(y) \\
&\quad = \int dx q(x) u H_u f(x) \\
&\quad = \int dx q(x) u\{H_v f(x) - (u - v) G_u^D H_v f(x)\} \\
(5.18)\qquad &\quad = \int v\pi_v q\,(dy)f(y) + (u - v)\int dx q(x) H_v f(x) \\
&\qquad\qquad - u(u - v)\int dx q(x) G_u^D H_v f(x) \\
&\quad = \int v\pi_v q\,(dy)f(y) + (u - v)\int dx\{q(x) - uG_u^D q(x)\} H_v f(x).
\end{aligned}
$$

An elementary computation shows that $uG_u^D q \leq q$ and we conclude that

$$(5.19)\qquad u\pi_u q \geq v\pi_v q \quad \text{for } u > v.$$

Similarly $uG_u^D p = p$ and the computation (5.18) for p instead of q shows that

$$(5.20)\qquad u\pi_u p = \pi_1 p \quad \text{for } u > 0.$$

Note that by Proposition 3.12 we can be sure $\pi_u p$ and $\pi_u q$ belong to $\mathfrak{M}^e$, but not necessarily to $\mathfrak{M}$.

For $g \geq 0$ on $\mathbf{X}$ we have

$$H_u G_u g = H_u G_u \pi_u g = H_u R_{(u)} \pi_u g$$

and so (3.9) leads directly to

$$(5.21)\qquad G_u g = G_u^D g + H_u R_{(u)} \pi_u g$$

Applying (5.21) to the special case $g = u1$ and restricting to M we get

$$R_{(u)} u\pi_u 1 \leq 1$$

which becomes

$$(5.22)\qquad R_{(u)}\{U_{0,u} 1 + \pi_1 p + u\pi_u q\} \leq 1.$$

Now we introduce the harmonic measures $H_u(x, dy)$ on M via the formula

$$\int H_u(x, dy)\varphi(y) = H_u \varphi(x)$$

and we introduce the bilinear form

$$
\begin{aligned}
&U_{v,u}\langle\varphi, \psi\rangle \\
(5.23)\qquad &\quad = (u - v)\int dx \int H_u(x, dy)\int H_v(x, dz)[\varphi(y) - \varphi(z)][\psi(y) - \psi(z)]
\end{aligned}
$$

which, since $\pi_u 1$ is Radon, converges when φ, ψ are restrictions to M of functions in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. An elementary computation using symmetry in y and z in (5.23) establishes the relationship

$$\frac{1}{2} U_{v,u}\langle \varphi, \varphi \rangle = U_{v,u}(1, \varphi^2) - U_{v,u}(\varphi, \varphi) \tag{5.24}$$

and therefore

$$Q_{(u)}(\varphi, \varphi) = Q_{(v)}(\varphi, \varphi) + U_{v,u}(1, \varphi^2) - \frac{1}{2} U_{v,u}\langle \varphi, \varphi \rangle. \tag{5.25}$$

Now for φ as above and for $\psi = T\varphi$ with T a normalized contraction we have,

$$\begin{aligned} a \int \nu\,(dy)\{\varphi^2(y) - aR_{(v)a}\varphi^2(y)\} - a \int \nu\,(dy)\{\psi^2(y) - aR_{(v)a}\psi^2(y)\} \\ = a \int \nu\,(dy)\{\varphi^2(y) - aR_{(u)a}\varphi^2(y)\} - a \int \nu\,(dy)\{\psi^2(y) - aR_{(u)a}\psi^2(y)\} \\ - a^2 U_{v,u}(R_{(u)a}\varphi^2, R_{(v)a}1) + a^2 U_{v,u}(R_{(u)a}\psi^2, R_{(v)a}1). \end{aligned}$$

The left side equals

$$a \int \nu\,(dy)\{1 - aR_{(v)}1\}(y)\{\varphi^2 - \psi^2\}(y)$$

and therefore it is always ≥ 0. The right side equals

$$\begin{aligned} a \int \nu\,(dy)\varphi(y)\{\varphi(y) - aR_{(u)a}\varphi(y)\} \\ - a \int \nu\,(dy)\psi(y)\{\psi(y) - aR_{(u)a}\psi(y)\} \\ - \frac{1}{2} a^2 R_{(u)a}\langle \varphi, \varphi \rangle + \frac{1}{2} a^2 R_{(u)a}\langle \psi, \psi \rangle \\ - a^2 U_{v,u}(R_{(u)a}\varphi^2, R_{(v)a}1) + a^2 U_{v,u}(R_{(u)a}\psi^2, R_{(v)a}1) \end{aligned}$$

where $R_{(u)a}\langle\ ,\ \rangle$ is defined by analogy with (5.23). Since φ^2 and ψ^2 both belong to $\mathbf{H}_{(+)}$, it follows that the last two terms have limit inferior which is less then or equal to

$$-U_{v,u}(1, \varphi^2 - \psi^2)$$

as $a \uparrow \infty$. The first two terms converge (Proposition 1.1) to $Q_{(u)}(\varphi, \varphi) - Q_{(u)}(\psi, \psi)$ and since the remainder is ≤ 0 we conclude that the bilinear form

$$Q_{(u)}(\varphi, \varphi) - U_{v,u}(1, \varphi^2), \quad 0 < v < u, \tag{5.26}$$

is contractive for such φ and equivalently (by (5.24)), that the bilinear form

$$Q_{(v)}(\varphi, \varphi) - \frac{1}{2} U_{v,u}\langle \varphi, \varphi \rangle, \quad 0 < v < u, \tag{5.27}$$

is contractive for such φ. Then it follows by a passage to the limit that actually (5.23) converges absolutely for general φ, ψ in $\mathbf{H}_{(+)}$ and then it is a simple matter to extend the contractivity of (5.26) and (5.27) to general φ in $\mathbf{H}_{(+)}$. From the contractivity of (5.27) it follows in particular that

(5.28) $$U_{v,\infty}\langle\varphi, \varphi\rangle = \text{Lim}\ U_{v,u}\langle\varphi, \varphi\rangle \quad (u \uparrow \infty)$$

is a well defined bilinear form on $\mathbf{H}_{(+)}$. Passing to the limit $v \downarrow 0$ in (5.26) and $u \uparrow \infty$ in (5.27), we have

LEMMA 5.6. *Assume the restriction* (5.2). *Then for* $u > 0$ *the forms*

$$Q_{(u)}(\varphi, \varphi) - U_{0,u}(\varphi, \varphi), \qquad Q_{(u)}(\varphi, \varphi) - \frac{1}{2} U_{u,\infty}\langle\varphi, \varphi\rangle$$

are contractive on $\mathbf{H}_{(+)}$.

Next we use (5.22) to strengthen Lemma 5.6. Applying (5.8), we have for $0 < v < u$,

$$\begin{aligned} U_{v,u}(\varphi, \varphi) &- U_{v,u}(R_{(u)} U_{v,u}\varphi, \varphi) \\ &= Q_{(u)}(\varphi, \varphi) - Q_{(v)}(\varphi, \varphi) - Q_{(u)}(R_{(u)} U_{v,u}\varphi, \varphi) + Q_{(v)}(R_{(u)} U_{v,u}\varphi, \varphi) \\ &= Q_{(v)}(R_{(u)} U_{v,u}\varphi, \varphi). \end{aligned}$$

But

$$R_{(u)} U_{v,u}\varphi = \gamma(u - v)G_u H_v \varphi$$

and so by Lemma 5.2, we have $R_{(u)} U_{v,u}\varphi \to \varphi$ strongly in $\mathbf{H}_{(+)}$ as $u \uparrow \infty$ and we have established

(5.30) $$Q_{(v)}(\varphi, \varphi) = \text{Lim}\ \{U_{v,u}(\varphi, \varphi) - U_{v,u}(R_{(u)} U_{v,u}\varphi, \varphi)\}$$

with the limit taken as $u \uparrow \infty$. (Note the analogy with (1.2).) But

$$\begin{aligned} U_{v,u}(\varphi, \varphi) - U_{v,u}(R_{(u)} U_{v,u}\varphi, \varphi) &= U_{v,u}(1, \varphi^2) - \frac{1}{2} U_{v,u}\langle\varphi, \varphi\rangle \\ &\quad - (U_{v,u} R_{(u)} U_{v,u})(1, \varphi^2) + \frac{1}{2}(U_{v,u} R_{(u)} U_{v,u})\langle\varphi, \varphi\rangle. \end{aligned}$$

Of course the last term is defined by analogy with (5.23). Reasoning as in the proof of (5.26) and applying (5.22), we deduce

LEMMA 5.7. *Assume the restriction* (5.2). *Then for* $u > 0$ *the form*

$$\begin{aligned} Q_{(u)}(\varphi, \varphi) + \frac{1}{2} U_{u,\infty}\langle\varphi, \varphi\rangle - U_{0,u}(1, \varphi^2) &- \int \pi_1 p\,(dy)\varphi^2(y) \\ &- u \int \pi_u q(dy)\varphi^2(y) \end{aligned}$$

is contractive on $\mathbf{H}_{(+)}$.

Added in proof. See [29, Section 3] for more details in a special case.

From Lemma 5.7 it follows in particular that

$$U_{0,\infty}\langle\varphi, \varphi\rangle = \text{Lim}\ U_{u,\infty}\langle\varphi, \varphi\rangle \quad (u \downarrow 0)$$

is a well defined bilinear form on $\mathbf{H}_{(+)}$ and that

$$\pi_* q(dy) = \operatorname{Lim} u\pi_u q(dy) \quad (u \uparrow \infty)$$

exists as a Radon measure on M and

$$\int \pi_* q\,(dy)\varphi^2(y) < +\infty$$

for all φ in $\mathbf{H}_{(+)}$. Replacing $Q_{(u)}(\varphi, \varphi)$ by

$$Q(\varphi, \varphi) + U_{0,u}(\varphi, \varphi)$$

and then $U_{0,u}(\varphi, \varphi) - U_{0,u}(1, \varphi^2)$ by

$$-\frac{1}{2}\, U_{0,u}\langle\varphi, \varphi\rangle$$

in Lemma 5.6 and passing to the limit $u \uparrow \infty$ and then approximating φ in $\mathbf{H}_{(0)}$ by φ_n in $\mathbf{H}_{(+)}$, we deduce (assuming (5.2))

THEOREM 5.8. *The bilinear form*

$$Q(\varphi, \varphi) - \frac{1}{2}\, U_{0,\infty}\langle\varphi, \varphi\rangle - \int \pi_1 p\,(dy)\varphi^2(y) - \int \pi_* q\,(dy)\varphi^2(y)$$

is contractive on $\mathbf{H}_{(0)}$.

We will continue our analysis of the time changed process in Section 8.

6. Modified Dirichlet spaces

Throughout this section ν is a fixed Radon measure in $\mathfrak{M}^e$ satisfying the following two conditions.

6.1. $\int \nu(dx) < +\infty$.
6.2. ν is equivalent to $\pi_1 1$.

Condition 6.1 is of a purely technical nature and will be removed in Section 8. The assumption in 6.2 that ν has no piece which is singular relative to $\pi_1 1$ is natural since in any case this piece would make no contribution to the Dirichlet space on $\mathbf{X}$ which we eventually construct. Moreover in light of our final results it seems unlikely that we have lost generality by insisting that ν be equivalent rather than merely absolutely continuous.

Throughout this section we work with functions φ on M which are specified up to ν equivalence, but not necessarily up to quasi-equivalence. It follows from 6.2 and from Theorem 3.6 (iv) that $H_u\varphi$, $u \geq 0$ is well defined quasi-everywhere on D for such φ which are either bounded or nonnegative.

6.3. *Notation.* For $g \geq 0$ on $\mathbf{X}$ the balayaged measure $\pi_u g$ is absolutely continuous relative to ν. Throughout this section we use the symbol $\pi_u^\circ g$ to denote the density of $\pi_u g$ relative to ν. Thus

$$\pi_u g = (\pi_u^\circ g)\cdot\nu. \tag{6.1}$$

We begin with

6.4. DEFINITION. The universal Dirichlet form on M is the bilinear form

$$N(\varphi, \psi) = \frac{1}{2} U_{0,\infty}\langle\varphi, \psi\rangle + \int \nu\,(dy)\pi_1^\circ p(y)\varphi(y)\psi(y) + \int \nu\,(dy)\pi_*^\circ q(y)\varphi(y)\psi(y)$$

defined when it converges for φ, ψ on M (specified up to ν equivalence). The universal Dirichlet space on M is the collection $\mathbf{H}^N$ of functions φ on M such that $N(\varphi, \varphi) < +\infty$. The $L^2(\nu)$- universal Dirichlet space is the subset $\mathbf{H}^N_{(0)}$ of φ in $\mathbf{H}^N$ which also belong to $L^2(\nu)$.

