ISOMETRIES OF FUNCTION ALGEBRAS

BY

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Let X and Y be compact Hausdorff spaces. A and B will denote sub-algebras of $C(X)$ and $C(Y)$ respectively. ($C(X)$ indicates the space of continuous complex-valued functions on X.) It will be assumed that A and B are equipped with the sup-norm, are point separating, and contain the constant functions. In this paper, we give a description of the linear isometries from A to B in the case where $A = C(X)$ and $B = C(Y)$, and under certain restrictions on the pair (X, Y).

Operators of the form

$$(*) \qquad Tf = g(f \circ \psi),$$

where g is a fixed function in $C(Y)$ of norm 1 and ψ is a continuous map from Y into X such that $\psi(|g|^{-1}(1)) = X$, constitute a class of isometries from $C(X)$ into $C(Y)$. In fact, if T is an isometry of $C(X)$ onto $C(Y)$, then T must be of the form $(*)$ (see, e.g., [1, p. 442]). It is not true, in general, that all isometries from $C(X)$ into $C(Y)$ are of the form $(*)$. For example: let $\phi_i : [0, 1] \to [0, 1]$, $i = 1, 2$ be continuous functions having the following properties: $\phi_1 = \phi_2$ on $[0, 1/2]$, $\phi_1([0, 1/2]) = [0, 1]$, and $\phi_1(1) \neq \phi_2(1)$. Define isometries $T_i : C[0, 1] \to C[0, 1]$ by $T_i f = f \circ \phi_i$, $i = 1, 2$. Let

$$T = (1/2)T_1 + (1/2)T_2 .$$

Then T is an isometry, but T is not of the form $(*)$.

Let S_A and S_B denote the unit balls in the dual spaces of A and B respectively. Suppose $T : A \to B$ is an isometry. It follows from the Hahn-Banach theorem, that the adjoint T^* of T maps S_B onto S_A. Let l be an element of the set ex S_A of extreme points of S_A. Then $(T^*)^{-1}(l) \cap S_B$ is a non-empty weak* closed face of S_B. (A *face* F of a convex set K is a convex subset of K such that

$$cf_1 + (1 - c)f_2 \in F \quad \text{and} \quad (c, f_1, f_2) \in (0, 1) \times K \times K$$

implies that $f_1, f_2 \in F$.) It follows from the Krein-Milman Theorem that there is an extreme point e of S_B such that $T^*(e) = l$. It is known (see, e.g., [3, Prop. 6.2]) that l is an extreme point of S_A iff it is of the form $e^{i\alpha} l_x$, where $\alpha \in [0, 2\pi]$ and l_x denotes evaluation at a point x of the Choquet boundary of X with respect to A. Thus, we have the following:

PROPOSITION 1. *Let T be an isometry from A into B. Let $Y(T) = \{y \in Y \mid |T1(y)| = 1$ and there is a $\hat{T}(y) \in X$ such that $Tf(y) =$*

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$T1(y)f(\hat{T}(y))$ for all $f \in A\}$. Then the mapping $\hat{T}: Y(T) \to X$ is continuous and $\hat{T}(Y(T))$ contains the Choquet boundary of X with respect to A. ▮

COROLLARY. *$|Tf|$ assumes its maximum on $Y(T)$ for every $f \in A$.* ▮

Let $\mathcal{B}$ denote the space of bounded linear operators from A to B. We will make use of the *weak operator topology* on $\mathcal{B}$ (see [1, p. 476]).

We will use $\mathcal{B}_1$ to designate the set of operators in $\mathcal{B}$ having norm ≤ 1.

DEFINITION. Let $T: A \to B$ be a linear isometry. $\mathcal{E}(T)$ will denote the set $\{U \in \mathcal{B}_1 \mid Uf(y) = Tf(y)$ for every $f \in A$ and every $y \in Y(T)\}$.

THEOREM 2. *Let $T \in \mathcal{B}$ be an isometry. Then $\mathcal{E}(T)$ is a face of $\mathcal{B}_1$ and is closed in the weak operator topology. Furthermore, every member of $\mathcal{E}(T)$ is an isometry.*

Proof. The only part of the theorem that is not immediate from the above definition is the assertion that $\mathcal{E}(T)$ is a face. Suppose

$$cU_1 + (1 - c)U_2 \in \mathcal{E}(T) \quad \text{where } (c, U_1, U_2) \in (0, 1) \times \mathcal{B}_1 \times \mathcal{B}_1 .$$

Let $y \in Y(T)$. Then the mapping $f \to Tf(y)$ is in ex S_A. Since the mappings $f \to U_i f(y)$, $i = 1, 2$ are in S_A, it follows that $Tf(y) = U_1 f(y) = U_2 f(y)$ for all $f \in A$. ▮

COROLLARY. ex $\mathcal{E}(T) \subseteq$ ex $\mathcal{B}_1$. ▮

It is natural to ask for a description of the extreme points of $\mathcal{E}(T)$. One might try to find conditions under which the extreme points of $\mathcal{E}(T)$ are of the form $f \to gMf$, where $g \in B$ and M is an algebra monomorphism. Another appropriate question is whether or not $\mathcal{E}(T)$ is the weak operator closed convex hull of operators of the form $f \to gMf$.

