THE UNBOUNDED BIDUAL OF $C(X)$

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1. Introduction

In a series of papers, we have studied the Riesz space $C(X)$ of real continuous functions on a compact space X, its dual $L(X)$—the space of Radon measures on X—and its bidual $M(X)$. For each Radon measure μ, $\mathfrak{L}^1(\mu)$ appears as a band (hence direct summand) of $L(X)$, and $\mathfrak{L}^\infty(\mu)$ appears as a band of $M(X)$. A good part of our work has been on the extension to $L(X)$ and $M(X)$ of the results in integration theory on $\mathfrak{L}^1(\mu)$ and $\mathfrak{L}^\infty(\mu)$ for fixed μ.

Now another important space determined by each μ is the space $\mathfrak{M}(\mu)$ of μ-measurable functions modulo those vanishing μ-almost everywhere. The question arises: what is the relation of $\mathfrak{M}(\mu)$ to $\mathfrak{L}^1(\mu)$ and $\mathfrak{L}^\infty(\mu)$, and what space plays the corresponding role to $L(X)$ and $M(X)$?. In [2] (cf. also [3]), Luxemburg and Masterson have given a general answer. For every archimedean Riesz space E, they define and study the space $\Gamma(E)$ of "unbounded" continuous linear functionals on E: each $\phi \epsilon \Gamma(E)$ is an order-continuous linear functional on an order-dense ideal of E, which is maximal in the sense that ϕ cannot be extended to a larger ideal. For $E = \mathfrak{L}^1(\mu)$, $\Gamma(E)$ can be identified with $\mathfrak{M}(\mu)$. They point out that $\Gamma(E)$ is isomorphic with Nakano's space of dilatators on E, but for our purposes, $\Gamma(E)$ is adequate and simpler.

In the present paper, we study $\Gamma(L(X))$, which we deonte by $\mathfrak{M}(X)$. $\mathfrak{M}(X)$ contains $M(X)$ as a dense ideal, and for each μ, $\mathfrak{M}(\mu)$ is the closure in $\mathfrak{M}(X)$ of the band $\mathfrak{L}^\infty(\mu)$ of $M(X)$. Most of the paper is devoted to obtaining the subspace $\mathfrak{U}(X)$ consisting of the "universally measurable" elements, and to establishing the following two characterizations of these elements: (1) they are the elements of $\mathfrak{M}(X)$ which are limits of nets of C; (2) they are the elements of $\mathfrak{M}(X)$ for which a general Lusin theorem holds.

Accomplishing this requires a surprising amount of work. One reason is that the standard order-convergence does not suffice for our purpose. This is already foreshadowed in the fact that on the Riesz space R^X of all functions on X (which appears as a band in $\mathfrak{M}(X)$), order-convergence does not include pointwise convergence. The appropriate order convergence is one defined by Nakano [4]. We can give it a relatively simple form in $\mathfrak{M}(X)$ because of the existence of the weak order unit **1**, and in this form it enables us to obtain the above theorems.

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2. Preliminaries

Let E be a Riesz space (that is, a vector lattice). A set A in E will be called *bounded* if it is contained in some interval $[a, b] = \{c \epsilon E \mid a \leq c \leq b\}$. E will be called *Dedekind complete* if the supremum $\bigvee A$ and the infimum $\bigwedge A$ exist for every bounded set A. It is called *universally complete* if every set of mutually disjoint positive elements has a supremum.

When we speak of a *net* $\{a_\alpha\}$ in E, we will mean that the set of indices is directed, and we will denote its order relation by $<$. A net $\{a_\alpha\}$ is *ascending* (resp. *descending*) if for every pair of indices α, β, $\alpha < \beta$ implies $a_\alpha \leq a_\beta$ (resp. $a_\alpha \geq a_\beta$). The notation $a_\alpha \uparrow a$ means that $\{a_\alpha\}$ is ascending and $a = \bigvee_\alpha a_\alpha$; and similarly for $a_\alpha \downarrow a$. A net $\{a_\alpha\}$ *converges* to a if there exists a net $\{b_\alpha\}$ such that $b_\alpha \downarrow 0$ and $| a - a_\alpha | \leq b_\alpha$ for all α. We denote this convergence by $a = \lim_\alpha a_\alpha$ or $a_\alpha \rightarrow a$.

A subset A of E will be called *closed* if for every net $\{a_\alpha\}$ in A, $\lim_\alpha a_\alpha = a$ implies $a \epsilon A$. Given *any* set A, the smallest closed set containing A will be called the *closure* of A and denoted by $\bar{A}$.

An ideal I of E is a linear subspace with the property that $a \epsilon I$, $| b | \leq | a |$ implies $b \epsilon I$. The closure $\bar{I}$ of an ideal I is related to I in a simple manner: every $a \epsilon \bar{I}_+$ (the positive cone of $\bar{I}$) is the supremum of the set of elements in I_+ below it. It follows every $a \epsilon \bar{I}$ is the limit of some net in I. If $\bar{I} = E$, I is said to be *dense* in E.

If an ideal I in E has a complementary ideal J, that is, $E = I \oplus J$, then I will be called a *band*. If J exists, it is uniquely determined; hence in the decomposition $a = a_I + a_J$, the component a_I of a is uniquely determined by I. Otherwise stated, if I is a band, then we have a canonical projection of E onto I. We will denote the image of any set A under this projection by A_I: $A_I = \{a_I \mid a \epsilon A\}$. The projection preserves suprema (and infima): $c = \bigvee A$ implies $c_I = \bigvee A_I$. In particular, $(a \vee b)_I = a_I \vee b_I$, whence $(a^+)_I = (a_I)^+$, $(a^-)_I = (a_I)^-$, and $| a |_I = | a_I |$.

Given a set A in E, we denote by A' the set of elements disjoint from A: $A' = \{b \epsilon E \mid | b | \wedge | a | = 0 \text{ for all } a \epsilon A\}$. A' is a closed ideal. If $E = I \oplus J$, then $J = I'$ and $I = J'$. Thus a band is closed. The Riesz Theorem states that *if E is complete*, then, conversely, every closed ideal I in E is a band: $E = I \oplus I'$. Moreover, for $a \epsilon E_+$,

$$a_I = \bigvee\{b \epsilon I \mid 0 \leq b \leq a\}.$$

If a band I is generated by a single element b (that is, I is the smallest closed ideal containing b), then in place of the symbols a_I and A_I we will often use the symbols a_b and A_b.

A (real) linear functional ϕ on E will be called *bounded* if it is bounded on every bounded set of E. It will be called *continuous* if for every net $\{a_\alpha\}$ in E, $a = \lim_\alpha a_\alpha$ implies $\langle a, \phi \rangle = \lim_\alpha \langle a, \phi \rangle$.

Throughout the present paper, X is a fixed compact space, C is the Banach lattice of (real) continuous functions on X, L is its dual, and M its bidual.

M is not only the set of norm-continuous linear functionals on L, it is in fact the set of continuous linear functionals in the order sense defined above. Both L and M are complete Riesz spaces, hence a closed ideal in either one is a band. Moreover, for every decomposition $L = I \oplus I'$ of L into bands, the corresponding decomposition $M = (I')^{\perp} \oplus I^{\perp}$ is also into bands. We will call $(I')^{\perp}$ the band in M *dual* to I. It is of course the dual of I in the ordinary Banach space sense.

L is the space of Radon measures on X. Given $\mu \in L$, if I is the band generated by μ, then as we stated above, for any $\nu \in L$ or $A \subset L$, ν_I and A_I will be denoted by ν_μ and A_μ. In particular, $I = L_\mu$, and we will write it in the latter form. Moreover, we will denote the band $(I^{\perp})'$ in M dual to I by M_μ; and for any $f \in M$ or $A \subset M$, f_{M_μ} and A_{M_μ} will be written f_μ and A_μ. This is because in integration theory, the properties of L_μ and M_μ are studied entirely in terms of μ. Specifically, by the Radon-Nikodym theorem, L_μ can be identified with $\mathcal{L}^1(\mu)$, hence M_μ can be identified with $\mathcal{L}^\infty(\mu)$.

We always consider X as a subset of L. It is easily shown that the linear subspace generated by X is an ideal, and that its *norm*-closure is a band. This band is the space of atomic Radon measures on X; we will denote it by L_a. L_a' is the space of diffuse Radon measures on X; we will denote it by L_d. So $L = L_a \oplus L_d$. (In our previous papers, we used the notation L_0 and L_1.) We denote the bands in M dual to L_a and L_d by M_a and M_d respectively. Thus $M = M_a \oplus M_d$. L_a is isomorphic to $l^1(X)$, hence M_a, as its dual, is isomorphic to $l^\infty(X)$.

For any $\mu \in L$ or $A \subset L$, we will write μ_a and A_a for μ_{L_a} and A_{L_a}. And for any $f \in M$ or $A \subset M$, we will write f_a and A_a for f_{M_a} and A_{M_a}.

The function $\mathbf{1}$ with constant value 1 on X is a strong unit not only for C, but in fact for M (under the canonical imbedding of C in M); that is, given $f \in M$, $\|f\| \leq 1$ if and only if $|f| \leq \mathbf{1}$.

3. The space $\mathfrak{M}$

We assume a knowledge of [2]. In this section, we give those results of that paper—stated for the Riesz space L—which we will need.

The letter J will always denote a dense ideal of L. L itself is considered a J. Given two J's, J_1 and J_2, $J_1 \cap J_2$ is again a J. Of importance to us, and easily verified, is the property that every J contains all of X.

We denote the space $\Gamma(L)$ [2] by $\mathfrak{M}$. Each element f of $\mathfrak{M}$ is a continuous linear functional on some $J = J_f$—its *domain*—which is maximal in the sense that f cannot be extended to a continuous linear functional on any larger ideal (indeed, not even to a bounded linear functional). Given f, $g \in \mathfrak{M}$, $f + g$, $\lambda f (\lambda \in R)$, $f \vee g$, and $f \wedge g$ are all defined on $J_f \cap J_g$, then extended, each to its domain (which is uniquely determined). We then have:

(3.1) *$\mathfrak{M}$ is a complete and universally complete Riesz space, containing M as a dense ideal, and with* $\mathbf{1}$ *for a weak unit.*

Since $\mathfrak{M}$ is complete, the statement that $\mathbf{1}$ is a weak unit means in effect that for every $f \in \mathfrak{M}_+$, $f = \bigvee_n(f \wedge n\mathbf{1})$.

Given $f \in \mathfrak{M}$ and $\mu \in L$, the notation $\langle f, \mu\rangle$ will be used only when $\mu \in J_f$. Even with this proviso, we have the same decomposition relation that exists between L and M:

(3.2) *For every decomposition $L = I \oplus I'$ into bands, we have the decomposition $\mathfrak{M} = (I')^{\perp} \oplus I^{\perp}$ into bands. And $(I')^{\perp} = \Gamma(I)$.*

We will call $(I')^{\perp}$ the band in $\mathfrak{M}$ *dual* to I. Given $\mu \in L$, we will denote by $\mathfrak{M}_\mu$ the band in $\mathfrak{M}$ dual to L_μ. $\mathfrak{M}_\mu$ can be identified with the space $\mathfrak{M}(\mu)$ of all μ-measurable functions modulo those vanishing on μ-null sets. Clearly $\mathfrak{M}_\mu \cap M = M_\mu$, and the latter is dense in $\mathfrak{M}_\mu$.

Remark. Note that in the present approach, $\mathfrak{M}(\mu)$ is a superspace of $\mathfrak{L}^\infty(\mu)$ not of $\mathfrak{L}^1(\mu)$. (Since $\mathfrak{L}^\infty(\mu)$ is the dual of $\mathfrak{L}^1(\mu)$, we never consider it a subspace of the latter.)

We will denote the bands in $\mathfrak{M}$ dual to L_a and L_d by $\mathfrak{M}_a$ and $\mathfrak{M}_d$ respectively. Thus $\mathfrak{M} = \mathfrak{M}_a \oplus \mathfrak{M}_d$. Again $\mathfrak{M}_a \cap M = M_a$ and the latter is dense in $\mathfrak{M}_a$; and similarly for $\mathfrak{M}_d$ and M_d. From the fact that every J in L contains X, it is not hard to show that $\mathfrak{M}_a$ can be identified with R^X (communicated by Masterson).

We recall that for two elements f, g of $\mathfrak{M}$ (say), g_f denotes the component of g in the band generated by f. Given $f \in \mathfrak{M}$ and $\lambda \in R$, we will often be concerned with the element $\mathbf{1}_{(f-\lambda\mathbf{1})^+}$. This is because it corresponds in ordinary function theory to the set of points on which the value of the function is (strictly) greater than λ. And similarly for $\mathbf{1}_{(f-\lambda\mathbf{1})^-}$.

A *component of* $\mathbf{1}$ is also defined as an element e such that $e \wedge (\mathbf{1} - e) = 0$. This definition is consistent with the above meaning of component. A component of $\mathbf{1}$ in either sense is always a component in the other sense.

We will need various properties of the components of $\mathbf{1}$, and we collect them here. The letters e and d will consistently denote such components.