Theorem 5.8 states that $\mathbf{H}_{(0)}$ is contained in $\mathbf{H}^N_{(0)}$ and that $Q - N$ is contractive on $\mathbf{H}_{(0)}$. The purpose of this section is to establish a partial converse. Our starting point is a pair $(\mathbf{H}^*_{(0)}, Q^*)$ where

6.5.1. $\mathbf{H}^*_{(0)}$ is a linear subset of $\mathbf{H}^N_{(0)}$,
6.5.2. $Q^* - N$ is contractive on $\mathbf{H}^*_{(0)}$,
6.5.3. $(\mathbf{H}^*_{(0)}, Q^*)$ is a Dirichlet space relative to $L^2(\nu)$.

The most important candidate for such a pair is provided in

LEMMA 6.1. *The pair* $(\mathbf{H}^N_{(0)}, N)$ *satisfies* 6.5.

Proof. Conditions 6.5.1 and 6.5.2 are empty. The only nontrivial part of 6.5.3 is completeness relative to the norms N_a and this is easily established with the help of Fatou's lemma and Fubini's theorem.

Now fix a pair $(\mathbf{H}^*_{(0)}, Q^*)$ satisfying 6.5 and define

$$\mathbf{H}^*_{(+)} = \{\varphi \text{ in } \mathbf{H}^*_{(0)} : U_{0,1}(\varphi, \varphi) < +\infty\}$$

$$Q^*_{(u)}(\varphi, \psi) = Q^*(\varphi, \psi) + U_{0,u}(\varphi, \psi), \quad \varphi, \psi \text{ in } \mathbf{H}^*_{(+)}. \tag{6.2}$$

Of course $U_{0,u}$ is always dominated by a multiple of $U_{0,1}$ and so (6.2) makes sense. From 6.5.2 and

$$Q^*_{(u)}(\varphi, \varphi) = Q^*(\varphi, \varphi) - \frac{1}{2} U_{0,u}\langle\varphi, \varphi\rangle + \frac{1}{2} U_{0,u}(1, \varphi^2) \tag{6.3}$$

it follows that $Q^*_{(u)}$ is contractive on $\mathbf{H}^*_{(+)}$. Now consider a sequence φ_n in $H^*_{(+)}$ such that as $m, n \uparrow \infty$

$$\int \nu\,(dy)\,|\,\varphi_m(y) - \varphi_n(y)\,|^2 \to 0 \tag{6.4.1}$$

$$Q^*(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \to 0 \tag{6.4.2}$$

$$U_{0,u}(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \to 0. \tag{6.4.3}$$

Since $(\mathbf{H}^*_{(0)}, Q^*)$ is a Dirichlet space relative to $L^2(\nu)$, it follows from (6.4.1) and (6.4.2) that there exists a unique function φ in $\mathbf{H}^*_{(0)}$ such that $\varphi_n \to \varphi$

strongly in $\mathbf{H}^*_{(0)}$. Replacing φ_n by an appropriate subsequence such that $\varphi_n(y) - \varphi_n(z)$ is Cauchy on $M \times M$, we conclude further that $U_{0,u}(\varphi, \varphi) < +\infty$ and that

$$Q^*_{(u)}(\varphi - \varphi_n, \varphi - \varphi_n) \to 0.$$

We have proved

LEMMA 6.2. *For each $u > 0$ the pair $(\mathbf{H}^*_{(+)}, Q^*_{(u)})$ is a Dirichlet space relative to $L^2(\nu)$.*

Denote the corresponding resolvent operators by $R^*_{(u)a}$. These act initially on square integrable functions and extend in a natural way to general nonnegative functions on M (of course specified up to ν equivalence).

From the contractivity on $\mathbf{H}^*_{(0)}$ of $Q^* - N$ and from (6.2), (5.19) and (5.20) follows the contractivity on $\mathbf{H}^*_{(+)}$ of the bilinear form

$$Q^*_{(u)}(\varphi, \varphi) - u \int \nu\,(dy)\pi^\circ_u 1(y)\varphi^2(y).$$

Now consider g on $\mathbf{X}$ satisfying $0 \le g \le m < 1$ so that the bilinear form

$$Q_{\widetilde{(u)}}(\varphi, \varphi) = Q^*_{(u)}(\varphi, \varphi) - u \int \nu\,(dy)\pi^\circ_u g(y)\varphi^2(y)$$

makes $\mathbf{H}^*_{(+)}$ into a Hilbert space and therefore $(\mathbf{H}^*_{(+)}, Q_{\widetilde{(u)}})$ is also a Dirichlet space relative to $L^2(\nu)$. Denote the corresponding resolvent by $R_{\widetilde{(u)}a}$. Then an argument exactly analogous to (5.10) establishes the identity

$$R_{\widetilde{(u)}a} = R^*_{(u)a} + uR_{\widetilde{(u)}a}(\pi^\circ_u g)R^*_{(u)a} \tag{6.5}$$

Since $(b - a)R_{\widetilde{(u)}b}1(y)\varphi(y)$ increases to φ [a.e. ν] as $b \uparrow \infty$ for $\varphi \ge 0$ in $\mathbf{H}^*_{(+)}$, we have for $\psi \ge 0$ in $L^2(\nu)$,

$$\begin{aligned} u \int \nu\,(dy)&\pi^\circ_u g(y)R^*_{(u)a}\psi(y) \\ &= \operatorname{Lim} u \int \nu\,(dy)\pi^\circ_u g(y)(b - a)R_{\widetilde{(u)}b}1(y)R^*_{(u)a}\psi(y) \\ &= \operatorname{Lim} u \int \nu\,(dy)\pi^\circ_u g(y)(b - a)R_{\widetilde{(u)}a}\{1 - (b - a)R_{\widetilde{(u)}b}\}1(y)R^*_{(u)a}\psi(y) \end{aligned}$$

which because of (6.5) is

$$\begin{aligned} &\le \operatorname{Lim} \int \nu\,(dy)\psi(y)(b - a)R_{\widetilde{(u)}a}\{1 - (b - a)R_{\widetilde{(u)}b}\}1(y) \\ &\le \int \nu\,(dy)\psi(y). \end{aligned}$$

Letting $g \uparrow 1$ and $a \downarrow 0$, we deduce

$$uR^*_{(u)}\pi^\circ_u 1(y) \le 1. \tag{6.6}$$

Of course $R^*_{(u)}$ is defined by analogy with $R_{(u)}$ in Section 5. Now we define

$$G^*_u g = G^D_u g + H_u R^*_{(u)} \overset{\circ}{\pi}_u g \tag{6.7}$$

for $u > 0$ and $g \geq 0$ on X and we prove

LEMMA 6.3. *The operators G^*_u, $u > 0$ defined by* (6.7) *form a symmetric submarkovian resolvent on $L^2(dx)$*

Proof. Symmetry and the submarkovian character of the individual G^*_u follow directly from (6.6) and from the symmetry of $R^*_{(u)}$ on $L^2(\nu)$. To establish the resolvent identity we use (3.9), (3.13) and

$$R^*_{(u)} - R^*_{(v)} = (v - u)R^*_{(u)} \overset{\circ}{\pi}_u H_v R^*_{(v)} \tag{6.8}$$

which can be established in the same way as (5.11) and compute

$$\begin{aligned} G^*_u &- G^*_v \\ &= G^D_u - G^D_v + H_u R^*_{(u)} \overset{\circ}{\pi}_u - H_v R^*_{(v)} \overset{\circ}{\pi}_v \\ &= (v - u)G^D_u G^D_v + H_u R^*_{(u)} \overset{\circ}{\pi}_v \{1 + (v - u)G^D_u\} \\ &\quad - \{1 - (v - u)G^D_v\} H_u R^*_{(v)} \overset{\circ}{\pi}_v \\ &= (v - u)G^D_u G^D_v + (v - u)H_u R^*_{(u)} \overset{\circ}{\pi}_u H_v R^*_{(v)} \overset{\circ}{\pi}_v \\ &\quad + (v - u)H_u R^*_{(u)} \overset{\circ}{\pi}_u G^D_v + (v - u)G^D_v H_u R^*_{(v)} \overset{\circ}{\pi}_v \\ &= (v - u)G^*_u G^*_v . \end{aligned}$$

We turn our attention now to the Dirichlet space $(\mathbf{F}^*, E^*)$ which corresponds to the resolvent operators G^*_u. We begin by establishing the analogues of Theorem 3.6 (i) and Theorem 3.6 (iii).

For f, $g \geq 0$ in $L^2(dx)$ we have

$$\begin{aligned} \int dx v G^*_{u+v} H_u G^*_u f(x) g(x) &= \int dx f(x) H_u G^*_u v G^*_{u+v} g(x) \\ &= \int dx f(x) H_u \{G^*_u - G^*_{u+v}\} g(x) \end{aligned}$$

which increases to

$$\int dx f(x) H_u G^*_u g(x)$$

as $v \uparrow \infty$. Next

$$\begin{aligned} v \int & dx \{H_u G^*_u f(x) - v G^*_{u+v} H_u G^*_u f(x)\} G^D_u g(x) \\ &= v \int dx \{H_u G^*_u f(x) - v G^D_{u+v} H_u G^*_u f(x)\} G^D_u g(x) \\ &\qquad\qquad - v^2 \int dx H_{u+v} G^*_{u+v} H_u G^*_u f(x) G^D_u g(x) \\ &= v \int dx G^D_{u+v} H_u G^*_u f(x) g(x) - v^2 \int dx G^D_{u+v} H_u G^*_{u+v} H_u G^*_u f(x) g(x). \end{aligned}$$

From the previous result and from the fact that

$$vG^D_{u+v}H_u h \uparrow 1_D H_u h$$

for $h \geq 0$ (see (3.10)) it follows that

$$\text{(6.9)} \qquad \operatorname{Lim} v \int dx \{H_u G^*_u f(x) - vG^*_{u+v} H_u G^*_u f(x)\} G^D_u g(x) = 0$$

with the limit taken as $v \uparrow \infty$. From (6.9) follows that

$$\begin{aligned}
\operatorname{Lim} v \int dx \{G^D_u f(x) &- vG^*_{u+v} G^D_u f(x)\} G^D_u g(x) \\
&= \operatorname{Lim} v \int dx \{G^D_u f(x) - vG^*_{u+v} G^D_u f(x)\}\{G^D_u g(x) + H^u_u G^*_u g(x)\} \\
&= \operatorname{Lim} v \int dx G^D_u f(x) \{G^*_u g(x) - vG^*_{u+v} G^*_u g(x) \\
&= \int dx G^D_u f(x) g(x).
\end{aligned}$$

These results together with (6.7) prove

LEMMA 6.4. *$\mathbf{F}^D$ is contained in $\mathbf{F}^*$ and*

$$E^D(f, g) = E^*(f, g)$$

for f, g in $\mathbf{F}^D$. Moreover if f belongs to $\mathbf{F}^*$, then $H_u f$ is the projection of f onto the E^*_u-orthogonal complement of $\mathbf{F}^D$ in $\mathbf{F}^*$.