For the remainder of the paper, it will be assumed that $A = C(X)$ and $B = C(Y)$.

DEFINITION. Let $T \in \mathcal{B}$ be an isometry. $\mathcal{F}(T)$ will denote the set

$$\{S \in \mathcal{E}(T) \mid S \text{ is of the form } (*)\}.$$

Note that the members of $\mathcal{F}(T)$ need not be extreme points of $\mathcal{E}(T)$.

DEFINITION. The pair (X, Y) is said to have the *weak Tietze property* if, whenever ϕ is a continuous map from a closed subset F of Y onto X, then ϕ has a continuous extension to all of Y.

Let Y be arbitrary. Let $\mathcal{C}$ be a collection of spaces such that (C, Y) has the weak Tietze property for each $C \in \mathcal{C}$. If X is the Cartesian product of $\mathcal{C}$, then (X, Y) has the weak Tietze property. It follows from the previous statement and from the Tietze Extension Theorem that, if X is an absolute retract, then (X, Y) has the weak Tietze property.

Let X be arbitrary. If (X, Z) has the weak Tietze property and Y is a closed subset of Z, then (X, Y) has the weak Tietze property. Suppose Y is totally disconnected and metric. Then Y can be looked upon as a closed subset of the Cantor set K. If it can be shown that (X, K) has the weak Tietze property, then it will follow that (X, Y) has the weak Tietze property. Suppose F is a closed subset of K and h maps F continuously onto X. It can be shown that there is a retraction $r : K \to F$. Thus, it follows that $h \circ r$ extends h.

THEOREM 3. *Let (X, Y) have the weak Tietze property. Suppose T is a linear isometry from $C(X)$ into $C(Y)$. Then $\mathcal{E}(T)$ is the weak operator closed convex hull of $\mathcal{F}(T)$.*

Proof. The following argument is an adaptation of one due to P. Morris and R. Phelps [2, Th. 2.1].

Suppose $U \in \mathcal{E}(T) \setminus \overline{\mathrm{cov}}\, \mathcal{F}(T)$. Then there are regular Borel measures $\mu_1, \mu_2, \cdots, \mu_n$, functions $f_1, f_2, \cdots, f_n \in C(X)$, and a real number $r > 0$, such that

$$(\dagger) \qquad \mathrm{Re}\left(\sum_{i=1}^n \int Uf_i \, d\mu_i\right) > \mathrm{Re}\left(\sum_{i=1}^n \int Ff_i \, d\mu_i\right) + r$$

for every $F \in \mathcal{F}(T)$. It can be assumed without loss of generality that $\mu_i \geq 0$ for $i = 1, 2, \cdots, n$. We can also assume without loss of generality that $\mu_i(Y(T)) = 0$ for $i = 1, 2, \cdots, n$, since $Uf_i = Ff_i$ on $Y(T)$ for $i = 1, 2, \cdots, n$. Let $\nu = \sum_{i=1}^n \mu_i$. Given $\varepsilon > 0$, there is a closed subset Z of $Y \setminus Y(T)$ such that $\nu(Y \setminus Z) < \varepsilon$. For $i = 1, 2, \cdots, n$, let h_i denote the Radon-Nikodym derivative of μ_i with respect to ν. Choose $h_i' \in C(Y)$ such that $0 \leq h_i' \leq 1$ and $\int | h_i - h_i' | \, d\nu < \varepsilon$ for $i = 1, 2, \cdots$.

Let $g = \sum_{i=1}^n h_i' \, Uf_i$. For each $y \in Y$, define $k_y = \sum_{i=1}^n h_i'(y) f_i$. Then $g(y) = Uk_y(y)$. $g(y)$ is also equal to $(U^* \delta_y)(k_y)$ where U^* is the adjoint of U and δ_y represents the unit point measure at y. The function $w(\rho) = \mathrm{Re} \int k_y \, d\rho$ is weak* continuous on $C(X)^*$. Since $U^* \delta_y \in S_{C(X)}$, it follows that $\sup w(S_{C(X)}) \geq \mathrm{Re}\, g(y)$. By the Krein-Milman Theorem, there is a $\lambda \in \mathrm{ex}\, S_{C(X)}$ such that $w(\lambda) > \mathrm{Re}\, g(y) - \varepsilon$. But $\lambda = e^{i\alpha} \delta_x$ for some $x \in X$ and some $\alpha \in [0, 2\pi)$. It follows that, for each $y \in Y$, we may choose a $\phi(y) \in X$ and a complex number $c(y)$ with $| c(y) | = 1$, such that

$$\mathrm{Re}\left(\sum_{i=1}^n c(y) h_i'(y) f_i(\phi(y))\right) > \mathrm{Re}\, g(y) - \varepsilon.$$