Consider $f \in \mathfrak{M}$. Given $\lambda \geq 0$, if we set

$$e = \mathbf{1}_{(f-\lambda\mathbf{1})^+},\ d = \mathbf{1}_{(f-\lambda\mathbf{1})^-}, \quad \text{and} \quad c = \mathbf{1} - e - d,$$

then e, d, c are mutually disjoint, and $f_e \geq \lambda e$, $f_d \leq \lambda d$, $f_c = \lambda c$. Given $0 \leq \lambda < \kappa$, $\mathbf{1}_{(f-\lambda\mathbf{1})^+} = \mathbf{1}_{(f\wedge\kappa\mathbf{1}-\lambda\mathbf{1})^+}$; in particular, for $\kappa > 0$, $\mathbf{1}_f = \mathbf{1}_{f\wedge\kappa\mathbf{1}}$. Note also that for $0 \leq \lambda < \kappa$,

$$\mathbf{1}_{(f-\lambda\mathbf{1})^+} + \mathbf{1}_{(f-\kappa\mathbf{1})^-} \geq \mathbf{1}.$$

Finally, $\bigwedge_{\lambda\in R}\mathbf{1}_{(f-\lambda\mathbf{1})^+} = 0$ and $\bigvee_{\lambda\in R}\mathbf{1}_{(f-\lambda\mathbf{1})^-} = \mathbf{1}$.

(3.3) *Let $A \subset \mathfrak{M}$ be bounded above, and for each $\lambda \in R$, set*

$$e_\lambda = \bigwedge_{f\in A}\mathbf{1}_{(f-\lambda\mathbf{1})^-}.$$

Then $\bigvee_\lambda e_\lambda = \mathbf{1}$.

By hypothesis, there exists $g \in \mathfrak{M}$ such that $f \leq g$ for all $f \in A$. Fixing λ, it follows that for each $f \in A$, $(f - \lambda\mathbf{1})^{-} \geq (g - \lambda\mathbf{1})^{-}$, hence $\mathbf{1}_{(f-\lambda 1)^-} \geq \mathbf{1}_{(g-\lambda 1)^-}$; this gives us $e_\lambda \geq \mathbf{1}_{(g-\lambda 1)^-}$. The last equality preceding the proposition now gives the desired result.

Now consider a set $\{e_\alpha\}$ of components of $\mathbf{1}$. If $\bigwedge_\alpha e_\alpha = 0$, then for every $f \in \mathfrak{M}_+$, $\bigwedge_\alpha f_{e_\alpha} = 0$. It follows that if $\bigvee_\alpha e_\alpha = \mathbf{1}$, then for every $f \in \mathfrak{M}_+$, $\bigvee_\alpha f_{e_\alpha} = f$. One consequence of this last is that $f_{e_\alpha} = 0$ for all α implies $f = 0$. Another consequence is that if $\{e_\alpha\}$ is a *net* and $e_\alpha \uparrow \mathbf{1}$, then for every $f \in \mathfrak{M}$ (not just $\mathfrak{M}_+$), $\lim_\alpha f_{e_\alpha} = f$.

Given a set $\{f_\alpha\}$ in $\mathfrak{M}$, if $\bigwedge_\alpha f_\alpha = 0$, it does not follow that $\bigwedge_\alpha \mathbf{1}_{f_\alpha} = 0$. However:

(3.4) *Suppose* $\bigwedge_\alpha f_\alpha = 0$ *in* $\mathfrak{M}$. *Then for* $\lambda > 0$, *setting* $e_\alpha = \mathbf{1}_{(f-\lambda 1)^+}$, *we have* $\bigwedge_\alpha e_\alpha = 0$.

Since $\bigwedge_\alpha f_\alpha = 0$, we have $\bigwedge_\alpha (\mathbf{1}/\lambda) f_\alpha = 0$. But for every α, $\lambda e_\alpha \leq f_\alpha$, hence $e_\alpha \leq (\mathbf{1}/\lambda) f_\alpha$. It follows $\bigwedge_\alpha e_\alpha = 0$.

4. The limsup and liminf of a net in $\mathfrak{M}$

Given a bounded net $\{f_\alpha\}$ in $\mathfrak{M}$, then $g = \operatorname{limsup}_\alpha f_\alpha$ and $h = \operatorname{liminf}_\alpha f_\alpha$ are defined respectively by $g = \bigwedge_\alpha \bigvee_{\beta > \alpha} f_\beta$ and $h = \bigvee_\alpha \bigwedge_{\beta > \alpha} f_\beta$. And if they coincide, that is, $\operatorname{limsup}_\alpha f_\alpha = \operatorname{liminf}_\alpha f_\alpha = f$, then $f = \lim_\alpha f_\alpha$ in the sense defined earlier (§2). Now these definitions require that $\{f_\alpha\}$ be bounded. Thus they do not include ordinary convergence in R, or pointwise convergence in R^X, or even in $\mathfrak{L}^\infty(X)$.

Nakano's more general *individual convergence* [4] eliminates this deficiency. In the present section and the following one, we develop his convergence in the simpler form made possible by the existence of a weak order unit in $\mathfrak{M}$.

Given $f \in \mathfrak{M}$ and $\lambda \geq 0$, we set

$$f^{(\lambda)} = (f \wedge \lambda\mathbf{1}) \vee (-\lambda\mathbf{1}) = (f \vee (-\lambda\mathbf{1})) \wedge \lambda\mathbf{1}.$$

$f^{(\lambda)}$ will be called a *truncation* of f.

The verification of the following is routine.

(4.1) (a) $-f^{-} \leq f^{(\lambda)} \leq f^{+}$.
(b) *For* $0 \leq \lambda \leq \kappa$, $f^{(\lambda)} = (f^{(\kappa)})^{(\lambda)}$.
(c) $(-f)^{(\lambda)} = -f^{(\lambda)}$.
(d) $(f \vee g)^{(\lambda)} = f^{(\lambda)} \vee g^{(\lambda)}$, $(f \wedge g)^{(\lambda)} = f^{(\lambda)} \wedge g^{(\lambda)}$.
(e) (*Particular case of* (d)) $(f^{+})^{(\lambda)} = (f^{(\lambda)})^{+}$, $(f^{-})^{(\lambda)} = (f^{(\lambda)})^{-}$, $|f|^{(\lambda)} = |f^{(\lambda)}|$.
(f) *For every band* I, $(f^{(\lambda)})_I = (f_I)^{(\lambda)}$.

Since $f^{+} \geq 0$, $(f^{+})^{(\lambda)} = f^{+} \wedge \lambda\mathbf{1}$ for all $\lambda \geq 0$, hence $(f^{+})^{(\lambda)} \uparrow f^{+}$ as $\lambda \to \infty$. Similarly, $(f^{-})^{(\lambda)} \uparrow f^{-}$ as $\lambda \to \infty$. Combining these with (e) above, we obtain

(4.2) $$f = \lim_{\lambda \to \infty} f^{(\lambda)}.$$

Remark. We clearly also have that $f = \lim_n f^{(n)}$.

As one consequence of (4.2), we have:

(4.3) *$f \leq g$ if and only if $f^{(\lambda)} \leq g^{(\lambda)}$ for all $\lambda \geq 0$. In particular, $f = g$ if and only if $f^{(\lambda)} = g^{(\lambda)}$ for all $\lambda \geq 0$.*

We extend (d) in (4.1) to an arbitrary collection.

(4.4) *Given $\{f_\alpha\} \subset \mathfrak{M}$ and $f \in \mathfrak{M}$, the following are equivalent:*
(1) $f = \bigvee_\alpha f_\alpha$.
(2) $f^{(\lambda)} = \bigvee_\alpha f_\alpha^{(\lambda)}$ *for all* $\lambda \geq 0$.
And similarly with $\bigvee$ replaced by $\bigwedge$.

Proof. That (1) implies (2) is elementary. Assume (2) holds. Then in particular, for each α, $f_\alpha^{(\lambda)} \leq f^{(\lambda)}$ for all $\lambda \geq 0$, whence by (4.3), $f_\alpha \leq f$. Thus $\{f_\alpha\}$ is bounded above, and therefore $g = \bigvee_\alpha f_\alpha$ exists. Then for every $\lambda \geq 0$, $g^{(\lambda)} = \bigvee_\alpha f_\alpha^{(\lambda)} = f^{(\lambda)}$, hence by (4.3) again, $f = g$, and we are through.

(4.5) *Let $\{f_\lambda\}$ be a bounded net in $\mathfrak{M}$ indexed by the non-negative real numbers and satisfying:*

$$\lambda \leq \kappa \text{ implies } f_\lambda = f_\kappa^{(\lambda)}.$$

Then there exists $f \in \mathfrak{M}$ such that $f_\lambda = f^{(\lambda)}$ for all $\lambda \geq 0$.

Proof. Set $f = \limsup_{\lambda\to\infty} f_\lambda$. Then for each λ_0,

$$f^{(\lambda_0)} = (\limsup_{\lambda\to\infty} f_\lambda)^{(\lambda_0)} = (\bigwedge_{\lambda\geq 0} \bigvee_{\kappa\geq\lambda} f_\kappa)^{(\lambda_0)} = \bigwedge_{\lambda\geq 0}(\bigvee_{\kappa\geq\lambda} f_\kappa)^{(\lambda_0)}$$
$$= \bigwedge_{\lambda\geq 0} \bigvee_{\kappa\geq\lambda}(f_\kappa)^{(\lambda_0)}$$

(routine computation). For all $\lambda \geq \lambda_0$, the element $(f_\kappa)^{(\lambda_0)}$ in the last expression is the fixed element f_{λ_0}, so the entire expression is f_{λ_0}, and we are through.

We turn to limsup and liminf. Given a net $\{f_\alpha\}$ in $\mathfrak{M}$ and $f \in \mathfrak{M}$, then by $f = \limsup_\alpha f_\alpha$, we will mean

$$f^{(\lambda)} = \limsup_\alpha f_\alpha^{(\lambda)} \quad \text{for all } \lambda \geq 0,$$

and by $f = \liminf_\alpha f_\alpha$, we will mean

$$f^{(\lambda)} = \liminf_\alpha f_\alpha^{(\lambda)} \quad \text{for all } \lambda \geq 0.$$

It is easily verified that for a bounded net in $\mathfrak{M}$, these definitions are equivalent to the ordinary ones. For this reason, we are using the same notation. We emphasize that now, however, the statements $f = \limsup_\alpha f_\alpha$ and $f = \liminf_\alpha f_\alpha$ no longer imply that $\{f_\alpha\}$ is bounded.

The extended definitions coincide on $\mathfrak{M}_a$ ($\cong R^X$) with the pointwise limsup and liminf. We state this formally.

(4.6) *Given a net $\{f_\alpha\}$ in $\mathfrak{M}_a$ and $f \in \mathfrak{M}_a$, the following are equivalent:*
(1) $f = \limsup_\alpha f_\alpha$.
(2) $\langle f, x\rangle = \limsup_\alpha \langle f_\alpha, x\rangle$ *for all* $x \in X$.

$\limsup_\alpha f_\alpha$ and $\liminf_\alpha f_\alpha$, when they exist, have most of the properties that they have under the ordinary definition. For example, if $f = \limsup_\alpha f_\alpha$, then $-f = \liminf_\alpha(-f_\alpha)$ and $f \vee g = \limsup_\alpha(f_\alpha \vee g)$ for all $g \in \mathfrak{M}$. The following theorems give additional examples.

(4.7) *If $f = \limsup_\alpha f_\alpha$, then $f^+ = \limsup_\alpha f_\alpha^+$ and $f^- = \liminf_\alpha f_\alpha^-$. Conversely, if $\limsup_\alpha f^+$ and $\liminf_\alpha f^-$ exist, then so does $\limsup_\alpha f_\alpha$.*

Proof. The first statement follows from the immediately preceding remarks. Conversely, suppose $g = \limsup f_\alpha^+$ and $h = \liminf_\alpha f_\alpha^-$. Then by straightforward computation, $g \wedge h = 0$; hence, setting $f = g - h$, we have $g = f^+$, $h = f^-$. It can then be verified that $f^{(\lambda)} = \limsup_\alpha f_\alpha^{(\lambda)}$ for all $\lambda \geq 0$, whence $f = \limsup_\alpha f_\alpha$.

(4.8) *Let $\{f_\alpha\}$, $\{g_\alpha\}$, $\{h_\alpha\}$ be nets in $\mathfrak{M}$ satisfying*

$$g_\alpha \leq f_\alpha \leq h_\alpha \quad \textit{for all } \alpha.$$

If $\limsup_\alpha g_\alpha$ and $\limsup_\alpha h_\alpha$ exist, then so does $\limsup_\alpha f_\alpha$, and

$$\limsup_\alpha g_\alpha \leq \limsup_\alpha f_\alpha \leq \limsup_\alpha h_\alpha .$$

Proof. Let $g = \limsup_\alpha g_\alpha$ and $h = \limsup_\alpha h_\alpha$. For each $\lambda \geq 0$, set $f_\lambda = \limsup_\alpha f_\alpha^{(\lambda)}$. We show $\{f_\lambda\}$ satisfies the conditions of (4.5).