Next we prove

LEMMA 6.5. *If $u > 0$ and If φ is in $\mathbf{H}^*_{(+)}$, then $H_u\varphi$ is in $\mathbf{F}^*$ and*

$$\text{(6.10)} \qquad E^*_u(H_u \varphi, H_u \varphi) = Q^*_{(u)}(\varphi, \varphi).$$

Proof. We consider first the special case when $\varphi = R^*_{(u)}\overset{\circ}{\pi}_u f$ with f bounded and integrable on $\mathbf{X}$. Then by (6.6) the function φ is bounded and therefore in $L^2(\nu)$. Also

$$\int \nu\,(dy) a\{\varphi - aR^*_{(u)a}\varphi\}(y)\varphi(y) = \int \nu\,(dy) aR^*_{(u)a}\overset{\circ}{\pi}_u f(y) R^*_{(u)}\overset{\circ}{\pi}_u f(y)$$

which increases as $a \uparrow \infty$ to

$$\begin{aligned}
\int \nu\,(dy)\overset{\circ}{\pi}_u f(y) R^*_{(u)}\overset{\circ}{\pi}_u f(y) &= \int dx f(x) H_u R^*_{(u)}\overset{\circ}{\pi}_u f(x) \\
&= E^*_u(H_u G^*_u f, H_u G^*_u f).
\end{aligned}$$

We conclude that φ belongs to $\mathbf{H}^*_{(u)}$, that $H_u\varphi = H_u G^*_u f$ belongs to $\mathbf{F}^*$ and that (6.10) is satisfied. Therefore the proof will be complete once we show that such φ are strongly dense in $\mathbf{H}^*_{(+)}$. Because of the analogue of (6.8) for fixed $a > 0$, it suffices to show that for ψ bounded on M, the functions $R^*_{(v)a}\psi \to 0$ strongly in $\mathbf{H}^*_{(+)}$ as $v \uparrow \infty$. For $\psi \geq 0$ the functions $R^*_{(v)a}\psi$ decrease as v increases and so

$$\psi^*(y) = \operatorname{Lim} R^*_{(v)a}\psi(y)$$

exists [a.e.ν] for any sequence of $v \uparrow \infty$. For $v > u$,

$$Q^*_{(u)a}(R^*_{(v)a}\psi, R^*_{(v)a}\psi) = \int \nu\,(dy)\psi(y)R^*_{(v)a}\psi(y) - U_{u,v}(R^*_{(v)a}\psi, R^*_{(v)a}\psi)$$

and we conclude on the one hand that

$$\text{(6.11)} \qquad \operatorname{Lim} Q^*_{(u)a}(R^*_{(v)a}\psi, R^*_{(v)a}\psi) \leq \int \nu\,(dx)\psi(x)\psi^*(x)$$

with the limit taken as $v \uparrow \infty$ and on the other hand that $U_{u,v}\ (\psi^*, \psi^*)$ is bounded independent of $v > u$. But this implies that

$$(v - u)\int dxH_u\psi^*(x)\{H_u\psi^*(x) - (v - u)G^D_vH_u\psi^*(x)\}$$

is bounded independent of $v > u$ which by Proposition 1.1 implies that $H_u\psi^*$ belongs to $\mathbf{F}^D$. Because of (6.11) it suffices to show that this is impossible unless $\psi^* = 0$ [a.e. ν]. If this is false choose $g \geq 0$ bounded and integrable on D such that

$$\int \nu\,(dy)\pi^\circ_u g(y)\psi^*(y) > 0.$$

It is easy to check that the functions

$$\text{(6.12)} \qquad H_u\psi(\sigma(D_n))\exp\{-u\sigma(D_n)\}$$

form a nonzero uniformly integrable martingale relative to the process $\mathcal{P}_{g\,dx}$. But since $H_u\psi^*$ belongs to $\mathbf{F}^D$ it can be approximated in the usual way by functions $G^D_u f$ and it follows easily that the random variables (6.12) converge to zero [a.e. $\mathcal{P}_{g\,dx}$] which is impossible.

We return to the Dirichlet space $(\mathbf{F}^*, E^*)$ in Section 8.

7. Probabilistic Interpretation of Dirichlet norms

Throughout this section we assume that the original regular Dirichlet space $(\mathbf{F}, E)$ corresponds to a conservative process. Our precise assumption is that

$$\text{(7.1)} \qquad uG_u 1 = 1$$

for all $u > 0$. We do this both to avoid technical complication and because we have not been able to resolve certain difficulties which arise in the nonconservative case. It follows easily that the life time $\zeta = +\infty$ quasi-surely on Ω. As in Section 4 (but with K replaced by $\mathbf{X}$) we introduce the truncated trajectory

$$\begin{aligned} \omega_u(t) &= \omega(t) \quad \text{for } 0 \le t < R_u \\ &= \partial \qquad \text{for } t \ge R_u . \end{aligned}$$

and the time reversed trajectory

$$\begin{aligned} J\omega(t) &= \omega(R_u - t - 0) \quad \text{for } 0 \le t < R_u \\ &= \partial \qquad \text{for } t \ge R_u . \end{aligned}$$

We also introduce the truncated trajectory variables

$$\begin{aligned} X_u(t) &= X(t) \quad \text{for } t < R_u \\ &= \partial \qquad \text{for } t \ge R_u \end{aligned}$$

and the special notation

$$f_u(t) = f[X_u(t)].$$

We introduce for $u > 0$ the measure $\mathcal{P}_u$ on Ω defined by 2.2 with $\nu = udx$. For typographical convenience we use the subscript u rather than udx to label this measure. A suitable modification of Theorem 4.2 establishes

Added in proof. The restriction (7.1) is removed in [29, Section 6].

THEOREM 7.1. *Let $u > 0$ and let ξ be a nonnegative Baire function on Ω. Then*

$$\int \mathcal{P}_u\,(d\omega)\xi(\omega) = \int \mathcal{P}_u\,(d\omega)\xi(J\omega_u).$$

For $f = G_u g$ with g in $C_{\text{com}}(\mathbf{X})$ define

$$M_u f(t) = f_u(t) + \int_0^t ds g_u(s). \tag{7.2}$$

It is easy to check that $M_u f(t)$, $t \ge 0$ is a right continuous square integrable martingale and that

$$M_u f(\infty) = \int_0^{R_u} ds g(s), \qquad \frac{1}{2}\,\mathcal{E}_u\{M_u f(\infty)\}^2 = E_u(f, f).$$

Now a passage to the limit using standard estimates [22, Chap. V.2] from martingale theory establishes

THEOREM 7.2. *For f in $\mathbf{F}$ there is a unique right continuous $\mathcal{P}_u$ square integrable martingale $M_u f(t)$, $t \ge 0$ on Ω such that*

(i) *$M_u f(t)$ is given by (7.2) whenever $f = G_u g$ with g in $C_{\text{com}}(\mathbf{X})$.*

(ii) *$E_u(f, f') = \frac{1}{2}\mathcal{E}_u M_u f(\infty) M_u f'(\infty)$ for f, f' in $\mathbf{F}$.*

Remark. Since $\mathcal{P}_u$ need not be bounded and $M_u f(t)$ need not be integrable, the formulation in [22] is not directly applicable. However the necessary modifications are obvious and we take them for granted.

By well known results of P. A. Meyer [22, Chap. VIII 3] there is for each f in $\mathbf{F}$ a unique continuous increasing process $\langle M_u f\rangle(t)$, $t \geq 0$, such that $\langle M_u f\rangle(0) = 0$ and such that

$$\{M_u f(t)\}^2 - \langle M_u f\rangle (t)$$

is an $\mathfrak{F}_t$ martingale. Now fix ν in $\mathfrak{M}^e$ and let M be as in Section 3 and let $D = \mathbf{X} - M$. We begin the serious work of this section by deriving a formula involving the quantity

$$\mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle (dt) 1_D(t) \tag{7.3}$$

Our analysis of (7.3) depends on the introduction of certain functions on Ω. First

$$r(0) = \inf \{t \geq 0 : X_u(t) \text{ or } X_u(t-0) \text{ is in } M\}$$

with the understanding that $r(0) = R_u$ when not otherwise specified. It is easy to check that $H_u 1$ agrees q.e. on every compact set with a u-potential and it follows with the help of Lemma 2.7 that $H_u 1(t)$ is quasi-surely right continuous and has left hand limits everywhere for $t \geq 0$. Since

$$M = \{x : H_u 1(x) = 1\},$$

it follows that the random set

$$\{t > r(0) : X_u(t) \text{ and } X_u(t-0) \text{ are in } D\}$$

can be represented as a finite or countable disjoint union of intervals of the form $I = (e, r)$. It is possible to index the intervals I_i (in general not the "natural" ordering) so that I_i and its corresponding endpoints $e(i)$, $r(i)$, $i = 1, 2, \cdots$ are Borel measurable on Ω.

In general $e(i)$ and $r(i)$ *are not stopping times* and so we introduce approximating functions $e(n; i)$, $r(n; i)$ which are. Also to facilitate eventual passage to the limit, we introduce at the same time a truncation at a constant time $T > 0$. Thus

$$e(n; 1) = \inf \{t > r(0) + (1/n) : t < T \text{ and } X_u(t) \text{ is in } D\}$$
$$r(n; 1) = \inf \{t > e(n; 1) : X_u(t) \text{ or } X_u(t-0) \text{ is in } M\}$$

with the understanding that $e(n; 1) = +\infty$ when not otherwise specified, that $r(n; 1) = +\infty$ whenever $e(n; 1) = +\infty$, and that $r(n; 1) = R_u$ when not otherwise specified. Similarly

$$e(n; i+1) = \inf \{t > r(n; i) + (1/n) : t < T \text{ and } X_u(t) \text{ is in } D\}$$
$$r(n; i+1) = \inf \{t > e(n; i+1) : X_u(t) \text{ or } X_u(t-0) \text{ is in } M\},$$

with the same understanding for $e(n; i+1)$ and $r(n; i+1)$ when not otherwise specified. We also introduce the truncated limiting variables

$$\begin{aligned} e(i; T) &= e(i) \quad \text{if } e(i) < T \\ &= +\infty \quad \text{if } e(i) \geq T \\ r(i; T) &= r(i) \quad \text{if } e(i) < T \\ &= +\infty \quad \text{if } e(i) \geq T \end{aligned}$$

and the special symbol

$$\begin{aligned} R_u(T) &= R_u & \text{if } R_u \leq T \\ &= \min [R_u, T + \theta(T)\sigma(M)] & \text{if } T < R_u . \end{aligned}$$

Consider again $f = G_u g$ with g in $C_{\text{com}}(\mathbf{X})$ and fix $\varepsilon > 0$ arbitrary. Clearly there exists N such that

$$\text{(7.4)} \qquad \mathcal{E}_u \sum_i \int_{e(n;i)}^{r(n;i)} \langle M_u \rangle \, (dt) \geq \mathcal{E}_u \sum_i \int_{e(i;T)}^{r(i;T)} \langle M_u f \rangle \, (dt) - \varepsilon$$

for $n \geq N$. The left side of (7.4)

$$\begin{aligned} = \sum_i \mathcal{E}_u \Big\{ &(f_u[r(n; i)] - f_u[e(n; i)])^2 \\ &+ 2 \int_{e(n;i)}^{r(n;i)} dt G_u^D g(t) g(t) \\ &+ 2(f_u[r(n; i)] - f_u[e(n; i)]) \int_{e(n;i)}^{r(n;i)} dt g(t) \Big\} \\ = \sum_i \mathcal{E}_u \Big\{ &(f[r(n; i)] - f_u[e(n; i) - 0])^2 \\ &- (f_u[e(n; i)] - f_u[e(n; i) - 0])^2 \\ &+ 2 \int_{e(n;i)}^{r(n;i)} dt G_u^D g(t) g(t) \\ &+ 2(f_u[r(n; i)] - f_u[e(n; i) - 0]) \int_{e(n;i)}^{r(n;i)} dt g(t) \Big\} . \end{aligned}$$

With the help of Fatou's lemma and [23, p. 82] we can guarantee that the last expression differs from

$$\begin{aligned} \mathcal{E}_u \Big\{ &\sum_i (f_u[r(T; i)] - f_u[e(T; i) - 0])^2 \\ &- \sum_i (f_u[e(T; i)] - f_u[e(T; i) - 0])^2 \\ &+ 2 \sum_i \int_{e(T;i)}^{r(T;i)} dt G_u^D g(t) g(t) \\ &+ 2 \sum_i (f_u[r(T; i)] - f_u[e(T; i) - 0]) \int_{e(T;i)}^{r(/T;i)} dt g(t) \Big\} \end{aligned}$$

by less than ε for $n \geq N$ (after possibly increasing N). This together with the corresponding argument for the time interval $0 \leq t \leq r(0)$ and then a passage to the limit $T \uparrow \infty$ establishes

$$\begin{aligned}
\mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle\, (dt) 1_D(t) &+ \mathcal{E}_u \{f_u[0]\}^2 \\
&+ \mathcal{E}_u \textstyle\sum_i \{f_u[e(i)] - f_u[e(i) - 0]\}^2 \\
= 2\mathcal{E}_u \int_0^{R_u} & dt G_u^D g(t) g(t) + \mathcal{E}_u \{f_u[r(0)]\}^2 \\
&+ \mathcal{E}_u \textstyle\sum_i \{f_u[r(i)] - f_u[e(i) - 0]\}^2 \\
&+ 2\mathcal{E}_u \{f_u[r(0)]\} \int_0^{r(0)} dt g(t) \\
&+ 2\mathcal{E}_u \textstyle\sum_i \{f_u[r(i)] - f_u[e(i) - 0]\} \int_{e(i)}^{r(i)} dt g(t).
\end{aligned}$$

The first term on the right can be written $2E_u(f - H_u f, f - H_u f)$ and Theorem 7.1 implies that the last two terms cancel. Thus, after passage to the limit in $\mathbf{F}$ we have

THEOREM 7.3. *For f in $\mathbf{F}$ and $u > 0$*

$$\begin{aligned}
\frac{1}{2} \mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle\, (dt) 1_D(t) &+ \frac{1}{2} \mathcal{E}_u \{f_u[0]\}^2 \\
&+ \frac{1}{2} \mathcal{E}_u \textstyle\sum_i \{f_u[e(i)] - f_u[e(i) - 0]\}^2 \\
&= \frac{1}{2} \mathcal{E}_u \{f_u[r(0)]\}^2 \\
&+ \frac{1}{2} \mathcal{E}_u \textstyle\sum_i \{f_u[r(i)] - f_u[e(i) - 0]\}^2 \\
&+ E_u(f - H_u f, f - H_u f).
\end{aligned}$$

The extra terms appearing on the left side can be best understood in the context of another increasing functional of Meyer associated with the square of the martingale $M_u f(t)$.