For each $y \in Z$, choose an open neighborhood V_y of y such that $V_y \cap Y(T) = \emptyset$ and

$$\mathrm{Re}\left(\sum_{i-1}^n c(y) h_i'(w) f_i(\phi(y))\right) > \mathrm{Re}\, g(w) - 2\varepsilon$$

for all $w \in V_y$. Let $\{V_{y_1}, V_{y_2}, \cdots, V_{y_p}\}$ be a finite collection of V_y's which covers Z. One can easily find another open cover $\{U_1, \cdots, U_p\}$ of Z such that $U_i \subseteq V_{y_i}$, $i = 1, 2, \cdots, p$ and

$$\nu(\{y \mid y \text{ is in more than one } U_j\}) < \varepsilon.$$

Consider the sets $H_j = (Z \cap U_j) \backslash \cup \{U_i \mid i \neq j\}, j = 1, 2, \cdots, p$. Then the sets H_j are closed and disjoint and $\nu(Z \backslash \bigcup_{j=1}^{p} H_j) < \varepsilon$.

Define a mapping $\theta : Y(T) \cup [\bigcup_{i=1}^{p} H_j] \to X$ by

$$\begin{aligned} \phi(y) &= \phi(y_j) \quad \text{if } y \in H_j \\ &= \hat{T}(y) \quad \text{if } y \in Y(T). \end{aligned}$$

Define a mapping $\psi : Y(T) \cup [\bigcup_{i=1}^{p} H_j] \to D$ where D is the closed unit disk by

$$\begin{aligned} \psi(y) &= C(y_j) \quad \text{if } y \in H_j \\ &= T1(y) \quad \text{if } y \in Y(T). \end{aligned}$$

Since (X, Y) has the weak Tietze property and θ is onto, it follows that θ has a continuous extension, which we shall also denote by θ, to all of Y. By the Tietze Extension Theorem, ψ also has an extension, denoted by ψ, to all of Y. Define an operator $F_1 : C(X) \to C(Y)$ by $F_1 f = \psi f \circ \theta$. Note that $F_1 \in \mathfrak{F}(T)$. By a straightforward argument (see [2, Th. 2.1]), one can find a constant $M > 0$ such that

$$\operatorname{Re}\left(\sum_{i=1}^{n} \int F_1 f_i \, d\mu_i\right) > \operatorname{Re}\left(\sum_{i=1}^{n} \int U f_i \, d\mu_i\right) - M\varepsilon.$$

Thus, by choosing ε sufficiently small we obtain a contradiction to (†).∎

COROLLARY 4. *Let Γ denote the unit circle and let*

$$T : C(X) \to C(Y)$$

be a linear isometry. Suppose that (X, Y) and (Γ, Y) have the weak Tietze property. Let $\mathfrak{F}_1(T) = \{U \in F(T) \mid |U1| \equiv 1\}$. Then $\mathcal{E}(T) = \overline{\text{cov}}\, \mathfrak{F}_1(T)$.

Proof. The proof is the same as that for Theorem 3, except that, in the present case, ψ can be extended as a mapping from $Y \to \Gamma$ instead of as a mapping from $Y \to D$.∎

Note that (Γ, Y) has the weak Tietze property iff any continuous map $\theta : F \to \Gamma$, where F is a closed subset of Y, has a continuous extension to all of Y.

COROLLARY 5. *Let (X, Y) and T be as in Theorem 3. Suppose that $T1 = 1$. Let $\mathfrak{M}$ be the set of algebra monomorphisms in $\mathcal{E}(T)$. Then $T \in \overline{\text{cov}}\, \mathfrak{M}$.*

Proof. Let $\mathcal{E}_1(T) = \{U \in \mathcal{E}(T) \mid U1 = 1\}$. Note that each $U \in \mathcal{E}_1(T)$ is positive, i.e., $f \geq 0 \Rightarrow Uf \geq 0$ (see [3, p. 36]). Note further, that $\mathfrak{M} = \mathcal{E}_1(T) \cap F(T)$. In the proof of Theorem 3, we made use of the fact that T^* mapped $S_{C(Y)}$ onto $S_{C(X)}$. In this case, T is positive, hence, T^* maps $P(Y)$ onto $P(X)$ where $P(Y)$ and $P(X)$ denote the sets of probability measures on Y and X respectively. It follows that, in the proof of Theorem 3, we can take $\psi \equiv 1$.∎

Example. Let $H : C(\Gamma) \to C(D)$ be defined by

$$\begin{aligned} Hf(z) &= (1/2\pi) \int_0^{2\pi} f(e^{it}) P_z(e^{it})\, dt \quad &&\textit{if } |z| < 1 \\ &= f(z) &&\text{if } |z| = 1, \end{aligned}$$

where $P_z(e^{it})$ denotes the Poisson kernel. Note that H is an isometry, that $D(H) = \Gamma$, and that $\hat{H}$ is the identity map. Since Γ is not a retract of D, it follows that $\mathfrak{F}(H) = \emptyset$. Thus, in Theorem 3, it is not possible to remove the condition that (X, Y) have the weak Tietze property.

It is interesting to note that ex $\mathcal{E}_1(H) = \emptyset$ by [2, Prop. 5.11.∎

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