The concluding inequalities in the theorem hold for the ordinary definition, so for each $\lambda \geq 0$, $\limsup_\alpha g_\alpha^{(\lambda)} \leq \limsup_\alpha f_\alpha^{(\lambda)} \leq \limsup_\alpha h_\alpha^{(\lambda)}$. But by definition, the first term here is $g^{(\lambda)}$ and the last $h^{(\lambda)}$; combining this with (a) of (4.1), we obtain

(i) $$-g^- \leq g^{(\lambda)} \leq f_\lambda \leq h^{(\lambda)} \leq h^+,$$

which gives us that $\{f_\lambda\}$ is bounded. Now suppose $0 \leq \lambda \leq \kappa$. Then

$$(f_\kappa)^{(\lambda)} = (\limsup_\alpha f_\alpha^{(\kappa)})^{(\lambda)} = \limsup_\alpha (f_\alpha^{(\kappa)})^{(\lambda)} = \limsup_\alpha f_\alpha^{(\lambda)} = f_\lambda .$$

(Here the second equality is by straightforward computation, and the third by (4.1).) We can thus apply (4.5) to obtain an $f \in \mathfrak{M}$, which is the desired $\limsup_\alpha f_\alpha$.

Remark. An immediate corollary of course is that if $0 \leq f_\alpha \leq h_\alpha$ for all α and $\limsup_\alpha h_\alpha$ exists, then so does $\limsup_\alpha f_\alpha$. It can be verified that under these assumption, also $\liminf_\alpha f_\alpha$ exists.

(4.9) *Given nets $\{g_\alpha\}$, $\{h_\alpha\}$ in $\mathfrak{M}_+$, if $\limsup_\alpha g_\alpha$ and $\limsup_\alpha h_\alpha$ exist, then $\limsup_\alpha(g_\alpha + h_\alpha)$ exists and satisfies the inequality*

$$\limsup_\alpha(g_\alpha + h_\alpha) \leq \limsup_\alpha g_\alpha + \limsup_\alpha h_\alpha .$$

Proof. It is easily verified that for each $\lambda \geq 0$,

(i) $$(g_\alpha + h_\alpha)^{(\lambda)} \leq g_\alpha^{(\lambda)} + h_\alpha^{(\lambda)} \quad \text{for all } \alpha.$$

Now let $g = \limsup_\alpha g_\alpha$ and $h = \limsup_\alpha h_\alpha$; and for each $\lambda \geq 0$, set

$$f_\lambda = \limsup_\alpha(g_\alpha + h_\alpha)^{(\lambda)}.$$

We show $\{f_\lambda\}$ satisfies the conditions of (4.5). By (i),

$$0 \leq f_\lambda \leq g^{(\lambda)} + h^{(\lambda)} \leq g + h,$$

and thus $\{f_\lambda\}$ is bounded. And given $0 \leq \lambda \leq \kappa$,

$$\begin{aligned}(f_\kappa)^{(\lambda)} = (\operatorname{limsup}_\alpha(g_\alpha + h_\alpha)^{(\kappa)})^{(\lambda)} &= \operatorname{limsup}_\alpha((g_\alpha + h_\alpha)^{(\kappa)})^{(\lambda)} \\ &= \operatorname{limsup}_\alpha(g_\alpha + h_\alpha)^{(\lambda)} = f_\lambda .\end{aligned}$$

Applying (4.5), there exists $f \in \mathfrak{M}$ such that $f_\lambda = f^{(\lambda)}$ for all $\lambda \geq 0$, whence $f = \operatorname{limsup}_\alpha(g_\alpha + h_\alpha)$. That $f \leq g + h$ follows from (i) and (4.2).

(4.10) *If* $g = \operatorname{limsup}_\alpha g_\alpha$, *then for every* $h \in \mathfrak{M}$,

$$g + h = \operatorname{limsup}_\alpha(g_\alpha + h).$$

Proof. Consider $\lambda \geq 0$; we have to show that

$$(g + h)^{(\lambda)} = \operatorname{limsup}_\alpha(g_\alpha + h)^{(\lambda)}.$$

Now

$$\begin{aligned}(g_\alpha + h)^{(\lambda)} &= ((g_\alpha + h) \wedge \lambda\mathbf{1}) \vee (-\lambda\mathbf{1}) = ((g_\alpha \wedge (\lambda\mathbf{1} - h) + h) \vee (-\lambda\mathbf{1}) \\ &= ((g_\alpha \wedge (\lambda\mathbf{1} - h)) \vee (-\lambda\mathbf{1} - h) + h.\end{aligned}$$

Thus

$$\begin{aligned}&\operatorname{limsup}_\alpha(g_\alpha + h)^{(\lambda)} \\ &\quad= \operatorname{limsup}_\alpha((g_\alpha \wedge (\lambda\mathbf{1} - h)) \vee (-\lambda\mathbf{1} - h) + h \\ &\quad= (g \wedge (\lambda\mathbf{1} - h)) \vee (-\lambda\mathbf{1} - h) + h \quad \text{(cf. the remarks preceding (4.7))} \\ &\quad= ((g + h) \wedge \lambda\mathbf{1}) \vee (-\lambda\mathbf{1}) \\ &\quad= (g + h)^{(\lambda)}.\end{aligned}$$

(4.11) *Corollary. Given a net* $\{g_\alpha\}$ *in* $\mathfrak{M}$ *and* $g \in \mathfrak{M}$, *the following are equivalent*:

(1) $\operatorname{limsup}_\alpha g_\alpha = g$.
(2) $\operatorname{limsup}_\alpha(g_\alpha - g) = 0$.

5. Convergence in $\mathfrak{M}$

Given a net $\{f_\alpha\}$ in $\mathfrak{M}$ and $f \in \mathfrak{M}$, then by $f = \lim_\alpha f_\alpha$, or $f_\alpha \to f$, we will mean $f^{(\lambda)} = \lim_\alpha f_\alpha^{(\lambda)}$ for all $\lambda \geq 0$. Equivalently, $f = \lim_\alpha f_\alpha$ if and only if $f = \operatorname{limsup}_\alpha f_\alpha = \operatorname{liminf}_\alpha f_\alpha$. If the net $\{f_\alpha\}$ is bounded, our definition reduces to the ordinary one, hence we again use the same notation.

In $\mathfrak{M}_a$, the definition coincides with pointwise convergence. We state this formally.

(5.1) *Given a net* $\{f_\alpha\}$ *in* $\mathfrak{M}_a$ *and* $f \in \mathfrak{M}_a$, *the following are equivalent:*
(1) $f = \lim_\alpha f_\alpha$
(2) $\langle f, x\rangle = \lim_\alpha\langle f_\alpha , x\rangle$ *for all* $x \in X$.

When it exists, $\lim_\alpha f_\alpha$ has all the properties it has under the ordinary definition. Thus (cf. the remarks preceding (4.7); also (4.9)):

(5.2) *If* $f = \lim_\alpha f_\alpha$ *and* $g = \lim_\alpha g_\alpha$, *then*

$$f \vee g = \lim_\alpha (f_\alpha \vee g_\alpha), \qquad f \wedge g = \lim_\alpha (f_\alpha \wedge g_\alpha),$$

$$f + g = \lim_\alpha (f_\alpha + g_\alpha).$$

Whence:

(5.3) *If* $f = \lim_\alpha f_\alpha$, *then* $f^+ = \lim_\alpha f_\alpha^+$ *and* $f^- = \lim_\alpha f_\alpha^-$.

Also:

(5.4) *Given a net* $\{f_\alpha\}$ *in* $\mathfrak{M}$ *and* $f \in \mathfrak{M}$, *the following are equivalent:*

(1) $\lim_\alpha f_\alpha = f$.

(2) $\lim_\alpha (f_\alpha - f) = 0$.

(3) $\lim_\alpha | f_\alpha - f | = 0$.

For sequences, the extended definition reduces to the ordinary one:

(5.5) *If* $f = \limsup_n f_n$ *in* $\mathfrak{M}$, *then* $\{f_n\}$ *is bounded above, hence the* limsup *holds in the ordinary sense.*

(5.6) COROLLARY. *If* $f = \lim_n f_n$ *in* $\mathfrak{M}$, *then* $\{f_n\}$ *is bounded and the limit holds in the ordinary sense.*

(5.5) and (5.6) follow essentially from Theorem 1.1 in [4] and the fact that $\mathfrak{M}$ is universally complete.

We next establish two properties equivalent to convergence. First:

(5.7) *Given* $\{g_\alpha\} \subset \mathfrak{M}_+$, *the following are equivalent:*

(1) $\bigwedge_\alpha g_\alpha = 0$.

(2) $\bigwedge_\alpha (g_\alpha \wedge \mathbf{1}) = 0$.

(3) *For every* $\lambda > 0$, $\bigwedge_\alpha \mathbf{1}_{(g_\alpha - \lambda \mathbf{1})^+} = 0$.

Proof. Assume (1) holds. That (2) then holds is trivial. To show (3) holds, consider $\lambda > 0$. (1) implies that $\bigwedge_\alpha (1/\lambda) g_\alpha = 0$. Since

$$\mathbf{1}_{(g_\alpha - \lambda \mathbf{1})^+} \leq (1/\lambda) g_\alpha,$$

we have (3).

Conversely, assume (1) does not hold. Then there exists $g \in \mathfrak{M}$ such that $0 < g \leq g_\alpha$ for all α. It follows $0 < g \wedge \mathbf{1} \leq g_\alpha \wedge \mathbf{1}$ for all α ($\mathbf{1}$ is a weak order unit for $\mathfrak{M}$), and thus (2) fails to hold. To show (3) fails to hold, choose $\lambda > 0$ such that $(g - \lambda \mathbf{1})^+ \neq 0$. Then $\mathbf{1}_{(g - \lambda \mathbf{1})^+} \neq 0$ (again, because $\mathbf{1}$ is a weak order unit). But $\mathbf{1}_{(g - \lambda \mathbf{1})^+} \leq \mathbf{1}_{(g_\alpha - \lambda \mathbf{1})^+}$ for all α, and so (3) fails to hold.

(5.8) *Given a net* $\{f_\alpha\}$ *in* $\mathfrak{M}_+$, *the following are equivalent:*

(1) $\lim_\alpha f_\alpha = 0$.

(2) $\lim_\alpha (f_\alpha \wedge \mathbf{1}) = 0$.

(3) *For every* $\lambda > 0$, $\lim_\alpha \mathbf{1}_{(f_\alpha - \lambda \mathbf{1})^+} = 0$.

Remark. It is obvious that for each of these statements to hold, it is sufficient that it hold for limsup.

Proof. Assume (1) holds. Then of course (2) holds. To show (3) holds, consider $\lambda \geq 0$. Choose any $\kappa > \lambda$. Then $\lim_\alpha f_\alpha \wedge \kappa\mathbf{1} = 0$ and for each α,

$$\mathbf{1}_{(f_\alpha - \lambda\mathbf{1})^+} = \mathbf{1}_{(f_\alpha \wedge \kappa\mathbf{1} - \lambda\mathbf{1})^+} .$$

Thus for simplicity, we can assume $\{f_\alpha\}$ is bounded. For each α, set $\bigvee_{\beta > \alpha} f_\alpha = g_\alpha$. Then $\bigwedge_\alpha g_\alpha = 0$, hence by (3.4), $\bigwedge_\alpha \mathbf{1}_{(g_\alpha - \lambda\mathbf{1})^+} = 0$. Since for every α, $\mathbf{1}_{(f_\alpha - \lambda\mathbf{1})^+} \leq \mathbf{1}_{(g_\alpha - \lambda\mathbf{1})^+}$, we have (3).

Assume (2) holds. That (1) holds follows from (5.7) and the identity $\bigvee_{\beta > \alpha}(f_\alpha \wedge \mathbf{1}) = g_\alpha \wedge \mathbf{1}$. Finally assume (3) holds. It is enough to show that this implies (3) in (5.7). But given $\lambda > 0$ and α, we have by straightforward computation that

$$\mathbf{1}_{(g_\alpha - \lambda\mathbf{1})^+} = \bigvee_{\beta > \alpha} \mathbf{1}_{(f_\beta - \lambda\mathbf{1})^+} .$$

(5.9) COROLLARY. *Given a net $\{f_\alpha\}$ in $\mathfrak{M}$, and $f \in \mathfrak{M}$, the following are equivalent:*

(1) $\lim_\alpha f_\alpha = f$.
(2) $\lim_\alpha |f_\alpha - f| \wedge \mathbf{1} = 0$.
(3) $\lim_\alpha (f_\alpha - f)^{(1)} = 0$.

Given a net $\{f_\alpha\}$ in $\mathfrak{M}$, and $f \in \mathfrak{M}$, we will say $\lim_\alpha f_\alpha = f$ *uniformly* if there exists α_0 such that $f_\alpha - f \in M$ for all $\alpha > \alpha_0$ and $\lim_{\alpha > \alpha_0} \| f_\alpha - f \| = 0$.