THEOREM 7.4. *If φ belongs to the universal Dirichlet space $\mathbf{H}^N$ then*

$$H_0 f(x) = \mathcal{E}_x f[\sigma(M)]$$

is well defined [a.e. dx] *on D and for every $u > 0$,*

$$\begin{aligned}
N(\varphi, \varphi) &= \frac{1}{2} \mathcal{E}_u \{H_0 \varphi[r(0)] - H_0 \varphi[0]\}^2 \\
&+ \frac{1}{2} \mathcal{E}_u \sum_i \{H_0 \varphi[r(i)] - H_0 \varphi[e(i) - 0]\}^2.
\end{aligned} \tag{7.5}$$

Conversely if φ is specified up to $\pi_1 1$ equivalence on M, if $H_0\varphi$ is well defined [a.e. dx] *on D and if the right side of* (7.5) *converges, then φ belongs to the Dirichlet space $\mathbf{H}^N$.*

Proof. Since the right side of (7.5) dominates

$$\frac{1}{2}u \int dx \int H_u(x, dy)\{H_0\,\varphi(x) - H_0\,\varphi(y)\}^2$$

it suffices to prove the theorem under the assumption that φ is bounded and therefore $H_0\varphi$ is well defined q.e. on M. We begin by fixing $v > u$ and computing

$$U_{0,v}\langle\varphi, \varphi\rangle + v\int \pi_v\, p(dy)\varphi^2(y)$$

$$= v\mathcal{E}_u \int_0^{r(0)} dt \int H_v(t, dy) \int H_0(t, dz)\{\varphi(y) - \varphi(z)\}^2$$

$$+ v\mathcal{E}_u \int_0^{r(0)} dt\; p(t)H_v(t, dy)\varphi^2(y)$$

$$+ v\mathcal{E}_u \textstyle\sum_i \displaystyle\int_{e(i)}^{r(i)} dt \int H_v(t, dy) \int H_0(t, dz)\{\varphi(y) - \varphi(z)\}^2$$

$$+ v\mathcal{E}_u \textstyle\sum_i \displaystyle\int_{e(i)}^{r(i)} dt\; p(t) \int H_v(t, dy)\varphi^2(y)$$

$$= v\mathcal{E}_u\, I[r(0) < R_u] \int_0^{r(0)} dt \int H_0(t, dz) \exp\left(-[v - u][r(0) - t]\right)$$

$$\cdot \{\varphi[r(0)] - \varphi[z]\}^2$$

$$+ v\mathcal{E}_u\, I[r(0) < R_u] \int_0^{r(0)} dt\; p(t) \exp\left(-[v - u][r(0) - t]\right)$$

$$\cdot \{\varphi[r(0)]\}^2$$

$$+ v\mathcal{E}_u \textstyle\sum_i I[r(i) < R_u] \displaystyle\int_{e(i)}^{r(i)} dt$$

$$\cdot \int H_0(t, dz) \exp\left(-[v - u][r(i) - t]\right) \{\varphi[r(i)] - \varphi[z]\}^2$$

$$+ v\mathcal{E}_u \textstyle\sum_i I[r(i) < R_u] \displaystyle\int_{e(i)}^{r(i)} dt\; p(t)$$

$$\cdot \exp\left(-[v - u][r(i) - t]\right)\{\varphi[r(i)]\}^2$$

Applying Theorem 7.1 and combining terms, we get

$$U_{0,v}\langle\varphi, \varphi\rangle + v\int \pi_v\, p(dy)\varphi^2(y)$$

$$= v\mathcal{E}_u \textstyle\sum_i \displaystyle\int_{e(i)}^{r(i)} dt \exp\left(-[v - u][t - e(i)]\right) \tag{7.6}$$

$$\cdot \{\varphi[r^*(i)] - \varphi[e(i) - 0]\}^2$$

where $r^*(i)$ is the time of next (after $e(i)$) return to M for the untruncated trajectory, with the convention $r^*(i) = \infty$ and therefore $\varphi[r^*(i)] = 0$ when no such return occurs. On the other hand

$$\{\varphi[r^*(i)] - \varphi[e(i) - 0]\}^2$$

can be replaced by

$$\{H_0\,\varphi[r(i)] - \varphi[e(i) - 0]\}^2 + \{\varphi[r^*(i)] - H_0\,\varphi[r(i)]\}^2$$

in the right side of (7.6) and so after computing the t-integrals and passing to the limit $v \uparrow \infty$ we get

$$U_{0,\infty}\langle\varphi, \varphi\rangle + \int \pi_1\, p(dy)\varphi^2(y)$$

$$= \mathcal{E}_u \textstyle\sum_i \{H_0\,\varphi[r(i)] - \varphi[e(i) - 0]\}^2 + \mathcal{E}_u \sum_i \{\varphi[r^*(i)] - H_0\,\varphi[r(i)]\}^2.$$

The theorem will be proved once we establish the identity

$$\mathcal{E}_u \textstyle\sum_i \{\varphi[r^*(i)] - H_0\,\varphi[r(i)]\}^2 + u \displaystyle\int dx\, p(x)H_u\,\varphi^2(x) = \mathcal{E}_u\{H_0\,\varphi[r(0)] - H_0\,\varphi[0]\}^2. \tag{7.7}$$

Before continuing we remark that it is relatively easy to give false proofs of (7.7) by expanding both sides and treating individual terms separately—ignoring the fact that these terms may not (and in general do not) converge. A legitimate proof seems to require some care.

First define

$$H(x) = \mathcal{E}_x\,\{\varphi[\sigma(M)] - H_0\varphi[0]\}^2$$

$$R(x) = \mathcal{E}_x\,\{H_0\varphi[r(0)] - H_0\varphi[0]\}^2$$

and note the identity

$$H(x) = \textstyle\sum_{n=0}^{\infty} \{uG_u^D\}^n R(x) \tag{7.8}$$

which can be established as follows. Let $R_u(1)$, $R_u(2)$, $\cdots$ be a sequence of independent copies of R_u and let $T(0) = 0$, $T(1) = R_u(1)$, $T(2) = R_u(1) + R_u(2)$ etc. Then

$$H(x) = \mathcal{E}_x \textstyle\sum_{n=0}^{\infty} [H_0\varphi\{\min[\sigma(M), T(n+1)]\}$$

$$- H_0\varphi\{\min[\sigma(M), T(n)]\}]^2$$

and it is easy to check that the n^{th} summand agrees with the n^{th} summand on the right in (7.8). A similar argument establishes

$$H_0\,1(x) = \textstyle\sum_{n=0}^{\infty} \{uG_u^D\}^n H_u\,1(x). \tag{7.9}$$

After substituting (7.8) into the left side of (7.7) and then applying (7.9), we see that (7.7) will be proved once we have established

$$(7.10)\qquad u\int dx\, p(x)H_u\,\varphi^2(x) = u\int dx\, p(x)R(x).$$

We first prove (7.10) under the assumption that the right side of (7.5) converges. Since $R(x)$ dominates

$$\int H_u(x, dy)\{\varphi(y) - H_0\,\varphi(x)\}^2$$

$$= \int H_u(x, dy)\{\varphi(y) - H_u\,\varphi(x)\}^2 + \{H_0\,\varphi(x) - H_u\,\varphi(x)\}^2$$

we conclude that

$$\int dx\, p(x)\{H_0\,\varphi(x) - H_u\,\varphi(x)\}^2$$

converges. On the other hand the earlier part of the proof shows that the right side of (7.5) dominates the left side of (7.10) and we conclude that

$$(7.11)\qquad \int dx\, p(x)\{H_0\,\varphi(x)\}^2$$

converges. But then the right side of (7.10)

$$= u\int dx\, p(x)[H_u\,\varphi^2(x) + H_u\,1(x)\{H_0\,\varphi(x)\}^2$$

$$- 2H_u\,\varphi(x)H_0\,\varphi(x) + uG_u^D\{H_0\,\varphi\}^2(x)$$

$$+ uG_u^D\,1(x)\{H_0\,\varphi(x)\}^2 - 2H_0\,\varphi(x)uG_u^D\,H_0\,\varphi(x)]$$

$$= u\int dx\, p(x)H_u\,\varphi^2(x)$$

$$+ \int dx\,[uG_u^D\{H_0\,\varphi\}^2(x) - \{H_0\,\varphi(x)\}^2]$$

and the last integral is zero since $p = uG_u^D p$ and since (7.11) converges. (The convergence of (7.11) is essential for this argument. For example, the last integral is not zero if we replace $\{H_0\varphi\}^2$ by $H_0\varphi^2$.) This proves (7.10) and therefore the theorem under the assumption that the right side of (7.5) converges. In fact our proof is valid under the assumption that both sides of (7.10) converge and therefore we will be done if we can show that convergence of the left side of (7.10) implies convergence of the right side. But then

$$\int dx\, p(x)H_v\,\varphi^2(x)$$

converges for all $v > 0$. Since it suffices to consider $\varphi \geq 0$, convergence of the right side of (7.10) follows from Fatou's lemma and the computation

$$u \int dx\, p(x)\mathcal{E}_x\{H_v\, \varphi[r(o)] - H_v\, \varphi[0]\}^2$$

$$= u \int dx\, p(x)[H_u\, \varphi^2(x) + H_u\, 1(x)\{H_v\, \varphi(x)\}^2$$

$$- 2H_u\, \varphi(x) H_v\, \varphi(x) + uG_u^D\{H_v\, \varphi\}^2(x)$$

$$+ uG_u^D\, 1(x)\{H_v\, \varphi(x)\}^2 - 2H_v\, \varphi(x) uG_u^D\, H_v\, \varphi(x)]$$

$$= u \int dx\, p(x) H_u\, \varphi^2(x)$$

$$- 2v \int dx\, p(x) H_v\, \varphi(x) uG_u^D\, H_v\, \varphi(x).$$

In the remainder of this section we apply the technique used in the first part of the proof of Theorem 7.4 to transform Theorem 7.3. As a preliminary step, fix $v > u$ and φ in the universal Dirichlet space $\mathbf{H}^N$ and compute

$$U_{u,v}\langle \varphi, \varphi \rangle$$

$$= (v - u)\mathcal{E}_u \int_0^{r(o)} dt \int H_u(t, dy) \int H_v(t, dz)\{\varphi[z] - \varphi[y]\}^2$$

$$+ (v - u)\mathcal{E}_u \sum_i \int_{e(i)}^{r(i)} dt \int H_u(t, dy) \int H_v(t, dz)\{\varphi[z] - \varphi[y]\}^2$$

$$= (v - u)\mathcal{E}_u\, I[r(0) < R_u] \int_0^{r(0)} dt \exp\left(-[v - u][r(0) - t]\right) \int H_u(t, dy)$$

$$\cdot \{\varphi[r(0)] - \varphi[y]\}^2$$

$$+ (v - u)\mathcal{E}_u \sum_i I[r(i) < R_u] \int_{e(i)}^{r(i)} dt \exp\left(-[v - u][r(i) - t]\right)$$

$$\cdot \int H_u(t, dy)\{\varphi[r(i)] - \varphi[y]\}^2$$

which, after applying Theorem 7.1,

$$= (v - u)\mathcal{E}_u \sum_i \int_{e(i)}^{r(i)} dt \exp\left(-[v - u][t - e(i)]\right) \int H_u(t, dy)$$

$$\cdot \{\varphi[e(i) - 0] - \varphi[y]\}^2$$

$$= (v - u)\mathcal{E}_u \sum_i I[r(i) < R_u] \int_{e(i)}^{r(i)} dt \exp\left(-[v - u][t - e(i)]\right)$$

$$\cdot \{\varphi[e(i) - 0] - \varphi[r(i)]\}^2.$$

As in the proof of Theorem 7.4 we compute the t-integrals and pass to the limit $v \uparrow \infty$ to get