(5.10) (Egorov) *Given a net $\{f_\alpha\}$ in $\mathfrak{M}$, and $f \in \mathfrak{M}$, if* $\lim_\alpha f_\alpha = f$, *then there exists a net $\{e_\gamma\}$ of components of* $\mathbf{1}$ *satisfying:*

(a) $e_\gamma \uparrow \mathbf{1}$.
(b) *For each* γ, $\lim_\alpha (f_\alpha)_{e_\gamma} = f_{e_\gamma}$ *uniformly.*

Proof. Consider first the case: $\{f_\alpha\} \subset \mathfrak{M}_+$ and $\lim_\alpha f_\alpha = 0$. Let $\{e_\gamma\}$ be the set of all components of $\mathbf{1}$ for which (b) holds, and order this set by $\leq$. It is easily verified that for $e_{\gamma_1}, e_{\gamma_2}$ in this set, $e_{\gamma_1} \vee e_{\gamma_2}$ is also in the set; so we have an ascending net. We show $\bigvee_\gamma e_\gamma = \mathbf{1}$, which will complete the proof for this first case.

LEMMA 1. *Given $\mu \in L_+$, $\delta > 0$, and $\lambda > 0$, there is a decomposition of* $\mathbf{1}$, $\mathbf{1} = d + e$, *and an α' such that*

$$\langle d, \mu \rangle \leq \delta,$$

$$(f_\alpha)_e \leq \lambda\mathbf{1} \quad \textit{for all } \alpha > \alpha'$$

By (5.8), *lim*$_\alpha$ $\mathbf{1}_{(f_\alpha - \lambda\mathbf{1})^+} = 0$. For each α, set $d_\alpha = \bigvee_{\beta > \alpha} \mathbf{1}_{(f_\beta - \lambda\mathbf{1})^+}$. Then $d_\alpha \downarrow 0$, hence $\langle d_\alpha, \mu \rangle \downarrow 0$, hence there exists α' such that $\langle d_\alpha, \mu \rangle \leq \delta$ for all $\alpha > \alpha'$. Set $d = d_{\alpha_0}$, $e = \mathbf{1} - d$. Then for $\alpha > \alpha'$, $\mathbf{1}_{(f_\alpha - \lambda\mathbf{1})^+} \leq d_\alpha \leq d$, whence $(f_\alpha)_e \leq \lambda\mathbf{1}$.

LEMMA 2. *Given $\mu \in L_+$ and $\delta > 0$, there is a decomposition of* $\mathbf{1}$, $\mathbf{1} = d + e$, *such that $\langle d, \mu\rangle \leq \delta$ and e is an e_γ.*

For each $n = 1, 2, \cdots$, take $(\frac{1}{2}^n)\delta$ and $1/n$ for the δ and λ of Lemma 1, and denote the resulting decomposition of $\mathbf{1}$ by $\mathbf{1} = d_n + e_n$, and the corresponding α' by α_n. Then $e = \bigwedge_n e_n$ has the property of e_γ in (b) (α_1 plays the role of α_0 in the definition of uniform convergence). This gives Lemma 2.

We can now show $\bigvee_\gamma e_\gamma = \mathbf{1}$. Suppose not; then there exists a component d_0 of $\mathbf{1}$ disjoint from all the e_γ's. Choose $\mu \in L_+$ such that $\langle d_0, \mu\rangle = \langle \mathbf{1}, \mu\rangle = 1$. Let $\mathbf{1} = d + e$ be the decomposition of $\mathbf{1}$ given by Lemma 2 for $\delta = \frac{1}{2}$. Then e is an e_γ not disjoint from d_0. We thus have a contradiction.

The general case follows from the above case immediately.

6. The space of semicontinuous elements

We will call $f \in \mathfrak{M}$ an l.s.c. element (resp. u.s.c. element) if $f = \bigvee A$ (resp. $f = \bigwedge A$) for some subset A of C. We have immediately that if f, g are l.s.c., then so are $f + g, f \vee g, f \wedge g$, and λf for $\lambda \geq 0$. Also if $\{f_\alpha\}$ are all l.s.c., and $f = \bigvee_\alpha f_\alpha$, then f is l.s.c.. Denoting by $\mathfrak{S}$ the linear subspace generated by the l.s.c. elements, it follows from the definition and the above properties that $\mathfrak{S}$ is a linear sublattice of $\mathfrak{M}$ and that every element of $\mathfrak{S}$ is the difference of two positive l.s.c. elements. The proofs are the same as those in [1]. Also, by the same argument as was used in [1], we can establish:

(6.1) *The projection mapping $\mathfrak{M} \to \mathfrak{M}_a$ maps $\mathfrak{S}$ isomorphically onto $\mathfrak{S}_a$.*

The subspace S of M is clearly contained in $\mathfrak{S}$. It might be expected that $\mathfrak{S} \cap M = S$, but in general this is not so. We give a partial description of $\mathfrak{S} \cap M$ in (6.4). First,

(6.2) *For each $f \in \mathfrak{S}$, there exists $\{f_n\} \subset S$ such that $f = lim_n f_n$.*

Proof. $f = g - h$, g and h positive l.s.c. elements. For each $n = 1, 2, \cdots$, set $g_n = g \wedge n\mathbf{1}$, $h_n = h \wedge n\mathbf{1}$. Then $g_n \uparrow g$ and $h_n \uparrow h$, hence $g_n - h_n \to f$. Since g_n and h_n are clearly l.s.c. elements of M, $g_n - h_n \in S$, so we are through.

(6.3) COROLLARY. $\mathfrak{S} \cap M \subset Bo$ (*the space of Borel elements of M* [1]).

Proof. Consider $f \in \mathfrak{S} \cap M$. By (6.2), there exists $\{f_n\} \subset S$ with $f = \lim_n f_n$. From $f \in M$, there exists $\lambda \geq 0$ such that $-\lambda\mathbf{1} \leq f \leq \lambda\mathbf{1}$. But then $f = \lim_n f_n^{(\lambda)}$. Since we still have $\{f_n^{(\lambda)}\} \subset S$ and the convergence takes place in M, we have the desired conclusion.

Note that for every $\lambda \geq 0$, $f^{(\lambda)} \in \mathfrak{S} \cap M$, hence, by the corollary, lies in Bo.

In M, every element is bounded above by an element of C—in fact, by a multiple of $\mathbf{1}$. This of course is no longer true in $\mathfrak{M}$. We give an example to show it is not even true if we replace C by $\mathfrak{S}$. Specifically we produce an element of $\mathfrak{M}$ which has no l.s.c. element above it; since every element of

$\mathcal{S}$ has an l.s.c. element above it (it is the difference of two positive l.s.c. elements), this will supply our example.

Let X be the closed interval $0 \leq x \leq 1$ of R, and $\{r_n\}$ the set of rationals in X. Define $f \in \mathfrak{M}_a$ by $f(r_n) = n$ $(n = 1, 2, \cdots)$, $f(x) = 0$ otherwise. Suppose there exists an l.s.c. element $g \geq f$. We obtain by induction a nest of closed intervals $K_1 \supset K_2 \supset \cdots$ such that $g(x) > k$ for all $x \in K_k$ $(k = 1, 2, \cdots)$. Since $g(r_2) > 1$ and g is l.s.c., we can find a closed interval K_1 such that $g(x) > 1$ for all $x \in K_1$. Suppose $K_1, \cdots, K_{k-1}$ have been chosen. The interior of K_{k-1} contains an r_n for which $g(r_n) > k$, hence we can find a closed subinterval K_k of K_{k-1} such that $g(x) > k$ for all $x \in K_k$. We thus have the nest $\{K_k\}$. Now $\bigcap_k K_k$ is non-empty. But for $x \in \bigcap_k K_k$, $g(x) > k$ for all k, giving us a contradiction.

7. The star elements

For the moment, let us consider M. S is isomorphic to its projection S_a in M_a, but the imbedding of S in M differs considerably from the imbedding of S_a in M_a. One important difference is the following. If (A, B) is a pair of subsets of S_a forming a Dedekind cut in S_a, then there is a unique $f \in M_a$ such that $f = \bigvee A = \bigwedge B$. It follows easily that M_a can be identified with the Dedeking completion of S_a (hence of S). In contradistinction to this, if (A, B) is a Dedekind cut in S, then in general $\bigvee A \neq \bigwedge B$, that is, there are many elements of M between A and B.

Of course, for some Dedekind cuts (A, B), there *is* a unique $f = \bigvee A = \bigwedge B$. We call the set of all such f's the *Dedekind closure* of S in M, and denote it by U. U is isomorphic to the space of functions on X which are integrable with respect to every Radon measure; consequently we call its elements the "universally integrable" elements of M.

We now want to define and study the corresponding space in $\mathfrak{M}$, the space of "universally measurable" elements. We recall the procedure followed in M. It turned out to be convenient to assign to *every* $f \in M$ a Dedekind cut (A, B) in S: the one determined by f_a; that is,

$$A = \{h \in S \mid h_a \leq f_a\}, \qquad B = \{h \in S \mid h_a \geq f_a\}.$$

We then denoted $\bigvee A$ by f_* and $\bigwedge B$ by f^*. Finally, U was defined as the set $\{f \in M \mid f = f_* = f^*\}$.

The first step in carrying out the above for $\mathfrak{M}$ is to extend the definition of the star elements to all $f \in \mathfrak{M}$, or to as many of them as possible. But in $\mathfrak{M}$—and indeed in $\mathfrak{M}_a$—as we saw at the end of the last §, not every element has an element of $\mathcal{S}$ above it (and one below it). Thus the above procedure cannot be paralleled in $\mathfrak{M}$ without modification. The natural modification, however, is at hand: the use of truncations.

Given $f \in \mathfrak{M}$ and $g \in \mathfrak{M}$, we write $g = f_*$ if $g^{(\lambda)} = (f^{(\lambda)})_*$ for all $\lambda \geq 0$, and we write $g = f^*$ if $g^{(\lambda)} = (f^{(\lambda)})^*$ for all $\lambda \geq 0$. Since the truncations are in M, the star elements used in the definitions are those defined above. Moreover, if $f \in M$, these definitions reduce to the above, hence we use the same notation.

Given $f \in \mathfrak{M}$, if f_* and f^* exist, then they have essentially all the properties proved in [1] for M, the proof in each case simply reducing to the corresponding one for M. Thus in the remainder of this section, we will usually only state these properties, and concern ourselves principally with questions of existence.

We note first that by the very definition:

(7.1) *If f_* exists, then for all $\lambda \geq 0$, $(f_*)^{(\lambda)} = (f^{(\lambda)})_*$; and if f^* exists, then for all $\lambda \geq 0$, $(f^*)^{(\lambda)} = (f^{(\lambda)})^*$.*

Some elementary properties:

(7.2) (a) *For $f \varepsilon \mathcal{S}$, $f = f_* = f^*$.*

(b) *If f^* exists, then $-f^* = (-f)_*$, and $\kappa f^* = (\kappa f)^*$ for all $\kappa \geq 0$. And similarly for f_*.*

(c) *If f_* and f^* exist, then $f_* \leq f^*$.*

(d) *f^* exists if and only if $(f_a)^*$ exists, and they are equal. And similarly for f_*.*

(e) *If f^* exists, then $(f^*)_a = f_a$; and similarly for f_*.*

(f) *If f^* and g^* exist, then $f \leq g$ implies $f^* \leq g^*$; and similarly for f_* and g_*.*

Remark. Because of (d), it will often simplify matters, in studying f_* and f^* for some f, to assume that $f \varepsilon \mathfrak{M}_a$.

(7.3) *Given $f \varepsilon \mathfrak{M}$, the following are equivalent:*

(1) *f^* exists.*

(2) *$(f^+)^*$ and $(f^-)_*$ exist.*

And if such is the case, then

$$(f^*)^+ = (f^+)^*, \qquad (f^*)^- = (f^-)_*.$$

Similarly for f_*.

Proof. We first show the following:

(7.4) LEMMA. *If f^* exists, then for $g \in \mathcal{S}$,*

$$f^* \vee g = (f \vee g)^*, \qquad f^* \wedge g = (f \wedge g)^*.$$

And similarly for f_*.

We show the second of these. Given $\lambda \geq 0$,

$$(f^* \wedge g)^{(\lambda)} = (f^*)^{(\lambda)} \wedge g^{(\lambda)} = (f^{(\lambda)})^* \wedge g^{(\lambda)}.$$

From (6.3), $g^{(\lambda)} \in U$, hence by (8.13) of [1],

$$(f^{(\lambda)})^* \wedge g^{(\lambda)} = (f^{(\lambda)} \wedge g^{(\lambda)})^* = ((f \wedge g)^{(\lambda)})^*.$$

Turning to the proof of the theorem, if f^* exists, then by the lemma (and (7.2)), $(f^*)^+ = (f^+)^*$ and $(f^*)^- = (f^-)_*$. Conversely, assume $(f^+)^*$ and $(f^-)_*$ exist. We show first that

(i) $$(f^+)^* \wedge (f^-)_* = 0.$$

It is enough to show $((f^+)^* \wedge (f^-)_*) \wedge \mathbf{1} = 0$.
$((f^+)^* \wedge (f^-)_*) \wedge \mathbf{1} = ((f^+)^* \wedge \mathbf{1}) \wedge ((f^-)_* \wedge \mathbf{1}) = (f^+ \wedge \mathbf{1})^* \wedge (f^- \wedge \mathbf{1})_*$ by the lemma. Now in M, $g \wedge h = 0$ implies $g^* \wedge h_* = 0$, so we have (i).

Setting $g = (f^+)^* - (f^-)_*$, it follows from (i) that $g^+ = (f^+)^*$ and $g^- = (f^-)_*$. Straightforward computation then gives that $g^{(\lambda)} = (f^{(\lambda)})^*$ for all $\lambda \geq 0$, whence $g = f^*$.

Before continuing with existence problems, we give an example to show that f^* need not exist. By symmetry, also f_* need not exist. Let X be the interval $0 \leq x \leq 1$ in R, μ the Lebesgue measure, and $\{X_n\}$ disjoint subsets of outer μ-measure 1 whose union is X. Let e_n be the element of $\mathfrak{M}_a$ having value 1 on each point of X_n and 0 elsewhere $(n = 1, 2, \cdots)$. Finally set $f = \sum_n ne_n = \bigvee_n ne_n \in \mathfrak{M}_a$. We show f^* does not exist.

Suppose f^* exists. We show $f^* \geq n\mathbf{1}_\mu$ for all n ($\mathbf{1}_\mu$ the component of $\mathbf{1}$ in $\mathfrak{M}_\mu$), which will contradict the fact that $\mathfrak{M}$ is archimedean. We first establish

(i) $$e_n^* \geq \mathbf{1}_\mu \quad \text{for all } n.$$

In $\mathfrak{M}_\mu$, if $0 \leq g \leq h$ and $\langle g, \mu\rangle = \langle h, \mu\rangle$, then $g = h$. Now $e_n \leq \mathbf{1}$, hence $e_n^* \leq \mathbf{1}$, hence $(e_n^*)_\mu \leq \mathbf{1}_\mu$, while $\langle (e_n^*)_\mu, \mu\rangle = \langle e_n^*, \mu\rangle = 1$ (since this last is precisely the μ-outer measure of X_n). Thus $(e_n^*)_\mu = \mathbf{1}_\mu$. But $e_n^* \geq (e_n^*)_\mu$, so we have (i).

Returning to f^*, for every n, $f \geq ne_n$, hence $f^* \geq (ne_n)^* = ne_n^* \geq n\mathbf{1}_\mu$.

In contrast to this example, we show:

(7.5) *Given $f \in \mathfrak{M}_+$, f_* exists. Specifically, the set*

$$\{g \in \mathcal{S} \mid g_a \leq f_a\}$$

is bounded above and its supremum is f_*.

Proof. For simplicity, we assume $f = f_a$ (that is, $f \in \mathfrak{M}_a$)

LEMMA. *For each $\mu \in L_+$, there exists $\nu \in L_+$ satisfying:*

(a) $0 \leq \nu \leq \mu$.
(b) $L_\nu = L_\mu$.
(c) $\sup_{h \text{ u.s.c.}, \, h_a \leq f} \langle h, \nu\rangle < \infty$.

We can assume $\| \mu \| = 1$. For each $m = 1, 2, \cdots, f \wedge m\mathbf{1} \in M_a$, hence $(f \wedge m\mathbf{1})_*$ exists; choose a u.s.c. element $h_m \geq 0$ such that $(h_m)_a \leq f \wedge m\mathbf{1}$ and

(i) $$\langle h_m, \mu\rangle \geq \langle (f \wedge m\mathbf{1})_*, \mu\rangle - 1.$$

For each $n = 1, 2, \cdots$, set $e_n = \bigwedge_m \mathbf{1}_{(h_m - n\mathbf{1})^-}$. So

(ii) $$(h_m)_{e_n} \leq ne_n \quad \textit{for all } n, m.$$

We show $e_n \uparrow \mathbf{1}$. Since the e_n's are in U, and U is isomorphic to U_a, it is enough to show $(e_n)_a \uparrow \mathbf{1}_a$. Now

$$(e_n)_a = \bigwedge_m (\mathbf{1}_{(h_m - n\mathbf{1})^-})_a = \bigwedge_m (\mathbf{1}_a)_{((h_m)_a - n\mathbf{1}_a)^-} .$$

It follows (cf. the statement preceding (3.3)) that $(e_n)_a \uparrow \mathbf{1}_a$.

Set $d_1 = e_1$, $d_n = e_n - e_{n-1}$ $(n = 2, 3, \cdots)$. Then the d_n's are mutually disjoint components of $\mathbf{1}$ with $\sum_n d_n = \mathbf{1}$. For each n, let I_n be the band generated by d_n, H_n its dual band in L, and $\mu_n = \mu_{H_n}$. Set $\nu_n = (1/n\ 2^n)\mu_n$ $(n = 1, 2, \cdots)$, and $\nu = \sum_n \nu_n$. That ν satisfies (a) and (b) of the lemma is clear; it remains to show it satisfies (c).

(iii) $\langle h_m, \nu\rangle \leq 1$ *for all* m.

In effect,

$$\begin{aligned}\langle h_m, \nu\rangle &= \langle h_m, \textstyle\sum_n \nu_n\rangle \\ &= \textstyle\sum_n \langle h_m, \nu_n\rangle \\ &= \textstyle\sum_n \langle (h_m)_{d_n}, (1/n\ 2^n)\mu_n\rangle \\ &\leq \textstyle\sum_n \langle n\, d_n, (1/n\ 2^n)\mu_n\rangle \\ &= \textstyle\sum_n (\tfrac{1}{2})^n \langle d_n, \mu_n\rangle \\ &\leq \textstyle\sum_n (\tfrac{1}{2})^n \\ &= 1.\end{aligned}$$

Here the first inequality follows from (ii) by taking the projection of both sides of (ii) on the band generated by d_n; and the second inequality from $\langle d_n, \mu_n\rangle = \| \mu_n \| \leq \| \mu \| = 1$.

To establish (c), we show that for every u.s.c. element h such that $h_a \leq f$, $\langle h, \nu\rangle \leq 2$. h, being u.s.c., is bounded above by $m\mathbf{1}$ for some m; hence $h_a \leq f \wedge m\mathbf{1}$. Also we can assume $h \geq h_m$ (else replace h by $h \vee h_m$). Then

$$\begin{aligned}\langle h, \nu\rangle &= \langle h_m, \nu\rangle + \langle h - h_m, \nu\rangle \\ &\leq \langle h_m, \nu\rangle + \langle h - h_m, \mu\rangle \\ &= \langle h_m, \nu\rangle + \langle h, \mu\rangle - \langle h_m, \mu\rangle \\ &\leq \langle h_m, \nu\rangle + \langle (f \wedge m\mathbf{1})_*, \mu\rangle - \langle h_m, \mu\rangle \\ &\leq \langle h_m, \nu\rangle + 1 && \text{(i)} \\ &\leq 2 && \text{(iii)}.\end{aligned}$$

This completes the proof of the lemma; we proceed to prove the theorem.

Every element of $\mathcal{S}$ is a supremum of u.s.c. elements, hence we can confine ourselves to the set $A = \{h \text{ u.s.c.} \mid h \geq 0, h_a \leq f\}$. We will produce a dense ideal J in L such that $\sup_{h \in A} \langle h, \omega\rangle < \infty$ for every $\omega \in J_+$. Since the h's in A form an ascending net, it will follow that the function ϕ on J_+ defined by $\phi(\omega) = \sup_{h \in A} \langle h, \omega \rangle$ extends to a continuous linear functional on all of J, hence is an element g of $\mathfrak{M}$ with $g = \bigvee A$.

For each $\mu \in L_+$, choose $\nu(\mu)$ by the Lemma. The ideal generated by $\{\nu(\mu) \mid \mu \in L_+\}$ is then the desired dense ideal J, and we thus have our $g \in \mathfrak{M}$ described above.

It remains to show $g = f_*$. As is easily verified, given $\lambda \geq 0$, the sets

$$\{h \wedge \lambda\mathbf{1} \mid h \in A\} \quad \text{and} \quad \{h \text{ u.s.c.} \mid h \geq 0,\ h_a \leq f \wedge \lambda\mathbf{1}\}$$

are identical. Hence for each $\lambda \geq 0$,

$$g \wedge \lambda\mathbf{1} = (\textstyle\bigvee_{h \in A} h) \wedge \lambda\mathbf{1} = \bigvee_{h \in A} (h \wedge \lambda\mathbf{1}) = (f \wedge \lambda\mathbf{1})_*.$$

Since this holds for all $\lambda \geq 0$, $g = f_*$.

Remark. We will later need the fact that f_* is a supremum of u.s.c. elements in M.

(7.6) *Given $g \leq f$, if f^* exists then g^* exists, and if g_* exists, then f_* exists.*

Proof. By (7.3) it is enough to show that $(g^+)^*$ and $(g^-)_*$ exist; and by (7.5) we need only show the former. Now $g^+ \leq f^+$, and by (7.3) again, $(f^+)^*$ exists. Thus for simplicity we can assume $0 \leq g \leq f$.

Set $g_n = g \wedge n\mathbf{1}, f_n = f \wedge n\mathbf{1}$ $(n = 1, 2, \cdots)$. Then $g_n \leq f_n$, hence

$$g_n^* \leq f_n^* = f^* \wedge n\mathbf{1} \leq f^*.$$

It follows $h = \bigvee_n g_n^*$ exists. We show that for every $k = 1, 2, \cdots$, $h \wedge k\mathbf{1} = g_k^*$, whence it will follow that $h = g^*$. Given k,

$$h \wedge k\mathbf{1} = (\textstyle\bigvee_n g_n^*) \wedge k\mathbf{1} = \bigvee_n (g_n^* \wedge k\mathbf{1}) = \bigvee_{n \geq k} (g_n^* \wedge k\mathbf{1}).$$

But for every $n \geq k$, $g_n^* \wedge k\mathbf{1} = g_k^*$, so we are through.

Remark. The above immediately gives the stronger conclusion: if f^* exists and $g_a \leq f_a$, then g^* exists.

The verification of the following corollary is straightforward.

(7.7) *Let $\{f_\alpha\}$ be a bounded set in $\mathfrak{M}$. If in the following chain, $(\bigvee_\alpha f_\alpha)^*$ exists, then so do all the star elements in the preceding terms, and the inequalities hold.*

$$(\textstyle\bigwedge_\alpha f_\alpha)^* \leq \bigwedge_\alpha f_\alpha^* \leq \bigvee_\alpha f_\alpha^* \leq (\bigvee_\alpha f_\alpha)^*.$$

Similarly for the following chain if $(\bigwedge_\alpha f_\alpha)_*$ exists.

$$(\textstyle\bigwedge_\alpha f_\alpha)_* \leq \bigwedge_\alpha (f_\alpha)_* \leq \bigvee_\alpha (f_\alpha)_* \leq (\bigvee_\alpha f_\alpha)_*.$$

For countable sets, the last inequality in the first chain and the first in the second chain become equalities:

(7.8) *For a countable bounded set* $\{f_n\}$ *in* $\mathfrak{M}$,
(a) *if* $(\bigvee_n f_n)^*$ *exists, then* $(\bigvee_n f_n)^* = \bigvee_n f_n^*$;
(b) *if* $(\bigwedge_n f_n)_*$ *exists, then* $(\bigwedge_n f_n)_* = \bigwedge_n (f_n)_*$.

This follows, via (4.3) and (4.4), from the corresponding theorem for M [1; (7.7)]. We remark, that (7.8) gives us in particular: if f^*, g^* exist then $f^* \vee g^* = (f \vee g)^*$, and if f_*, g_* exist then $f_* \wedge g_* = (f \wedge g)_*$.

By only slight modifications of the proof of (7.6), we obtain:

(7.9) *If f^* and g^* exist, then so does $(f + g)^*$, and*

$$(f + g)^* \leq f^* + g^*.$$

Similarly, if f_* and g_* exist, then so does $(f + g)_*$, and

$$f_* + g_* \leq (f + g)_*.$$

If $f \in \mathcal{S}$, then $f = f_* = f^*$. It follows from (7.6) that f^* exists for every $f \in \mathfrak{M}$ bounded above by an element of $\mathcal{S}$, and f_* exists for every $f \in \mathfrak{M}$ bounded below by an element of $\mathcal{S}$. Indeed these hold if f_a is bounded above, or respectively bounded below, by an element of $\mathcal{S}_a$. We can actually make a sharper statement. It is intuitive and what we would have expected, but the proof is not short.