LEMMA 7.5. *For φ in the universal Dirichlet space $\mathbf{H}^N$ and for $u > 0$*

$$U_{u,\infty}\langle\varphi, \varphi\rangle = \mathcal{E}_u \sum_i I[r(i) < R_u]\{\varphi[r(i)] - \varphi[e(i) - 0]\}^2.$$

Now we consider the special case $\varphi = \gamma f$ with f in $\mathbf{F}$ and apply Lemma 7.5 to Theorem 7.3 to get

$$\frac{1}{2}\mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle(dt)1_D(t) + \frac{1}{2}u \int dx\, f^2(x) + \frac{1}{2}\mathcal{E}_u \sum_i \{f[e(i)] - f[e(i) - 0]\}^2$$

$$= \frac{1}{2}u \int dx\, H_u f^2(x) + \frac{1}{2}U_{u,\infty}\langle f,f\rangle + \frac{1}{2}\mathcal{E}_u \sum_i I[r(i) = R_u]f^2[e(i) - 0] + E_u(f - H_u f, f - H_u f)$$

which, after applying Theorem 7.1 to the third term,

$$= \frac{1}{2}U_{u,\infty}\langle f,f\rangle + u \int dx\, H_u f^2(x) + E_u(f - H_u f, f - H_u f)$$

$$= N(f,f) + U_{0,u}(1,f^2) - \frac{1}{2}U_{0,u}\langle f,f\rangle + E_u(f - H_u f, f - H_u f)$$

and we have proved

THEOREM 7.6. *For f in $\mathbf{F}$ and $u > 0$,*

$$\frac{1}{2}\mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle\, (dt)1_D(t) + \frac{1}{2}u \int_D dx f^2(x) + \frac{1}{2}\mathcal{E}_u \sum_i \{f[e(i)] - f[e(i) - 0]\}^2$$

$$= N(f,f) + E_u(f - H_u f, f - H_u f) + u \int_D dx H_0 f(x) H_u f(x).$$

We will improve this result in Section 8.

8. Extension of Dirichlet norms

We begin with

8.1. DEFINITION. A function f specified and finite up to quasi-equivalence on $\mathbf{X}$ belongs to the extended Dirichlet space $\mathbf{F}_{(e)}$ if there exists a sequence f_n in $\mathbf{F}$ which is Cauchy relative to the Dirichlet norm E and such that $f_n \to f$ q.e. For such f put

$$E(f,f) = \operatorname{Lim} E(f_n, f_n).$$

In the next lemma we show in particular that $E(f, f)$ is well defined for f in $\mathbf{F}_{(e)}$ and that $\mathbf{F} = \mathbf{F}_{(e)} \cap L^2(dx)$.

LEMMA 8.1. *If f belongs to the extended Dirichlet space $\mathbf{F}_{(e)}$, then*

$$E^{(u)}(f,f) = u\int dx\{1 - uG_u1(x)\}f^2(x) + \frac{1}{2}u^2\int dx\int G_u(x,dy)\{f(x) - f(y)\}^2$$

converges for all $u > 0$ *and*

$$E^{(u)}(f,f) \uparrow E(f,f) \quad as\ u \uparrow \infty.$$

Proof. The lemma follows from Proposition 1.1 for f in $\mathbf{F}$. For f in $\mathbf{F}_{(e)}$ it suffices to choose f_n as in Definition 8.1 and to observe that the f_n are uniformly Cauchy relative to the $E^{(u)}$.

Important contractivity and completeness properties for $\mathbf{F}_{(e)}$ are established in

LEMMA 8.2. (i) *Let f belong to $\mathbf{F}_{(e)}$ and let T be a normalized contraction. Then Tf belongs to $\mathbf{F}_{(e)}$ and $E(Tf, Tf) \le E(f,f)$.*

(ii) *Let f be specified and finite up to dx equivalence on $\mathbf{X}$ and let f_n be a sequence in $\mathbf{F}_{(e)}$ which is Cauchy relative to E and such that $f_n \to f$* [a.e. dx]. *Then f has a quasi-everywhere defined refinement f^* belonging to $\mathbf{F}_{(e)}$ and, for a subsequence, $f_n \to f^*$* q.e.

Proof. To prove (i) it suffices to observe that if f_n in $\mathbf{F}$ form a Cauchy sequence relative to E and if $f_n \to f$ q.e. then, for the Cesáro means of a subsequence [24, p. 80], Tf_n is Cauchy relative to E and $Tf_n \to Tf$ q.e. In proving (ii) we consider first the special case when f and f_n are bounded by a fixed constant c. Fix n and choose $f_{n,m}$ in $\mathbf{F}$ bounded by c such that as m varies the $f_{n,m}$ form a Cauchy sequence relative to E and such that $f_{n,m} \to f_n$ q.e. as $m \uparrow \infty$. For open G with finite capacity the functions $p_1^G f_{n,m}$ are E_1 bounded in $\mathbf{F}$ (since $Tx = x^2$ is a contraction for $|x| \le \frac{1}{2}$ and since

$$p_1^G f_{n,m} = \frac{1}{2}(p_1^G + f_{n,m})^2 - \frac{1}{2}(p_1^G - f_{n,m})^2)$$

and therefore the Cesáro means of a subsequence are Cauchy relative to E_1. It follows that the function $p_1^G f_n$ belongs to $\mathbf{F}$ and that as n varies the $p_1^G f_n$ form a Cauchy sequence relative to E_1. The lemma for this case can now be established by approximating the f_n with functions in $\mathbf{F}$, by letting G expand to $\mathbf{X}$, and by applying the Tchebychev estimate (1.5) together with a diagonalization argument. The general case is easily established upon approximating f by truncations and verifying with the help of (i) that these truncations form a Cauchy sequence relative to E.

We now improve the results of Sections 5 through 7, in particular *dispensing with the restrictions* (5.2) *and* (6.1).

LEMMA 8.3. (i) *If φ belongs to $\mathbf{H}_{(0)} \cap \gamma\mathbf{F}_{(e)}$, then*

$$H_0\varphi(x) = \mathcal{E}_x\varphi[\sigma(M)] \tag{8.1}$$

is well defined q.e. *and belongs to $\mathbf{F}_{(e)}$ and*

$$E(H_0\varphi, H_0\varphi) = Q(\varphi,\varphi). \tag{8.2}$$

(ii) *If* φ *belongs to* $\gamma\mathbf{F}_{(e)}$, *then there exists a sequence* φ_n *in* $\mathbf{H}_{(0)} \cap \mathbf{F}$ *which is Cauchy relative to the Dirichlet norm* Q *and such that* $\varphi_n \to \varphi$ q.e. *on* M.

Proof. Consider first bounded φ belonging to $\mathbf{H}_{(0)} \cap \gamma\mathbf{F}$. For $v, u > 0$ the function $H_v\varphi - H_u\varphi$ belongs to $\mathbf{F}^D$ and therefore

$$
\begin{aligned}
E(H_v\varphi - H_u\varphi, H_v\varphi - H_u\varphi) \\
&= E_v(H_v\varphi - H_u\varphi, H_v\varphi) - E_u(H_v\varphi - H_u\varphi, H_u\varphi) \\
&\quad - v\int dx\{H_v\varphi(x) - H_u\varphi(x)\}H_v\varphi(x) \\
&\quad + u\int dx\{H_v\varphi(x) - H_u\varphi(x)\}H_u\varphi(x) \\
&= -v\int dx\{H_v\varphi(x) - H_u\varphi(x)\}H_v\varphi(x) \\
&\quad + u\int dx\{H_v\varphi(x) - H_u\varphi(x)\}H_u\varphi(x)
\end{aligned}
$$

which $\to 0$ by Lemma 5.5. Thus $H_0\varphi$ belongs to $\mathbf{F}_{(e)}$ and (8.2) follows upon passing to the limit $u \downarrow 0$ in (5.4). The restriction that φ be bounded is easily removed by considering truncations and (i) is proved. It follows from Proposition 5.4 (ii), with $\mathbf{H}_{(0)}$ playing the role of $\mathbf{F}$, that $\mathbf{H}_{(0)}$ increases if ν is replaced by an equivalent measure satisfying (5.2). This, together with (8.2), shows that in proving (ii) there is no loss of generality in assuming (5.2). But then (ii) follows easily from Lemma 5.2 and from (8.2).

8.2. *Extension.* For φ in $\gamma\mathbf{F}_{(e)}$ put

$$Q(\varphi, \varphi) = \operatorname{Lim} Q(\varphi_n, \varphi_n)$$

with φ_n as in Proposition 8.3 (ii). The analogue of Lemma 8.1 shows that $Q(\varphi, \varphi)$ is well defined.

Now it is a simple matter to prove the following two theorems.

THEOREM 8.4. (i) $\mathbf{H}_{(0)} = \gamma\mathbf{F}_{(e)} \cap L^2(\nu)$.

(ii) *If* f *belongs to* $\mathbf{F}_{(e)}$, *then* $H_0 f$ *(defined by* (8.1)*) belongs to* $\mathbf{F}_{(e)}$ *and* (8.2) *is valid. Moreover*

$$E(f, f) = E(H_0 f, H_0 f) + E(f - H_0 f, f - H_0 f). \tag{8.3}$$

THEOREM 8.5. φ *in* $\mathbf{H}_{(0)}$ *belongs to the domain of the time changed generator* B *if and only if there exists* ψ *in* $L^2(\nu)$ *such that*

$$E(H_0\varphi, f) = -\int \nu(dy)f(y)\psi(y)$$

for all f *in* $\mathbf{F}_{(e)} \cap L^2(\nu)$. *In this case*

$$B\varphi = \psi.$$

Next we put Theorem 7.6 into a more natural form.

THEOREM 8.6. *(Assume (7.1).) Then for f in* **F** *and $u > 0$*

$$\frac{1}{2}\mathcal{E}_u \int_0^{R_u} \langle M_u f\rangle (dt) 1_D(t) + \frac{1}{2}\mathcal{E}_u \sum_i \{f_u[e(i)] - f_u[e(i) - 0]\}^2 = N(f, f) + E(f - H_0 f, f - H_0 f) + \frac{1}{2} u \int_D dx f^2(x). \tag{8.4}$$

Proof. We remark first that (3.10) leads to

$$E_v(H_v f - H_u f, H_v f - H_u f) = E_v^D([u - v] G_v^D H_u f, H_v f - H_u f) = (u - v) \int_D dx H_u f(x) \{H_v f(x) - H_u f(x)\}$$

which upon passage to the limit $v \downarrow 0$ becomes

$$E(H_0 f - H_u f, H_0 f - H_u f) = u \int_D dx H_u f(x) \{H_0 f(x) - H_u f(x)\}. \tag{8.5}$$

A similar argument establishes

$$E(H_0 f - H_u f, f - H_0 f) = u \int_D dx H_u f(x) \{f(x) - H_0 f(x)\}.$$

Then

$$\begin{aligned} E_u(f - H_u f, f - H_u f) &+ u \int_D dx H_0 f(x) H_u f(x) \\ &= E(f - H_0 f, f - H_0 f) + E(H_0 f - H_u f, H_0 f - H_u f) \\ &\quad + 2E(f - H_0 f, H_0 f - H_u f) + u \int_D dx \{f(x) - H_u f(x)\}^2 \\ &\quad + u \int_D dx H_0 f(x) H_u f(x) \\ &= E(f - H_0 f, f - H_0 f) + u \int dx H_u f(x) \{H_0 f(x) - H_u f(x)\} \\ &\quad + 2u \int_D dx H_u f(x) \{f(x) - H_0 f(x)\} + u \int_D dx \{f(x) - H_u f(x)\}^2 \\ &\quad + u \int_D dx H_0 f(x) H_u f(x) \end{aligned}$$

and we are done.