(7.10) *If $f \in \mathfrak{M}$ is dominated by an element of $\mathcal{S}$—more generally, if f_a is dominated by an element of $\mathcal{S}_a$—then*

$$f^* = \bigwedge \{h \in \mathcal{S} \mid h_a \geq f_a\}.$$

Proof. For simplicity we assume $f \in \mathfrak{M}_a$. Also, as in the proof of (7.6), we can assume $f \geq 0$. But now, given $h \in \mathcal{S}$, $h_a \geq f$ if and only if $h \geq f$. Finally, every element of $\mathcal{S}$ is an infimum of l.s.c. elements, so the theorem reduces to the following: We have $f \in (\mathfrak{M}_a)_+$, the set $\{l \mid l$ an l.s.c. element, $l \geq f\}$ is nonempty, and we have to show the infimum of this set is f^*. In the remainder of the proof we denote l.s.c. elements by the letter l. Also we fix, once and for all, a particular $l_0 \geq f$.

Set $g = \bigwedge \{l \mid l \geq f\}$. Consider $\lambda \geq 0$; we have to show that $g \wedge \lambda \mathbf{1} = (f \wedge \lambda \mathbf{1})^*$. That $g \wedge \lambda \mathbf{1} \geq (f \wedge \lambda \mathbf{1})^*$ follows from the fact that $l \geq f$ implies $l \wedge \lambda \mathbf{1} \geq f \wedge \lambda \mathbf{1}$. To show equality, it is enough to show that for every $l \geq f \wedge \lambda \mathbf{1}$, $g \wedge \lambda \mathbf{1} \leq l$.

Consider $l \geq f \wedge \lambda \mathbf{1}$, and we can assume that $l \leq l_0 \wedge \lambda \mathbf{1}$. Choose a sequence of positive real numbers $\lambda_n \uparrow \lambda$, and for each n, set

$$e_n = \mathbf{1}_{(l - \lambda_n \mathbf{1})^+}, \quad d_n = \mathbf{1} - e_n, \quad l_n = l_{d_n} + (l_0)_{e_n}.$$

As the notation indicates, l_n is l.s.c. However this requires proof. By its very definition, e_n is l.s.c., hence also $(l_0)_{e_n}$. That l_n is l.s.c. then follows from

(i) $$l_n = (l \wedge \lambda_n \mathbf{1}) \vee (l_0)_{e_n}.$$

We show this. Note first that $l_n = l_{d_n} \vee (l_0)_{e_n}$. Since $l_{d_n} \leq l \wedge \lambda_n \mathbf{1}$, this gives us first that $l_n \leq (1 \wedge \lambda_n \mathbf{1}) \vee (l_0)_{e_n}$. It gives us secondly that for the opposite inequality, we need only show

$$l \wedge \lambda_n \mathbf{1} \leq l_{d_n} \vee (l_0)_{e_n}.$$

This follows from $(l \wedge \lambda_n \mathbf{1})_{d_n} \leq l_{d_n}$ and $(l \wedge \lambda_n \mathbf{1})_{e_n} \leq (l_0)e_n$.

(ii) $$l_n \geq f.$$

We of course have $(l_0)_{e_n} \geq f_{e_n}$; we show $l_{d_n} \geq f_{d_n}$.

$$(f \wedge \lambda \mathbf{1})_{d_n} \leq l_{d_n} \leq \lambda_n d_n .$$

Writing this $f_{d_n} \wedge \lambda d_n \leq \lambda_n d_n$, the strict inequality $\lambda > \lambda_n$, gives us $f_{d_n} \leq \lambda_n d_n$. But then

$$f_{d_n} = f_{d_n} \wedge \lambda_n d_n = (f \wedge \lambda_n \mathbf{1})d_n \leq (f \wedge \lambda \mathbf{1})_{d_n} \leq l_{d_n}.$$

It follows from (ii) that $g \leq \wedge_n l_n$, hence

$$g \wedge \lambda \mathbf{1} \leq \wedge_n (l_n \wedge \lambda \mathbf{1}).$$

We show $\wedge_n (l_n \wedge \lambda \mathbf{1}) = l$, which will complete the proof. Set $e = \wedge_m e_m$. Then $e \vee (\vee_m d_m) = \mathbf{1}$, hence it is enough to show that for every m,

$$\wedge_n (l_n \wedge \lambda \mathbf{1})_{d_m} = l_{d_m} \quad \text{and} \quad \wedge_n (l_n \wedge \lambda \mathbf{1})_e = l_e .$$

$\wedge_n (l_n \wedge \lambda \mathbf{1})_{d_m} = \wedge_n [(l_n)_{d_m} \wedge \lambda\, d_m] = (\wedge_n (l_n)_{d_m}) \wedge \lambda\, d_m = l_{d_m} \wedge \lambda\, d_m = l_{d_m}$. Here the second last equality follows from the fact that for $n < m$, $(l_n)_{d_m} \geq l_{d_m}$ and for $n \geq m$, $(l_n)_{d_m} = l_{d_m}$.

That $l_e = \lambda e$ is clear; we show $(l_n \wedge \lambda \mathbf{1})_e = \lambda e$ for every n. Given n, $(l_n \wedge \lambda \mathbf{1})_e = (l_n)_e \wedge \lambda e = (l_0)_e \wedge \lambda e$, so it is enough to show $(l_0)_e \geq \lambda e$. For every n, $(l_0)_e = ((l_0)_{e_n})e \geq (\lambda_n e_n)_e = \lambda_n e$; since $\lambda = \sup_n \lambda_n$, it follows that $(l_0)_e \geq \lambda e$.

f^* may exist for an $f \in \mathfrak{M}$ which is not dominated by an l.s.c. element. The $f \in \mathfrak{M}_a$ obtained in the example at the end of §6 has the property $f = f^*$ (since for every $\lambda \geq 0, f \wedge \lambda \mathbf{1} \in U$).

8. Convergence to the star elements

In §9 we define the universally measurable elements and establish the characterizations described in the Introduction. The present section is devoted to various preparatory propositions, culminating in (8.9).

(8.1) *Given $f = \vee_n f_n$ in M, if each f_n is an infimum of* l.s.c. *elements, then so is f.*

Proof. Since this of course holds for finite suprema, we can assume $f_1 \leq f_2 \leq \cdots$. $f \in M$, hence there are l.s.c. elements above it; hence $f' = \bigwedge \{h \text{ l.s.c.} \mid h \geq f\}$ exists. We show that for each $\mu \in L_+$ and $\epsilon > 0$, $[f', \mu\rangle \leq \langle f, \mu\rangle + 2\epsilon$. It will follow $f' = f$.

Consider such a μ and ϵ. It is enough to show there exists an l.s.c. element $h \geq f$ such that $\langle h, \mu\rangle \leq \langle f, \mu\rangle + 2\epsilon$.

(i) There exists an ascending sequence $h_1 \leq h_2 \leq \cdots$ of l.s.c. elements such that for each n, $h_n \geq f_n$ and $\langle h_n, \mu\rangle \leq \langle f_n, \mu\rangle + \epsilon$.

Each f_n is actually the limit of a descending net of l.s.c. elements, hence there exists an l.s.c. element $g_n \geq f_n$ such that

$$\langle g_n, \mu\rangle \leq \langle f_n, \mu\rangle + \epsilon/2^n.$$

Set $h_n = \bigvee_1^n g_i$ $(n = 1, 2, \cdots)$. We show by induction that

$$\langle h_n, \mu\rangle \leq \langle f_n, \mu\rangle + (\epsilon - \epsilon/2^n),$$

which will give us (i). The inequality of course holds for $n = 1$. Assume it holds for $n - 1$. Then

$$\begin{aligned}
\langle h_n, \mu\rangle &= \langle g_n \vee h_{n-1}, \mu\rangle \\
&= \langle g_n, \mu\rangle + \langle (h_{n-1} - g_n)^+, \mu\rangle \\
&\leq \langle g_n, \mu\rangle + \langle h_{n-1} - f_{n-1}, \mu\rangle \\
&= \langle g_n, \mu\rangle + \langle h_{n-1}, \mu\rangle - \langle f_{n-1}, \mu\rangle \\
&\leq \langle f_n, \mu\rangle + \epsilon/2^n + (\epsilon - \epsilon/2^{n-1}) \\
&= \langle f_n, \mu\rangle + (\epsilon - \epsilon/2^n).
\end{aligned}$$

(The first inequality follows from $g_n \geq f_n \geq f_{n-1}$.)

Now $f \leq \lambda\mathbf{1}$ for some λ, so we can assume $h_n \leq \lambda\mathbf{1}$ for all n; hence there exists h such that $h_n \uparrow h$. It follows we can find n such that

$$\langle h, \mu\rangle \leq \langle h_n, \mu\rangle + \epsilon \leq \langle f_n, \mu\rangle + 2\epsilon \leq \langle f, \mu\rangle + 2\epsilon.$$

(8.2) *If $f \in \mathfrak{M}_+$ is an infimum of* l.s.c. *elements, then $\mathbf{1}_f$ is an infimum of* l.s.c. *components of* $\mathbf{1}$.

Proof. By hypothesis, $f = \bigwedge_\alpha h_{\alpha'}$ the h's l.s.c. elements. Then for each $n = 1, 2, \cdots, nf = \bigwedge_\alpha nh_{\alpha'}$ hence $nf \wedge \mathbf{1} = \bigwedge_\alpha (nh_\alpha \wedge \mathbf{1})$. Since $\mathbf{1}_f = \bigvee_n (nf \wedge \mathbf{1})$, it follows from (8.1) that $\mathbf{1}_f$ is an infimum of l.s.c. elements. The proof thus reduces to:

(8.3) *If a component e of $\mathbf{1}$ is an infimum of* l.s.c. *elements, then it is an infimum of* l.s.c. *components of* $\mathbf{1}$.

We can assume there is a descending net $\{h_\alpha\}$ of l.s.c. elements such that $h_\alpha \downarrow e$. For each α, set $e_\alpha = \mathbf{1}_{(h_\alpha - (1/2)\mathbf{1})^+}$. We show first that $e_\alpha \geq e$ for all α.

In effect, $h_\alpha \geq e$, so

$$(h_\alpha - (\tfrac{1}{2})\mathbf{1})^+ \geq (e - (\tfrac{1}{2})\mathbf{1})^+ = (\tfrac{1}{2})e.$$

That $e_\alpha \geq e$ now follows from the simple property that $g \geq \lambda e$ for some $\lambda > 0$ implies $\mathbf{1}_g \geq e$.

It follows $\bigwedge_\alpha e_\alpha \geq e$. On the other hand, $e_\alpha \leq 2h_\alpha$, hence $\bigwedge_\alpha e_\alpha \leq 2 \bigwedge_\alpha h_\alpha = 2e$. Since $\bigwedge_\alpha e_\alpha$ is itself a component of $\mathbf{1}$, we must have $\bigwedge_\alpha e_\alpha = e$. This completes the proof of (8.3), and with it, (8.2).

Remark 1. The infimum of two l.s.c. elements is an l.s.c. element, and similarly for two l.s.c. components of $\mathbf{1}$. Thus an infimum of l.s.c. elements (resp. components of $\mathbf{1}$) is the limit of a descending net of l.s.c. elements (resp. components of $\mathbf{1}$).

Remark 2. Given a countable collection of nets on a space, we can, by a standard procedure in the theory of nets, replace them by subnets all having the same index system. We will be using this below.

(8.4) *Given $f \in \mathfrak{M}_+$, if for every $\lambda \geq 0$, $f \wedge \lambda\mathbf{1}$ is an infimum of* l.s.c. *elements, then there exists a net $\{g_\alpha\}$ of* l.s.c. *elements in M_+ such that $f = \lim_\alpha f_\alpha$.*

Proof. For each $n = 1, 2, \cdots$, set

$$f_n = f \wedge n\mathbf{1} - f \wedge (n-1)\mathbf{1} = (f \wedge n\mathbf{1} - (n-1)\mathbf{1})^+,$$

and $d_n = \mathbf{1}_{f_n}$. Then $d_n = \mathbf{1}_{(f-(n-1)\mathbf{1})^+}$ and therefore $d_n \downarrow 0$ (cf. the paragraph preceding (3.3)). Also, by the definition of the f_n's, $\sum_n f_n = f$, hence $\lim_n f_n = 0$.

Fix n. By hypothesis, $f \wedge n\mathbf{1}$ is an infimum of l.s.c. elements, hence f_n is also, hence in turn, by (8.2), d_n is an infimum of l.s.c. components of $\mathbf{1}$. Thus f_n is the limit of a descending net of l.s.c. elements, and d_n of a descending net of l.s.c. components of $\mathbf{1}$.

Now this holds for each n. Applying Remark 2 above, we can thus assume there exists a countable collection $\{h_{n,\beta} \mid \beta\}$ $(n = 1, 2, \cdots)$ of descending nets of l.s.c. elements, and one $\{d_{n,\beta} \mid \beta\}$ $(n = 1, 2, \cdots)$ of l.s.c. components of $\mathbf{1}$, all with the same index system $\{\beta\}$, such that for each n,

$$f_n = \lim_\beta h_{n,\beta} \quad \text{and} \quad d_n = \lim_\beta d_{n,\beta}.$$

Even more,

(i) *the above nets can be chosen so that $d_{n+1,\beta} \leq d_{n,\beta}$ and $h_{n,\beta} \leq d_{n,\beta}$ for all n and β.*

Consider the nets obtained in the preceding paragraph. For each n and β set $d'_{n,\beta} = \bigwedge_{j=1}^n d_{j,\beta}$. The $d'_{n,\beta}$'s satisfy the first inequality in (i). Moreover, they have all the properties of the $d_{n,\beta}$'s. Thus we can assume the $d_{n,\beta}$'s themselves already satisfy the inequality. Now for each n and β, set $h'_{n,\beta} =$

$h_{n,\beta} \wedge d_{n,\beta}$. The $h'_{n,\beta}$'s then satisfy the second inequality in (i). It is also easily verified that they have all the properties of the $h_{n,\beta}$'s. (That $h'_{n,\beta} \geq f_n$ follows from $d_{n,\beta} \geq d_n \geq f_n$, this last since $0 \leq f_n \leq \mathbf{1}$.) So again we can assume the $h_{n,\beta}$'s already satisfy the inequality. We thus have (i).

We can now obtain the net $\{g_\alpha\}$ of the theorem. Endow the product set $\{(n, \beta)\}$ with the product order. For each (n, β), set $g_{(n,\beta)} = \sum_{j=1}^{n} h_{j,\beta}$. We show $\{g_{(n,\beta)}\}$ is the desired net $\{g_\alpha\}$.

For this, it is enough to show that for every $k = 1, 2, \cdots$,

$$f \wedge k\mathbf{1} = \lim_{(n,\beta)} (g_{(n,\beta)} \wedge k\mathbf{1}).$$

Since $\{(n, \beta) \mid n \geq k\}$ is terminal in $\{(n, \beta)\}$, we need only show that

$$\text{(ii)} \qquad f \wedge k\mathbf{1} = \lim_{(n,\beta),n\geq k} (g_{(n,\beta)} \wedge k\mathbf{1}).$$

In the following computation, we relegate the justification for some of the steps to the Appendix, §11.

Consider $g_{(n,\beta)}$ with $n \geq k$.
$g_{(n,\beta)} \wedge k\mathbf{1} = (\sum_{j=1}^{n} h_{j,\beta}) \wedge k\mathbf{1} \leq (\sum_{j=1}^{k} h_{j,\beta}) \wedge k\mathbf{1} + (\sum_{j=k+1}^{n} h_{j,\beta}) \wedge k\mathbf{1}$. Since $h_{j,\beta} \leq \mathbf{1}$ for all j, β, the first of these two terms is simply $\sum_{j=1}^{k} h_{j,\beta}$. We consider the second.

$$\begin{aligned}
(\textstyle\sum_{j=k+1}^{n} h_{j,\beta}) \wedge k\mathbf{1} &\leq (\textstyle\sum_{j=k+1}^{n} d_{j,\beta}) \wedge k\mathbf{1} \\
&= (\textstyle\bigvee_{j=k+1}^{n} (j-k) d_{j,\beta}) \wedge k\mathbf{1} \qquad (11.1)\\
&= \textstyle\bigvee_{j=k+1}^{n} ((j-k) d_{j,\beta}) \wedge k\mathbf{1}) \\
&\leq \textstyle\bigvee_{j=k+1}^{n} (k d_{j,\beta}) \\
&= k \textstyle\bigvee_{j=k+1}^{n} d_{j,\beta} \\
&= k\, d_{k+1,\beta}.
\end{aligned}$$

Thus $g_{(n,\beta)} \wedge k\mathbf{1} \leq \sum_{j=1}^{k} h_{j,\beta} + k d_{k+1,\beta}$. This gives the second inequality in

$$\text{(iii)} \qquad f \wedge k\mathbf{1} \leq g_{(n,\beta)} \wedge k\mathbf{1} \leq (\textstyle\sum_{j=1}^{k} h_{j,\beta} + k d_{k+1,\beta}) \wedge k\mathbf{1}.$$

The first follows from

$$g_{(n,\beta)} = \textstyle\sum_{j=1}^{n} h_{j,\beta} \geq \sum_{j=1}^{n} f_j = f \wedge n\mathbf{1} \geq f \wedge k\mathbf{1}.$$

We show finally that the (descending) net on the right of (iii) converges to $f \wedge k\mathbf{1}$ with β. This will give us (ii).

$$\begin{aligned}
\lim_\beta (\textstyle\sum_{j=1}^{k} h_{j,\beta} + k d_{k+1,\beta}) &= \textstyle\sum_{j=1}^{k} f_j + k d_{k+1} \\
&= f \wedge k\mathbf{1} + k d_{k+1} \\
&= (f \wedge k\mathbf{1}) \vee (2k d_{k+1}) \qquad (11.3)
\end{aligned}$$

whence,

$$\begin{aligned}\lim_\beta \left(\textstyle\sum_{j=1}^{k} h_{j,\beta} + kd_{k+1,\beta}\right) \wedge k\mathbf{1} &= ((f \wedge k\mathbf{1}) \vee (2kd_{k+1})) \wedge k\mathbf{1} \\ &= (f \wedge k\mathbf{1}) \vee (kd_{k+1}) \\ &= f \wedge k\mathbf{1}.\end{aligned}$$

For the last equality, see the opening remark in the proof of (11.1).

We have actually proved more than the statement of (8.4). Denote each $d_{n,\beta}$ in the proof by $d_{(n,\beta)} = d_\alpha$; we thus have a net $\{d_\alpha\}$, and clearly $d_\alpha \downarrow \mathbf{0}$.

(8.5) *Together with the net* $\{g_\alpha\}$ *obtained in* (8.4), *there exists a net* $\{e_\alpha\}$ *of* u.s.c. components of $\mathbf{1}$ *such that:*

(1) $e_\alpha \uparrow \mathbf{1}$.

(2) *For each* α_0, *there exists* $n(\alpha_0)$ *with the property that*

$$(g_\alpha)_{e_{\alpha_0}} \leq n(\alpha_0)\mathbf{1} \quad \textit{for all } \alpha.$$

Proof. For each α, set $e_\alpha = \mathbf{1} - d_\alpha$. We need only prove (2). $\alpha_0 = (n_0, \beta_0)$; we claim n_0 is the desired $n(\alpha_0)$. In effect, given α,

$$\begin{aligned}(g_{(n,\beta)})_{e_{(n_0,\beta_0)}} &= \textstyle\sum_{j=1}^{n} (h_{j,\beta})_{e_{(n_0,\beta_0)}} \\ &\leq \textstyle\sum_{j=1}^{n} (d_{j,\beta})e_{(n_0,\beta_0)}) \\ &= \textstyle\sum_{j=1}^{n} (d_{j,\beta} \wedge e_{(n_0,\beta_0)}) \\ &\leq \textstyle\sum_{j=1}^{n_0} (d_{j,\beta} \wedge e_{(n_0,\beta_0)}) \\ &\leq n_0\mathbf{1}.\end{aligned}$$

We turn specifically to the star elements. From (8.4) and (8.5), we have:

(8.6) *Given* $f \in \mathfrak{M}_+$, *if* f^* *exists, then there exist a net* $\{g_\alpha\}$ *of* l.s.c. *elements in* M_+, *and a net* $\{e_\alpha\}$ *of* u.s.c. *components of* $\mathbf{1}$, *such that:*

(1) $f^* = \lim_\alpha g_\alpha$.

(2) $e_\alpha \uparrow \mathbf{1}$.

(3) *For each* α_0, *there exists* $n(\alpha_0)$ *with the property that* $(g_\alpha)_{e_{\alpha_0}} \leq n(\alpha_0)\mathbf{1}$ for all α.

For $f \in \mathfrak{M}_+$, f_* always exists and is the limit of an ascending net of positive u.s.c. elements of M ((7.5) and the remark following it). Moreover, if f^* exists, this net can be chosen in a well-defined relation to the nets in (8.6):

(8.7) *Given* $f \in \mathfrak{M}_+$, f^* *exists, then there exist nets* $\{g_\alpha\}$ *and* $\{e_\alpha\}$ *as described in* (8.6), *and a net* $\{q_\alpha\}$ *of* u.s.c. *elements* (*with the same index system*) *such that:*

(1) $0 \leq q_\alpha \leq g_\alpha$ *for all* α.

(2) $f_* = \lim_\alpha q_\alpha$.

Proof. As in the proof of (8.4), for each $n = 1, 2, \cdots$, set

$$(f^*)_n = f^* \wedge n\mathbf{1} - f^* \wedge (n-1)\mathbf{1}$$

and

$$(f_*)_n = f_* \wedge n\mathbf{1} - f_* \wedge (n-1)\mathbf{1}.$$

Let $\{h_{n,\beta}\}$, $\{d_{n,\beta}\}$ $(n = 1, 2, \cdots)$ be the nets obtained in that proof, this time for the $(f^*)_n$'s.

Now every $(f_*)_n$ is the limit of an ascending net of positive u.s.c. elements. By remark 2 preceding (8.4), we can assume that these nets all have the index system $\{\beta\}$. Thus for each n, we have $\{p_{n,\beta}\}$, positive u.s.c. elements, such that $p_{n,\beta} \uparrow (f_*)_n$. Since $(f_*)_n \leq (f^*)_n$, $p_{n,\beta} \leq h_{n,\beta'}$ for all n and all β, β'.

Setting $g_{(n,\beta)} = \sum_{j=1}^{n} h_{n,\beta'}\, q_{(n,\beta)} = \sum_{j=1}^{n} p_{n,\beta'}$ (and $e_{(n,\beta)} = \mathbf{1} - d_{(n,\beta)}$), it is easily verified that (8.7) is satisfied.

Remark. From (1) above, it follows of course that (3) in (8.6) holds for the q_α's also.

We now extend (8.6) to elements of $\mathfrak{M}$ not necessarily in $\mathfrak{M}_+$.

(8.8) *Given $f \in \mathfrak{M}$, if f^* exists, then there exist a net $\{l_\alpha\}$ of* l.s.c. *elements in M, and a net $\{e_\alpha\}$ of* u.s.c. *components of* $\mathbf{1}$, *such that:*

(1) $f^* = \lim_\alpha l_\alpha$.

(2) $e_\alpha \uparrow \mathbf{1}$.

(3) *For each α_0, there exists a natural number $n(\alpha_0)$ with the property that*

$$(l_\alpha)_{e_{\alpha_0}} \leq n(\alpha_0)\mathbf{1} \quad \textit{for all } \alpha.$$

Proof. By (7.3), $(f^+)^*$ and (of course) $(f^-)_*$ exist, and $f^* = (f^+)^* - (f^-)_*$. Applying (8.6) and the remarks preceding (8.7) (we do not use (8.7) itself)—and, as usual, Remark 2 preceding (8.4)—we have nets $\{g_\alpha\}$, $\{q_\alpha\}$, $\{e_\alpha\}$, consisting respectively of positive l.s.c. elements, positive u.s.c. elements, and u.s.c. components of $\mathbf{1}$, such that $\{g_\alpha\}$ and $\{e_\alpha\}$ satisfy (8.6) for $(f^+)^*$ and $q_\alpha \uparrow (f^-)_*$. For each α, set $l_\alpha = g_\alpha - q_\alpha$. Then $\{l_\alpha\}$ is a net of l.s.c. elements converging to f^* (5.2). That (3) holds for $\{l_\alpha\}$ follows from the fact that it holds for $\{g_\alpha\}$ and that $l_\alpha \leq g_\alpha$ for all α.

Finally:

(8.9) *Given $f \in \mathfrak{M}$. if f^* and f_* exist, then there exist a net $\{l_\alpha\}$ of* l.s.c. *elements, a net $\{u_\alpha\}$ of* u.s.c. *elements, and a net $\{e_\alpha\}$ of* u.s.c. *components of* $\mathbf{1}$, *all in M, such that:*

(1) $u_\alpha \leq l_\alpha$ *for all* α.

(2) $f^* = \lim_\alpha l_\alpha$.

(3) $f_* = \lim_\alpha u_\alpha$.

(4) $e_\alpha \uparrow \mathbf{1}$.

(5) *For each α_0, there exists a natural number $n(\alpha_0)$ with the property that for all α,*

$$-n(\alpha_0)\mathbf{1} \leq u_\alpha \leq l_\alpha \leq n(\alpha_0)\mathbf{1}.$$

Proof. Choose nets $\{g_\gamma^{(1)}\}$, $\{q_\alpha^{(2)}\}$, $\{e_\alpha^{(1)}\}$ to satisfy (8.7) for f^+, and nets $\{g_\alpha^{(2)}\}$, $\{q_\alpha^{(2)}\}$, $\{e_\alpha^{(2)}\}$ to satisfy it for f^-, and as always we arrange for all of these to have the same index system. For each α, set $l_\alpha = g_\alpha^{(1)} - q_\alpha^{(2)}$, $u_\alpha = q_\alpha^{(1)} - g_\alpha^{(2)}$, and $e_\alpha = e_\alpha^{(1)} \wedge e_\alpha^{(1)}$. It is easily verified that these have the desired properties.