Remark. It would be interesting to pass to the limit $u \downarrow 0$ on the left side of (8.4) and then extend Theorem 8.6 to general f in $\mathbf{F}_{(e)}$. We can do this in special cases, but we are not sure what is the correct formulation (if any) in general. It seems clear, for example, that the recurrent and transient cases must be handled differently.

In the remainder of the section we assume the following.

8.3. *Condition of Universality.* This condition is satisfied if

(i) Every function φ in the universal Dirichlet space $\mathbf{H}^N$ has a quasi-everywhere defined refinement which is the restriction to M of a function in $\mathbf{F}_{(e)}$.

(ii) There exist positive constants c, C such that

$$cN(\varphi, \varphi) \leq E(H_0\varphi, H_0\varphi) \leq CN(\varphi, \varphi) \quad \text{for } \varphi \text{ in } \mathbf{H}^N. \tag{8.6}$$

Remark. The left hand inequality in (8.6) actually follows from (i) because of Theorem 5.7 and Theorem 8.4. We do not know if the same is true for the right hand inequality. Standard results in this direction are not directly applicable since the relevant inner product spaces are not complete in the usual sense. It is clear from Theorem 6.6 that there exist Dirichlet spaces which fail to satisfy condition 8.3. In this case it is natural to replace the given Dirichlet space $(\mathbf{F}, E)$ by the one constructed in Section 6 with $\mathbf{H}^*_{(0)} = \mathbf{H}^N_{(0)}$ and $Q^* = N$. We discuss this point in Section 9 for Brownian motion.

8.4. *Convention.* Every φ in $\mathbf{H}^N$ is represented by the version specified up to quasi-equivalence in 8.3 (i). Thus $\mathbf{H}^N$ is identified with $\gamma\mathbf{F}_{(e)}$.

Finally we consider again the modified Dirichlet space of Section 6. Note that by convention 8.4, functions in $\mathbf{H}^*_{(0)}$ belong to $\gamma\mathbf{F}_{(e)}$ and are specified up to quasi-equivalence on M.

Lemmas 6.1 through 6.4 are unchanged, but before proceeding beyond that point we expand the function spaces $\mathbf{H}^*_{(0)}$ and $\mathbf{F}^*$.

8.5. Definition. (Assume condition 8.3.) A function φ in $\gamma\mathbf{F}_{(e)}$ belongs to $\mathbf{H}^*$ if there exists a sequence φ_n in $\mathbf{H}^*_{(0)}$ which is Cauchy relative to the Dirichlet norm Q^* and such that $\varphi_n \to \varphi$ q.e. For such φ put

$$Q^*(\varphi, \varphi) = \operatorname{Lim} Q^*(\varphi_n, \varphi_n).$$

Also put

$$Q^*_{(u)}(\varphi, \varphi) = Q^*(\varphi, \varphi) + U_{0,u}(\varphi, \varphi)$$

whenever $U_{0,u}(\varphi, \varphi) < +\infty$.

8.6. Definition. (Assume condition 8.3.) A function f in the extended Dirichlet space $\mathbf{F}_{(e)}$ belongs to $\mathbf{F}^*_{(e)}$ if there exists a sequence f_n in $\mathbf{F}^*$ which is Cauchy relative to the Dirichlet norm E^* and such that $f_n \to f$ q.e.

The extension of Lemma 8.1 and of Lemma 8.2 (i) to both $\mathbf{F}^*_{(e)}$ and $\mathbf{H}^*$ is immediate. The extension of Lemma 8.2 (ii) to $\mathbf{H}^*$ can easily be made by considering $H_0\varphi$ in place of φ and applying the right hand inequality in (8.6).

Now consider $\varphi = R^*_{(u)}\pi^\circ_u f$ with f bounded and integrable. After approximating $\pi^\circ_u f$ from below by functions in $L^2(\nu)$, we conclude from the proof of Lemma 6.5 that φ belongs to $\mathbf{H}^*$, that $H_u\varphi$ belongs to $\mathbf{F}^*$ and that (6.10) is valid. Since φ also belongs to $\gamma\mathbf{F}_{(e)}$ by condition 8.3, it follows easily from the square integrability of $H_u\varphi$ that actually φ belongs to $\gamma\mathbf{F}$ and $H_u\varphi$ belongs to **F**. Then the proof of Lemma 8.3 together with Lemma 6.4 implies that the $H_u\varphi$ are Cauchy relative to E^* as $u \downarrow 0$. Therefore $H_0\varphi$ belongs to $\mathbf{F}^*_{(e)}$ and then a passage to the limit in (6.10) establishes

$$E^*(H_0\varphi, H_0\varphi) = Q^*(\varphi, \varphi) \tag{8.7}$$

and a passage to the limit in Lemma 6.4 establishes

$$E^*(H_0\varphi, h) = 0 \tag{8.8}$$

for h in $\mathbf{F}^D$. Now we are ready to prove

THEOREM 8.7. *(Assume* 6.2 *and* 8.3.*) Let* $(\mathbf{F}^*, E^*)$ *be the Dirichlet space corresponding to the resolvent of Lemma* 6.3. *Then:*

(i) *Lemma* 8.2 (ii) *is valid for* $\mathbf{F}^*_{(e)}$

(ii) F^* *is the subset of* f *in* **F** *such that* γf *belongs to* $\mathbf{H}^*$.

(iii) $\mathbf{F}^*_{(e)}$ *is contained in the subset of* f *in* $\mathbf{F}_{(e)}$ *such that* γf *belongs to* $\mathbf{H}^*$.

(iv) *If* f *belongs to* $\mathbf{F}^*_{(e)}$, *then so does* $H_0 f$ *and* (8.2) *and* (8.3) *are valid with* E *replaced by* E^* *and* Q *by* Q^*. *Also*

$$E^*(f - H_0f, f - H_0f) = E(f - H_0f, f - H_0f).$$

(v) *For* f *in* $\mathbf{F}_{(e)}$,

$$E^*(f, f) = E(f - H_0f, f - H_0f) + Q^*(\gamma f, \gamma f). \tag{8.9}$$

Proof. Equation (8.9) follows from (8.7) and (8.8) for the special case $f = G^*_u g$ with g bounded and integrable and then it is routine to prove (i), (iii), (iv) and (v) with the help of the right hand inequality in (8.6). To prove (ii), consider bounded φ in $\mathbf{H}^* \cap \gamma\mathbf{F}$. Then for $u > 0$, the function $H_u\varphi$ belongs to **F** and from 1.4.1 it follows that $R^*_{(v)a}\varphi$ belongs to $\mathbf{H}^*$ and that

$$U_{0,u}(R^*_{(v)a}\varphi, R^*_{(v)a}\varphi) < +\infty$$

for v, $a > 0$. The proof of Lemma 6.5 shows that for fixed $a > 0$,

$$Q^*_{(u)}(R^*_{(v)a}\varphi, R^*_{(v)a}\varphi) \to 0$$

as $v \uparrow \infty$ and it follows from the analogue of (6.8) for $a > 0$ that

$$(v - u)R^*_{(u)a}\pi^\circ_u H_v R^*_{(v)a}\varphi$$

converges to $R^*_{(u)a}\varphi$ as $v \uparrow \infty$ relative to the $Q^*_{(u)}$ norm. After approximating $H_v R^*_{(v)a}\varphi$ from below by integrable functions on **X**, we conclude from (6.10) and from the first part of the proof that $H_u R^*_{(u)a}\varphi$ belongs to $\mathbf{F}^*$ and that (6.10) is valid with φ replaced by $R^*_{(u)a}\varphi$. It follows upon approximating φ by $aR^*_{(u)a}\varphi$, again with the help of 1.4.1, that $H_u\varphi$ belongs to $\mathbf{F}^*$. But this implies (ii) since it suffices to consider f bounded and since $f - H_u f$ belongs to $\mathbf{F}^D$ and therefore to $\mathbf{F}^*$.

One important consequence of Theorem 8.7 is that the Dirichlet space $(\mathbf{F}^*, E^*)$ depends only on $\mathbf{H}^*$ and Q^* and not on the measure ν. We do not know if $\mathbf{F}^*_{(e)}$ is actually equal to the subset of f in $\mathbf{F}^*_{(e)}$ such that γf belongs to $\mathbf{H}^*$. This would follow if we knew that general φ in $\mathbf{H}^*$ could be approximated by φ_n in $\mathbf{H}^* \cap \gamma\mathbf{F}$. Proposition 5.4 (i) implies that this is true for $\mathbf{H}^* = \mathbf{H}$.

We complete this section by describing the generator A^* which corresponds to the Dirichlet space $(\mathbf{F}^*, E^*)$. To facilitate comparison with the work of other authors, we introduce

8.7 DEFINITION. f in $\mathbf{F}$ belongs to the domain of the local generator $\mathfrak{A}$ if there exists g in $L^2(D, dx)$ such that

$$E(f, h) = -\int_D dx g(x) h(x)$$

whenever h belongs to $\mathbf{F}^D$. In this case

$$\mathfrak{A} f = g.$$

THEOREM 8.8. *f in $\mathbf{F}$, belongs to the domain of the generator A^* if and only if:*

8.8.1. *γf belongs to the extended Dirichlet space $\mathbf{H}^*$.*

8.8.2. *f belongs to the domain of the local generator $\mathfrak{A}$.*

8.8.3. *There exists ψ in $L^2(M, dx)$ such that for φ in $\mathbf{H}^* \cap \gamma\mathbf{F}$,*

$$Q^*(\gamma f, \varphi) = -\int_M dx\psi(x)\varphi(x) - \operatorname{Lim} \int_D dx \mathfrak{A} f(x) H_u \varphi(x) \tag{8.11}$$

with the limit taken as $u \downarrow 0$.

In this case

$$\begin{aligned} A^* f(x) &= \mathfrak{A} f(x) \quad \text{on } D \\ &= \psi(x) \quad \text{on } M. \end{aligned}$$

Proof. The direct part of the theorem follows from Theorem 8.7 (i) and from

$$E^*(f, h) = -\int dx A^* f(x) h(x), \quad h \text{ in } \mathbf{F}^*,$$

upon applying (8.9) with h in $\mathbf{F}^D$ and then with $h = H_u \varphi$ and passing to the limit with the help of (8.5) and Lemma 5.5. To establish the converse, note that by Theorem 8.7 (ii) f belongs to $\mathbf{F}^*$ and then for h in $\mathbf{F}^*$,

$$\begin{aligned} E^*(f, h) &= E(f - H_0 f, h - H_0 h) + Q^*(\gamma f, \gamma h) \\ &= \operatorname{Lim} E(f, h - H_u h) - \int_M dx\psi(x)h(x) - \operatorname{Lim}\int_D dx \mathfrak{A} f H_u h(x) \\ &= -\operatorname{Lim}\int_D dx \mathfrak{A} f(x)\{h(x) - H_u h(x)\} \\ &\qquad - \int_M dx\psi(x)h(x) - \operatorname{Lim}\int_D dx\mathfrak{A} f(x) H_u h(x) \\ &= -\int_D dx \mathfrak{A} f(x) h(x) - \int_M dx\psi(x) h(x) \end{aligned}$$

and the proof is complete.

Remark. We cannot justify passing to the limit under the integral sign in (8.11) since in general we cannot restrict our attention to the case when $\mathfrak{A}f \geq 0$.

9. Brownian motion

Throughout this section, **X** is d-dimensional Euclidean space $\mathbf{R}^d$ and dx is standard Lebesgue measure. We introduce the familiar functions

$$P^t(x) = (2\pi t)^{-d/2} \exp\{-|x|^2/2t\}$$

$$G_u(x) = \int_0^\infty dte^{-ut}P^t(x)$$

for u, $t > 0$ and for x in $\mathbf{R}^d$. Of course $|x|$ denotes the standard Euclidean norm. The operators

$$G_u f(x) = \int dyG_u(x-y)f(y)$$

form a symmetric Markovian resolvent on $L^2(dx)$. The corresponding transition operators P^t are

$$P^t f(x) = \int dyP^t(x-y)f(y).$$

The generator A is the L^2 closure of $\frac{1}{2}\Delta$ applied to any reasonable class of smooth functions. Of course Δ is the usual Laplacian operator on $\mathbf{R}^d$.