9. The universally measurable elements

We denote by $\mathfrak{U}$ the set $\{f \in \mathfrak{M} \mid f = f_* = f^*\}$, and call its elements *universally measurable.* We will show they are precisely the elements of $\mathfrak{M}$ which are limits of nets of C, and for which our general Lusin theorem (9.9) holds.

First for their elementary properties.

(9.1) *An element of $\mathfrak{M}$ lies in $\mathfrak{U}$ if and only if $f^{(\lambda)}$ lies in U for every $\lambda \geq 0$.*

This follows from the very definition.

(9.2) *$\mathfrak{U}$ is a σ-closed linear sublattice of $\mathfrak{M}$.*

Proof. Given $f, g \in \mathfrak{U}$, then by (7.9) and (7.2),

$$(f + g)^* \leq f^* + g^* = f + g = f_* + g_* \leq (f + g)_* \leq (f + g)^*.$$

We thus have equality, giving $f + g \in \mathfrak{U}$. Th√t $f \in \mathfrak{U}$ implies $\kappa f \in \mathfrak{U}$ for all $\kappa \in R$ is straightforward.

Again, given $f, g \in \mathfrak{U}$, then by (7.7) and the remark following (7.8),

$$(f \vee g)^* = f^* \vee g^* = f \vee g = f_* \vee g_* \leq (f \vee g)_* \leq (f \vee g)^*.$$

Thus we have equality, giving $f \vee g \in \mathfrak{U}$. Finally, the σ-closedness of $\mathfrak{U}$ follows from (9.1) and the σ-closedness of U in M [1; (8.2)].

Since for every $f \in \mathfrak{M}$, $f = \lim_n f^{(n)}$, (9.1) and (9.2) give:

(9.3) *$\mathfrak{U}$ is the σ-closure in $\mathfrak{M}$ of U, and moreover, every element of $\mathfrak{U}$ is the limit of a sequence in U.*

The next three propositions are easily verified.

(9.4) *$\mathfrak{U}$ is isomorphic with its projection $\mathfrak{U}_a$ in $\mathfrak{M}_a$.*

(9.5) *Given a sequence $\{f_n\}$ in $\mathfrak{U}$, and $f \in \mathfrak{U}$, then $f = \lim_n f_n$ if and only if $f_a = \lim_n (f_n)_a$.*

(9.6) *$\mathfrak{U}$ is Dedekind-closed in $\mathfrak{M}$.*

We now proceed to our two characterizations of $\mathfrak{U}$.

(9.7) *An element of $\mathfrak{M}$ lies in $\mathfrak{U}$ if and only if it is the limit of a net of C.*

Proof. Suppose $f = f_* = f^*$. Let $\{l_\alpha\}$ and $\{u_\alpha\}$ be the nets given by (8.9). As is well known, for each α, we can choose $f_\alpha \in C$ such that $u_\alpha \leq f_\alpha \leq l_\alpha$. Then $f = \lim_\alpha f_\alpha$, and we thus have the necessity. The sufficiency follows from (9.1) and the corresponding theorem for M [1; (9.6)].

For the Lusin theorem, we need the following lemma. We emphasize that in both (9.8) and (9.9), the relation $\bigvee_\gamma e_\gamma = \mathbf{1}$ is referring to M (thus, for example, the set of "characteristic functions" of finite subsets of X does *not* have $\mathbf{1}$ for its supremum).

(9.8) *Let* $\{e_\gamma\}$ *be a set of components of* $\mathbf{1}$ *such that* $\bigvee_\gamma e_\gamma = \mathbf{1}$. *Given a net* $\{f_\alpha\}$ *in* $\mathfrak{M}$, *and* $f \in \mathfrak{M}$, *if for each* γ, $\lim_\alpha (f_\alpha)_{e_\gamma} = f_{e_\gamma}$, *then* $\lim_\alpha f_\alpha = f$.

Proof. Assume first that $\{f_\alpha\}$ is bounded, and set $g = \limsup_\alpha f_\alpha$. Since projection is (order) continuous, it follows that for each γ, $g_{e_\gamma} = \limsup_\alpha (f_\alpha)_{e_\gamma} = f_{e_\gamma}$. This gives in turn that $g = f$ (cf. the discussion following (3.3)). Thus $f = \limsup_\alpha f_\alpha$. A similar argument gives us that $f = \liminf_\alpha f_\alpha$, so we have $f = \lim_\alpha f_\alpha$.

We turn to the general case. We have to show that for each $\lambda \geq 0$, $\lim_\alpha f_\alpha^{(\lambda)} = f^{(\lambda)}$. Given $\lambda \geq 0$, and applying (f) in (4.1), we have that for each γ, $(f_\alpha^{(\lambda)})_{e_\gamma} = (f_\alpha)_{e_\gamma}^{(\lambda)}$ for all α, and $(f^{(\lambda)})_{e_\gamma} = f_{e_\gamma}^{(\lambda)}$. Thus, from the hypothesis, $\lim_\alpha (f_\alpha^{(\lambda)})_{e_\gamma} = (f^{(\lambda)})_{e_\gamma}$. It follows from the first part of the proof that $\lim_\alpha f_\alpha^{(\lambda)} = f^{(\lambda)}$, and we are through.

(9.9) *Given* $f \in \mathfrak{M}$, *let* A *be the set of components* e *of* $\mathbf{1}$ *each satisfying*

(a) e u.s.c.

(b) $f_e \in C_e$.

Then $f \in \mathfrak{U}$ *if and only if* $\bigvee A = \mathbf{1}$.

Proof. We show the condition of the theorem is equivalent to that of (9.7).

(i) *For each pair* e_1, e_2 *of components of* $\mathbf{1}$ *in* A, $e_1 \vee e_2$ *also lies in* A. *Thus the elements of* A *form an ascending net.*

This can be shown by means of the Tietze Extension Theorem and the fact that for a u.s.c. component e of $\mathbf{1}$, if K denotes the support of e in X, then C_e can be identified with $C(K)$.

Now assume $f \in \mathfrak{U}$, and let $\{l_\alpha\}$ and $\{u_\alpha\}$ be the nets given by (8.9). For each α, set $g_\alpha = l_\alpha - u_\alpha$. Then $\lim_\alpha g_\alpha = 0$, hence by (5.10), there exists a net $\{e_\gamma\}$ of components of $\mathbf{1}$, with $e_\gamma \uparrow \mathbf{1}$, such that for each γ, $\lim_\alpha (g_\alpha)_{e_\gamma} = 0$ uniformly. Moreover, since the g_α's are (non-negative) l.s.c. elements, the argument used in the proof of (5.10) gives us that the e_γ's can be chosen to be u.s.c.

For each α, choose $f_\alpha \in C$ such that $u_\alpha \leq f_\alpha \leq l_\alpha$. Then for each γ, $\lim_\alpha (f_\alpha)_{e_\gamma} = f_{e_\gamma}$ uniformly. Now it is easily verified that C_{e_γ} is norm-complete. It follows $f_{e_\gamma} \in C_{e_\gamma}$. We thus have the necessity.

Now assume $\bigvee A = \mathbf{1}$. From (i), we write A as a net $\{e_\gamma\}$, with $e_\gamma \uparrow \mathbf{1}$. From (b), for each γ, we can choose $g_\gamma \in C$ such that $(g_\gamma)_{e_\gamma} = f_{e_\gamma}$. We show $\lim_\gamma g_\gamma = f$. But this is immediate from (9.8) using the fact that for each γ, $(g_{\gamma'})e_\gamma = f_{e_\gamma}$ for all $\gamma' \succ \gamma$. We thus have the sufficiency.

10. Some final remarks

In the present section, the only convergence dealt with will be that of sequences, hence by Nakano's theorem (5.5), the convergence will be the ordinary one.

A subset A of a Riesz space is *σ-closed* if for every sequence $\{a_n\}$ in A, $\lim_n a_n = a$ implies $a \in A$. Given *any* set A, the smallest σ-closed set containing A will be called the *σ-closure* of A. If A is a linear sublattice, then so is its σ-closure.

We denote by $\mathcal{B}a$ (respectively $\mathcal{B}o$), the σ-closure of C (respectively $\mathcal{S}$) in $\mathfrak{M}$, and call its elements the *Baire* (respectively *Borel*) elements. By (9.2), we have $\mathcal{B}a \subset \mathcal{B}o \subset \mathcal{U}$. The set of elements of $\mathcal{B}a$ which are each the limit of a sequence in C will be denoted by $\mathcal{B}a^1$ and called the *first Baire class*. The set of elements which are each the limit of a sequence in $\mathcal{B}a^1$ will be denoted by $\mathcal{B}a^2$ and called the *second Baire class.*

Now let μ be a fixed (positive) element of L. The theorems (9.7) and (9.9) are extensions to $\mathfrak{M}$ of standard theorems on $\mathfrak{M}_\mu$. We give the latter here in the form they assume in the present context. Note that all of $\mathfrak{M}_\mu$ plays the role in $\mathfrak{M}_\mu$ that $\mathcal{U}$ plays in $\mathfrak{M}$.

(10.1) *Every element of $\mathfrak{M}_\mu$ is the limit of a sequence in C_μ.*

(10.2) (Lusin) *Given $f \in \mathfrak{M}_\mu$, there exists a sequence $\{e_n\}$ of* u.s.c. *components of* $\mathbf{1}$ *such that:*

(a) $(e_n)_\mu \uparrow \mathbf{1}_\mu$.

(b) $f_{e_n} \in C_{e_n}$ $(n = 1, 2, \cdots)$.

(b) is equivalent to (b′) $f_{(e_n)_\mu} \in (C_\mu)_{(e_n)_\mu}$. Finally, a third standard theorem on $\mathfrak{M}_\mu$ is that every μ-measurable function is equal μ-almost everywhere to some function of Baire class 2. In our context this takes the form:

(10.3) *Given $f \in \mathfrak{M}_\mu$, $f = h_\mu$ for some $h \in \mathcal{B}a^2$.*

Otherwise stated, the projection $\mathfrak{M} \to \mathfrak{M}_\mu$ maps $\mathcal{B}a^2$ onto $\mathfrak{M}_\mu$.

11. Appendix

In this section, as stated there, we establish some of the equalities and inequalities occurring in the proof of (8.4).

(11.1) *If the components $e_1, \cdots, e_n$ of* $\mathbf{1}$ *satisfy $e_1 \geq e_2 \geq \cdots \geq e_n$, then $\sum_1^n e_i = \bigvee_1^n ie_i$.*

Set $d_i = e_i - e_{i+1}$ $(i = 1, \cdots, n-1)$, $d_n = e_n$. For each i, $e_i = \sum_{j=i}^n d_j$, hence

$$\sum_{i=1}^n e_i = \sum_{i=1}^n \sum_{j=i}^n d_j = \sum_{i=1}^n i\, d_i = \bigvee_{i=1}^n i\, d_i \leq \bigvee_{i=1}^n ie_i.$$

For the opposite inequality, it is enough to show that for each i_0, $\sum_{i=1}^n e_i \geq i_0\, e_{i_0}$. But this is clear.

(11.2) *For a component e* of $\mathbf{1}$, *and λ, $\kappa \geq 0$,*

$$\lambda e \wedge \kappa \mathbf{1} \leq \kappa e.$$

This is clear.

In the following, f and d_{k+1} are those of (8.4)

(11.3) $f \wedge k\mathbf{1} + k\,d_{k+1} = (f \wedge k\mathbf{1}) \vee (2k\,d_{k+1})$

As we noted at the beginning of the proof of (8.4), $d_{k+1} = \mathbf{1}_{(f-k\mathbf{1})^+}$, so $k\,d_{k+1} \leq f$. So, setting $e = \mathbf{1} - d_{k+1}$, we have

$$f \wedge k\mathbf{1} = f \wedge (k\,d_{k+1} \vee ke) = (f \wedge k\,d_{k+1}) \vee (f \wedge ke) = (k\,d_{k+1}) \vee (f \wedge ke).$$

It follows that

$$\begin{aligned} f \wedge k\mathbf{1} + k\,d_{k+1} &= (k\,d_{k+1}) \vee (f \wedge ke) + k\,d_{k+1} \\ &= (2k\,d_{k+1}) \vee (f \wedge ke + k\,d_{k+1}) \\ &= (2k\,d_{k+1}) \vee (f \wedge ke + f \wedge k\,d_{k+1}) \\ &= (2kd_{k+1}) \vee (f \wedge k\mathbf{1}). \end{aligned}$$

BIBLIOGRAPHY

1. S. KAPLAN, *On the second dual of the space of continuous functions*, Trans. Amer. Math. Soc., vol. 86 (1957), pp. 70–90.
2. W. A. J. LUXEMBURG AND J. J. MASTERSON, *An extension of the concept of the order dual of a Riesz space*, Canad. J. Math., vol. 19 (1967), pp. 488–498.
3. J. J. MASTERSON, *A characterization of the Riesz space of measurable functions*, Trans. Amer. Math. Soc., vol. 135 (1969), pp. 193–197.
4. H. NAKANO, *Ergodic theorems in semi-ordered linear spaces*, Ann. of Math., vol. 49 (1948), pp. 538–556.

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