The probabilities $\mathcal{P}_x$ of Theorem 2.4 can be defined for all x in $\mathbf{R}^d$ and the regularity properties which are established in Section 2 using the potential theory of Section 1 can be verified by direct arguments involving only the properties of $P(x)$. Moreover the exceptional x-set is always empty and the probabilities $\mathcal{P}_x$ are concentrated on the subset of ω in Ω such that $X(t, \omega)$ is everywhere continuous and such that the life time $\zeta(\omega) = +\infty$. The collection of probabilities is generally referred to as *standard d-dimensional Brownian motion.* For verification of the above statements and other properties of Brownian motion used below, we refer the reader to any of the books [7, 17, or 21].

We begin by noting that for f in $C^2_{\text{com}}(\mathbf{R}^d)$ (twice continuously differentiable with compact support) the defining relation (7.2) becomes

$$M_u f(t) = f_u(t) + \int_0^t ds\left\{uf_u(s) - \frac{1}{2}\Delta f_u(s)\right\}.$$

Application of Ito's formula [21, p. 32] yields the alternative formula

$$\begin{aligned} M_u f(t) = f(0) - I[R_u < t]f(R_u) + u\int_0^{\min(R_u,t)} dsf(s) \\ + \int_0^{\text{mi }(R_u,t)} \operatorname{grad} f(s)\cdot dX(s). \end{aligned} \tag{9.1}$$

The last integral is an example of a *stochastic integral in the sense of Ito.* A tedious but routine calculation involving the calculus of stochastic integrals shows that for $s > 0$ and $t \geq 0$,

$$\mathfrak{F}_t\{M_u f(t+s) - M_u f(t)\}^2 = \mathfrak{F}_t \int_{\min(R_u, t)}^{\min(R_u, t+s)} dr\ \{[\operatorname{grad} f(r)]^2 + u[f(r)]^2\}. \tag{9.2}$$

Again we are using Hunt's notation for conditional expectation. Notice that this conditional expectation can be relative to any of the $\mathcal{P}_x$. Equation (9.2) implies

$$\langle M_u f\rangle(t) = \int_0^{\min(R_u, t)} ds\ \{[\operatorname{grad} f(s)]^2 + u[f(s)]^2\} \tag{9.3}$$

and in particular

$$\langle M_u f\rangle(\infty) = \int_0^{R_u} ds\ \{[\operatorname{grad} f(s)]^2 + u[f(s)]^2\} \tag{9.4}$$

and therefore

$$\frac{1}{2}\,\mathcal{E}_u\{M_u f(\infty)\}^2 = \int dx \left\{\frac{1}{2}[\operatorname{grad} f(x)]^2 + u[f(x)]^2\right\}. \tag{9.5}$$

Theorem 7.2 establishes

$$E(f, f) = \frac{1}{2}\int dx\ [\operatorname{grad} f(x)]^2 \tag{9.6}$$

for f in $C^2_{\text{com}}(\mathbf{R}^d)$. A passage to the limit using the fact that such functions are necessarily E_u dense in $\mathbf{F}$ (see 1.4.2) shows that $\mathbf{F}$ is the Sobolev space $\mathbf{W}$ of square integrable functions on $\mathbf{R}^d$ which have square integrable gradients in the distribution sense and that (9.1) through (9.6) are valid for f in $\mathbf{W}$. It would be interesting to establish in general the analogue of (9.5) for $u = 0$. Hunt's construction of approximate h-chains in [15, pp. 334, 335] shows how to do this for the transient cases $d \geq 3$, but the recurrent cases $d = 1$ or 2 seem to require new ideas.

In the remainder of the section we use the notation of Sections 5 through 8 corresponding to ν in $\mathfrak{M}^e$. For the sake of simplicity and to avoid certain technical complications, we introduce

9.1. *Restriction.* M is Cl (G), ∂G or $\mathbf{R}^d - G$ where G is an open subset of $\mathbf{R}^d$ whose Euclidean boundary ∂G has Lebesgue measure zero and such that both G and ext $(G) = \mathbf{R}^d -$ Cl (G) are connected.

From the continuity of trajectories it follows that the harmonic measures $H_u(x, dy)$ are, for x in G and for x in ext (G) concentrated on the boundary ∂G. From the easily verified fact that nonnegative u-harmonic functions in a connected open set can vanish at a point only if they vanish identically, it follows that the harmonic measures $H_u(x, dy)$ for $u \geq 0$ are equivalent as x runs over G and as x runs over ext (G). Therefore we fix reference points x_i

in G and x_e in ext (G) and introduce the notations

$$H^i(dy) = H_0(x_i, dy), \qquad H^e(dy) = H_0(x_e, dy)$$

$$\begin{aligned} H_u(x, dy) &= H_u(x, y)H^i(dy) \quad \text{for } x \text{ in } G \\ &= H_u(x, y)H^e(dy) \quad \text{for } x \text{ in ext } (G). \end{aligned}$$

The functions $H_u(x, y)$ are understood to be specified up to H^i or H^e equivalence on ∂G. It is known that H^e and H^i are equivalent if G has a smooth enough boundary. We doubt that this is true in general, but do not have a counterexample. We introduce the function

$$p(x) = \mathcal{E}_x[\sigma(\partial G) = +\infty]$$

and then

$$p^{*i}(y) = \int_G dx p(x) H_1(x, y)$$

$$p^{*e}(y) = \int_{\text{ext}(G)} dx p(x) H_1(x, y).$$

It follows from the analogue of (5.13) with $v = 0$ and from Theorem 5.8 that the functions

$$U^i_{0,\infty}(y, z) = \text{Lim } u \int_G dx H_0(x, y) H_u(x, z)$$

$$U^e_{0,\infty}(y, z) = \text{Lim } u \int_{\text{ext}(G)} dx H_0(x, y) H_u(x, z)$$

with the limit taken for any sequence of $u \uparrow \infty$, exist and are finite [a.e. $H^i \times H^i$] and [a.e. $H^e \times H^e$] respectively. The kernel $U_{0,\infty}(y, z)$ is the analogue of the one introduced by Feller in [9, p. 559] for the Kolmogorov equations and by L. Naïm in [27, Chap. III, 15] for Brownian motion on the Martin closure.

We now have all of the ingredients for a general formulation of Green's identity.

THEOREM 9.1. (i) *If f belongs to $\mathbf{W}_{(e)}$, then*

$$H_0 f(x) = \mathcal{E}_x f[\sigma(\partial G)] \tag{9.7}$$

is defined q.e. *and belongs to* $\mathbf{W}_{(e)}$ *and*

$$\begin{aligned} \frac{1}{2}\int_G dx \{\text{grad } f(x)\}^2 &= \frac{1}{2}\int_G dx \{\text{grad } f(x) - \text{grad } H_0 f(x)\}^2 \\ &+ \frac{1}{2}\int H^i(dy) \int H^i(dz) U^i_{0,\infty}(y, z)\{f(y) - f(z)\}^2 \\ &+ \int H^i(dy) p^{*i}(y) f^2(y), \end{aligned} \tag{9.8}$$

$$\frac{1}{2}\int_{\text{ext}(G)} dx\,\{\text{grad}\, f(x)\}^2 = \frac{1}{2}\int_{\text{ext}(G)} dx\,\{\text{grad}\, f(x) - \text{grad}\, H_0 f(x)\}^2$$

$$(9.9) \qquad + \frac{1}{2}\int H^e(dy)\int H^e(dz) U^e_{0,\infty}(y,z)\{f(y) - f(z)\}^2$$

$$+ \int H^e(dy) p^{*e}(y) f^2(y).$$

(ii) *If* φ *specified up to* H^i *equivalence on* ∂G *satisfies*

$$(9.10) \qquad \frac{1}{2}\int H^i(dy)\int H^i(dz) U^i_{0,\infty}(y,z)\{\varphi(y) - \varphi(z)\}^2 + \int H^i(dy) p^{*i}(y)\varphi^2(y) < +\infty$$

then

$$(9.11) \qquad H_0\varphi(x) = \mathcal{E}_x \varphi[\sigma(\partial G)]$$

is defined and harmonic for x in G *and*

$$\frac{1}{2}\int_G dx\,\{\text{grad}\, H_0\varphi(x)\}^2$$

$$(9.12) \qquad = \frac{1}{2}\int H^i(dy)\int H^i(dz) U^i_{0,\infty}(y,z)\{\varphi(y) - \varphi(z)\}^2$$

$$+ \int H^i(dy) p^{*i}(y)\varphi^2(y).$$

(iii) *If* φ *specified up to* H^e *equivalence on* ∂G *satisfies* (9.10) *with i replaced by e, then* (9.11) *is defined and harmonic for x in ext* (G) *and* (9.12) *is satisfied with G replaced by* ext (G) *and with i replaced by e.*

Proof. To prove (ii), choose ν in $\mathfrak{M}^e$ such that $M = \mathbf{R}^d - G$ and apply Theorem 7.4 (together with familiar facts about the connection between harmonic functions and Brownian motion) to conclude that $H_0\varphi$ is defined and harmonic on G and that the right side of (9.12) equals

$$\frac{1}{2}\mathcal{E}_u\{H_0\varphi[r(0)] - H_0\varphi[0]\}^2 + \frac{1}{2}\mathcal{E}_u \sum_i \{H_0\varphi[r(i)] - H_0\varphi[e(i)]\}^2$$

in the notation of Section 7. Now (9.12) can be established by applying a simple extension of (9.3) with $u = 0$ together with the techniques used to prove Theorem 7.3. Conclusion (iii) follows from (ii) and the symmetric role of G and ext (G). Finally (i) follows from Theorem 5.8, from Theorem 8.4 and from (ii) and (iii).

Theorem 9.1 suggests the following question. If φ is specified and finite up to $H^i + H^e$ equivalence on ∂G and if φ satisfies both (ii) and (iii), does φ have a quasi-everywhere defined refinement which is the restriction to ∂G of a function in $\mathbf{F}_{(e)}$. This is true if and only if φ in $\mathbf{H}^N$ can be approximated in the sense of Definition 8.1 by functions in $\gamma\mathbf{F}_{(e)}$ which is certainly the case when G is a half space. We do not know about the general situation. Notice that a yes answer is equivalent to the condition of universality 8.3 for the Dirichlet space $\mathbf{W}_{(e)}$.

9.2. *Convention.* Throughout this section the functions $H_0\varphi$ and $H_0 f$ are defined by (9.8) and (9.11) no matter what ν in $\mathfrak{M}^e$ is being considered.

Theorem 8.4 becomes in the present context.

THEOREM 9.2. (i) *Let ν in $\mathfrak{M}^e$ be such that $M = \partial G$. Then $\mathbf{H}_{(0)} = \gamma\mathbf{W}_{(e)} \cap L^2(\nu)$ and*

$$(9.13)\quad \begin{aligned} Q(\varphi, \varphi) &= \frac{1}{2}\int H^i(dy)\int H^i(dz)U^i_{0,\infty}(y,z)\{\varphi(y) - \varphi(z)\}^2 \\ &\quad + \frac{1}{2}H^e(dy)\int H^e(dz)U^e_{0,\infty}(y,z)\{\varphi(y) - \varphi(z)\}^2 \\ &\quad + \int H^i(dy)p^{*i}(y)\varphi^2(y) + \int H^e(dy)p^{*e}(y)\varphi^2(y) \end{aligned}$$

for φ in $\mathbf{H}_{(0)}$.

(ii) *Let ν in $\mathfrak{M}^e$ be such that $M = \mathbf{R}^d - G$. Then*

$$\mathbf{H}_{(0)} = \gamma\mathbf{W}_{(e)} \cap L^2(\nu)$$

and

$$(9.14)\quad \begin{aligned} Q(\varphi, \varphi) &= \frac{1}{2}\int_{\mathrm{ext}(G)} dx\,\{\operatorname{grad}\varphi(x)\}^2 \\ &\quad + \frac{1}{2}\int H^i(dy)\int H^i(dz)U^i_{0,\infty}(y,z)\{\varphi(y) - \varphi(z)\}^2 \\ &\quad + \int H^i(dy)p^{*i}(y)\varphi^2(y) \end{aligned}$$

for φ in $\mathbf{H}_{(0)}$.

(iii) *Let ν* in *$\mathfrak{M}^e$ be such that $M = \mathbf{R}^d$. Then*

$$\mathbf{H}_{(0)} = \mathbf{W}_{(e)} \cap L^2(\nu)$$

and

$$(9.15)\quad Q(\varphi, \varphi) = \frac{1}{2}\int dx\,\{\operatorname{grad}\varphi(x)\}^2 \quad \textit{for } \varphi \textit{ in } \mathbf{H}_{(0)}.$$

We finish by discussing the modified Dirichlet space and resolvent of Section 6, considering first the case $M = \partial G$. The universality condition 8.3 is satisfied and therefore Theorem 8.7 is directly applicable only if we can be sure that actually $\mathbf{H}^N = \gamma\mathbf{F}$. (See the discussion above.) In this case Theorem 8.7 characterizes Dirichlet spaces $(\mathbf{F}^*, E^*)$ relative to $L^2(\mathbf{R}^d, dx)$ which are formed by combining in the manner of Section 6 the Dirichlet space for

Brownian motion "killed" upon hitting ∂G with a Dirichlet space relative to $L^2(H^i + H^e)$. It follows from (6.7) that the corresponding resolvent operators G_u^* satisfy

$$G_u^* f(x) = G_u^D f(x) + \mathcal{E}_x \exp\{-u\sigma(M)\} G_u^* f[\sigma(M)]. \tag{9.16}$$

The converse is also true. Every symmetric resolvent $\{G_u^*, u > 0\}$ on $L^2(\mathbf{R}^d, dx)$ which satisfies (9.16) arises in this manner from a unique Dirichlet space relative to $L^2(H^i + H^e)$. This follows from Theorem 5.8 applied to $(\mathbf{F}^*, E^*)$ rather than to $(\mathbf{F}, E)$. Of course we cannot apply Theorem 5.8 directly since $(\mathbf{F}^*, E^*)$ need not be regular, but there is no difficulty modifying our arguments to establish an appropriate version. Thus our results classify symmetric resolvents on $L^2(\mathbf{R}, dx)$ which satisfy (9.16). If condition 8.3 fails, then there is still a classification with $\gamma \mathbf{F}_{(e)}$ replaced by $\mathbf{H}^N$. But our arguments provide only a weakened version of Theorem 8.7.

To classify symmetric resolvents on $L^2(G, dx)$ satisfying (9.16) it is necessary to replace the Dirichlet space $\mathbf{W}$ by a different one.

9.3. A function f specified and finite up to quasi-equivalence on G belongs to the Dirichlet space $\mathbf{W}(G)$ if it satisfies the following three conditions.

(i) f is square integrable on G.

(ii) f has a square integrable gradient on G in the distribution sense.

(iii) There exists a function φ specified up to H^i equivalence on ∂G such that $f = H_0 \varphi$ q.e. on G.

According to Theorem 7.4 a function f belongs to $\mathbf{W}(G)$ if and only if $f = H_0\varphi$ q.e. on G where φ satisfies (9.10) and in this case (9.12) is true. It follows that $\mathbf{W}(G)$ together with the bilinear form

$$\frac{1}{2}\int_G dx\, \{\operatorname{grad} f(x)\}^2$$

is indeed a Dirichlet space relative to $L^2(G, dx)$. For the Martin closure this follows from results of Doob in [5]. Consideration of special cases suggests that the corresponding process should be regarded as reflecting Brownian motion on the Euclidean closure cl (G). Our results, with this Dirichlet space playing the role of $(\mathbf{F}, E)$, classify symmetric resolvents on $L^2(G, dx)$ which satisfy (9.16). Again we must modify our arguments since in general we cannot be sure that this Dirichlet space is regular.

Condition (iii) can be removed and (9.16) can be replaced by the more intrinsic requirement that $G_u^* f - G_u^D f$ be u-harmonic on G if we replace the Euclidean closure of G by the Martin closure. This is done for bounded G by Fukushima in [14] using different techniques.

Appendix

We consider a Dirichlet space $(\mathbf{F}, E)$ relative to $L^2(dx)$ such that $\mathbf{F}$ is dense in $L^2(dx)$ and show how to replace it by a regular Dirichlet space without changing any of the relevant structure. Our construction is only slightly different from the one given by Fukushima in [12].

Note first that if f in **F** is [a.e. dx] bounded then f^2 belongs to **F** and so the subcollection of f in **F** which are integrable and [a.e. dx] bounded form an algebra. From contractivity it follows easily that this algebra is dense in **F**. Moreover **F** itself (being the domain of a self adjoint operator) is separable and, putting everything together, we conclude that there exists a subset $\mathbf{B}_0$ of **F** satisfying:

A.1.1. $\mathbf{B}_0$ is countable.

A.1.2. $\mathbf{B}_0$ is a subalgebra over the rationals.

A.1.3. Every member of $\mathbf{B}_0$ is integrable and bounded [a.e. dx].

A.1.4. $\mathbf{B}_0$ is dense in **F** (and therefore in $L^2(dx)$).

A.1.5. If 1 belongs to **F**, then also 1 belongs to $\mathbf{B}_0$. Otherwise there exists a sequence $g_n \geq 0$ in $\mathbf{B}_0$ such that $g_n \uparrow 1$ [a.e. dx].

Fix one such $\mathbf{B}_0$ and let **B** be the uniform closure (in the [a.e. dx] sense) of $\mathbf{B}_0$. Then **B** is in a natural way a commutative Banach algebra. We now introduce the Gelfand transform of **B**. For the reader's convenience we develop the relevant part of the theory for this simple special case rather than appeal to the general theory.

Let **Y** be the collection of real valued functions γ on **B**, *not identically zero*, which satisfy:

A.2.1. $\gamma(f) \leq \|f\|_\infty$.

A.2.2. $\gamma(fg) = \gamma(f)\gamma(g)$.

A.2.3. $\gamma(af + bg) = a\gamma(f) + b\gamma(g)$.

We give **Y** the weakest topology which makes continuous all of the real-valued functions F on **Y** which can be represented $F(\gamma) = \gamma(f)$ for some f in **B**. It is well known and easy to verify directly that **Y** is then a separable locally compact Hausdorff space and is compact if and only if 1 belongs to **B**.

Once and for all choose everywhere defined measurable versions for f in $\mathbf{B}_0$. Let $\mathbf{X}_0$ be the subset of x in **X** satisfying:

A.3.1. $|f(x)| \leq \|f\|_\infty$ for f in $\mathbf{B}_0$.

A.3.2. $(fg)(x) = f(x)g(x)$ for f, g in $\mathbf{B}_0$.

A.3.3. $(f + g)(x) = f(x) + g(x)$ for f, g in $\mathbf{B}_0$.

A.3.4. $(af)(x) = af(x)$ for a rational and f in $\mathbf{B}_0$.

Then $\mathbf{X}_0$ is a Borel measurable subset of **X** of full measure. That is,

$$\int_{\mathbf{X}-\mathbf{X}_0} dx = 0.$$

Clearly for each x in **X** there exists a unique γ in **Y** such that $\gamma(f) = f(x)$ for f in $\mathbf{B}_0$. Thus there exists a unique mapping $J : \mathbf{X}_0 \to \mathbf{Y}$ such that

$$(Jx)(f) = f(x) \tag{A.1}$$

for f in $\mathbf{B}_o$ and x in $\mathbf{X}$. Clearly J is Borel measurable and so there exists a unique measure $d\gamma$ on $\mathbf{Y}$ such that

$$\int_{\mathbf{Y}} d\gamma\ \varphi(\gamma) = \int_{\mathbf{X}} dx\ \varphi(Jx) \tag{A.2}$$

for nonnegative Borel φ in $\mathbf{Y}$. We use the same symbol J to denote the natural mapping of $\mathbf{B}$ onto $C(\mathbf{Y})$ (continuous functions vanishing at ∞) given by

$$Jf(\gamma) = \gamma(f). \tag{A.3}$$

It follows from (A.3) that for f in $\mathbf{B}$ and for any polynomial P

$$JP(f) = P(Jf). \tag{A.4}$$

Since any normalized contraction T is continuous and therefore can be uniformly approximated by polynomials on compact sets, we have also

$$JT(f) = T(Jf). \tag{A.5}$$

Using $\max(f, g) = \frac{1}{2}|f + g| + \frac{1}{2}|f - g|$ and a similar formula for $\min(f, g)$, we conclude from (A.5) that

$$J \max(f, g) = \max(Jf, Jg) \tag{A.6}$$

$$J \min(f, g) = \min(Jf, Jg) \tag{A.7}$$

for f, g in $\mathbf{B}$. Next we show that J can be extended from $\mathbf{B} \cap L^2(dx)$ to an isometry from $L^2(dx)$ onto $L^2(\gamma)$. For this purpose note that the relation

$$\int d\gamma\ Jf(\gamma)Jg(\gamma) = \int dx\ f(x)g(x) \tag{A.8}$$

for f, g in $\mathbf{B}_0$ is an immediate consequence of $(A.4)$ and (A.2). Denote by J^* the unique isometry of $L^2(dx)$ into $L^2(d\gamma)$ which agrees with J on $\mathbf{B}_o$. Our problem then is to show that $J = J^*$ on $L^2(dx) \cap \mathbf{B}$. Equations (A.1) and (A.3) imply that $\| Jf \|_\infty = \| f \|_\infty$ for f in $\mathbf{B}_0$ and therefore $J(\mathbf{B}_o)$ is uniformly dense in $C(\mathbf{Y})$. With g_n as in condition A.1.5 (put $g_n = 1$ for all n if 1 is in $\mathbf{B}_0$) the functions Jg_n increase to 1 [a.e. $d\gamma$] on $\mathbf{Y}$. Therefore functions of the form $Jg_n(\gamma)\varphi(\gamma)$ with φ in $C(\mathbf{Y})$ are dense in $L^2(d\gamma)$ and it follows by the dominated convergence theorem that $J\mathbf{B}_0$ is dense in $L^2(d\gamma)$. Now fix f in $\mathbf{B} \cap L^2(dx)$ and let f_n in $\mathbf{B}_0$ converge [a.e. dx] uniformly to f. The dominated convergence theorem implies that

$$\int dx\ f(x)h(x) = \text{Lim} \int dx\ f_n(x)h(x) \quad (n \uparrow \infty)$$

and similarly

$$\int d\gamma\ Jf(\gamma)Jh(\gamma) = \text{Lim} \int d\gamma\ Jf_n(\gamma)Jh(\gamma) \quad (n \uparrow \infty)$$

for h in $\mathbf{B}_0$. Thus (A.8) is valid for f and h and since $J\mathbf{B}_0$ is dense in $L^2(d\gamma)$, certainly $J^*f = Jf$ for f in $\mathbf{B} \cap L^2(dx)$ as required. It is easy to check that the pair $(J\mathbf{F}, JE)$ is a Dirichlet space relative to $L^2(d\gamma)$. Of course JE is the bilinear form on $J\mathbf{F}$ determined by

$$JE(Jf, Jg) = E(f, g).$$

To establish regularity, it only remains to check that $d\gamma$ is dense. For this it suffices to show that

$$\text{meas} \bigcap_{i=1}^{n} \{x : |f_i(x) - \gamma(f_i)| < \varepsilon\} > 0 \tag{A.9}$$

for any choice of $\varepsilon > 0$, of $f_1, \cdots, f_n$ in $\mathbf{B}$ and of γ in $\mathbf{Y}$. If (A.9) is false, then there exist polynomials P_m in n indeterminates such that $P_m(f_1, \cdots, f_n)$ converges uniformly to

$$\min_{i=1}^{n} |f_i - \gamma(f_i)|^{-1}$$

and so there exists g in $\mathbf{B}$ such that

$$g \min_{i=1}^{n} |f_i - \gamma(f_i)| = 1$$

and therefore

$$hg \min_{i=1}^{n} |f_i - \gamma(f_i)| = h$$

for all h in $\mathbf{B}$. Since the function min $(x_1, \cdots, x_n)$ can be uniformly approximated on bounded sets by polynomials which do not contain the constant term

$$\gamma(\min_{i=1}^{n} |f_i - \gamma(f_i)|) = 0$$

and therefore $\gamma(h) = 0$ for all h in $\mathbf{B}$, which possibility has been ruled out by hypothesis. This proves regularity and we are done except for checking that (A.5) through (A.7) can be extended to square integrable f and g.

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University of Southern California
Los Angeles, California