AN INVERSION FORMULA AND COHOMOLOGY OPERATIONS

BY

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1. Introduction

For each integer $p > 1$ we shall consider the inverse relations in p variables $(1.2)_p$ and $(1.3)_p$ below. To find explicit expressions for these relations is interesting from the point of view of combinatory analysis and also because of its possible applications to algebraic topology. In fact, they can be used to obtain formulas that generalize Wu's formula. As is well known, Wu's formula relates Steenrod squares (in ordinary mod 2 cohomology) and Stiefel-Whitney classes. Ricabarra shows in [7] that using $(1.2)_p$ it is possible to get similar formulas for other primes ($\neq 2$) and for any cohomology theory which admits characteristic classes. Carlitz in [2] and Ricabarra in [7] obtain explicit formulas for $(1.2)_2$ and $(1.3)_2$. In this paper we find formulas for $(1.2)_3$, $(1.3)_3$ and the corresponding generalized Wu's formula.

Before we can state the main results we must introduce some definitions and notation.

We shall use the following combinatorial functions:

(a) $C(x, y) = \binom{x}{y}$ (ordinary binomial coefficients with the usual conventions)

(b) $E(x, y, z, t) = C(x + y + z, y + t)$

(c) $E(x, y) = E(x, y, 0, 0) = C(x + y, y)$

(d) $A(x, y) = (-1)^y[E(x, y) + E(x, y - 1)]$ so that

$$A(x, y) = 0 \qquad \text{if} \quad x < 0 \quad \text{or} \quad y < 0$$

$$= 1 \qquad \text{if} \quad x = y = 0$$

$$= (-1)^y \frac{x + 2y}{x + y} \binom{x + y}{y} \quad \text{otherwise}$$

(e) $B(x, y, z) = (-1)^{y-z}[C(z - x, y - z) + C(z - x - 1, y - z - 1)]$ so that

$$B(x, y, z) = 0 \qquad \text{if} \quad z < x \quad \text{or} \quad y < z \quad \text{or} \quad 2z < x + y$$

$$= 1 \qquad \text{if} \quad y = z \geq x$$

$$= (-1)^{y-z} \frac{y - x}{z - x} \binom{z - x}{y - z} \quad \text{otherwise}$$

(f) $D(x, y, z, t) = A(y, z - t)E(x, t) + (-1)^{z-t}E(y, z - t - 1)E(x, t - 1)$.

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It is easily verified that

$$(1.1) \qquad B(x, y, z) = A(2z - x - y, y - z) \quad \text{if} \quad z \geq x.$$

We shall denote by $\oplus$ the symmetric sum, that is, the sum of non-identical terms.

Let $P(n)$ denote the set of partitions of the positive integer n, so that if $w \in P(n)$ then $w = (i_m, \cdots, i_1)$ with $n = i_m + \cdots + i_1$ and $i_m \geq i_{m-1} \geq \cdots \geq i_1 > 0$. Given $w \in P(n)$ we shall also write $w = [a_p, \cdots, a_1]$ where $n = p \cdot a_p + \cdots + 1 \cdot a_1$ and $a_p \neq 0$.

We fix the notation $w = (i_m, \cdots, i_1) = [a_p, \cdots, a_1] \in P(n)$ and write $p(w) = p$ and $m(w) = m$.

We denote by $F(w)$ the symmetrized polynomial of $t_1^{i_1} \cdots t_m^{i_m}$ in $Z[t_1, \cdots, t_n]$; by $S(w)$ the polynomial in $Z[s_1, \cdots, s_n]$ representing $F(w)$ in terms of the elementary symmetric functions $s_0, s_1, \cdots, s_n$ $(s_0 = 1)$ of $t_1, \cdots, t_n$; and by $s(w)$ the monomial $s_{i_1} \cdots s_{i_m} = s_1^{a_1} \cdot s_2^{a_2} \cdots s_p^{a_p}$ in $Z[s_1, \cdots, s_n]$

It is known (see [4]) that

(i) $S(w)$ is a polynomial of degree $p = p(w)$,
(ii) $S(w) = \sum_{w' \in P(n)} T(w, w') s(w')$,
(iii) $s(w) = \sum_{w' \in P(n)} T'(w, w') S(w')$

where $T(w, w')$, $T'(w, w')$ are integers and $T'(w, w') = 0$ if $p(w') > m(w)$.

Then, for each integer $p \geq 1$, there exist relations

$$(1.2)_p \quad S[a_p, \cdots, a_1] = \sum_{0 \leq r_1 \leq \cdots \leq r_p, r_1 + \cdots + r_p = n} V_w(r_1, \cdots, r_p) s_{r_1} \cdots s_{r_p}$$

and

$$(1.3)_p \quad s_{i_1} \cdots s_{i_p} = \sum_{p b_p + \cdots + b_1 = n} V'_w(b_p, \cdots, b_1) S[b_p, \cdots, b_1]$$

where $V_w(r_1, \cdots, r_p)$ and $V'_w(b_p, \cdots, b_1)$ are integers.

It is clear that $(1.2)_p$ and $(1.3)_p$ are inverse relations in p variables.

In the case $p = 1$, $(1.2)_1$ and $(1.3)_1$ reduce to $S[n] = s_n$.

In the case $p = 2$, Carlitz shows (see [2, formulas 1.12 and 1.11])

$$(1.2)_2 \qquad S[a_2, a_1] = \sum_{h=0}^{a_2} A(a_1, h) s_{a_2 - h} \cdot s_{a_2 + a_1 + h}$$

and

$$(1.3)_2 \qquad s_{i_1} s_{i_2} = \sum_{k=0}^{i_1} C(i_1 + i_2 - 2k, i_1 - k) S[k, i_1 + i_2 - 2k].$$

Independently Ricabarra obtains in [7] the equivalent expression

$$S[a_2, a_1] = \sum_{0 \leq r_1 \leq r_2, r_1 + r_2 = n} B(r_1, r_2, m) s_{r_1} s_{r_2}$$

where $n = 2a_2 + a_1$ and $m = a_2 + a_1$, for the relation $(1.2)_2$.

In this paper we get, for $p = 3$, the following results:

THEOREM 1.1. *If $w = [c, b, a] \in P(n)$ with $m(w) = m$, then*

$$S(w) = S[c, b, a] = \sum_{r_1=0}^{c} \sum_{r_2=r_1}^{d(r_1)} V_w(r_1, r_2, r_3) s_{r_1} s_{r_2} s_{r_3}$$

where $d(r_1) = \min\{n - m - r_1, [(n - r_1)/2]\}$, $r_3 = n - r_1 - r_2$ *and*

$$V_w(r_1, r_2, r_3) = V_{r_1} \oplus V_{r_2}$$

with

$$V_{r_i} = \sum_{H=0}^{c-r_i} (-1)^H B(r_{3-i}, r_3, m + H) D(a, b, c - r_i, H) \quad (i = 1, 2).$$

THEOREM 1.2. *If* $w = (i_3, i_2, i_1) \in P(n)$ *then*

$$s(w) = s_{i_1} s_{i_2} s_{i_3} = \sum_{c=0}^{i_1} \sum_{b=0}^{i-c} V'_w(c, b, a) S[c, b, a]$$

where $i = \min\{i_3, i_1 + i_2\}$, $a = n - 3c - 2b$ *and*

$$V'_w(c, b, a) = \sum_{t=0}^{b} \binom{a + c - i_3 + 2t}{i_1 - b - c + t}\binom{b}{t}\binom{a}{i_3 - c - t}.$$

In §2 we get the generalized Wu-formula for the primes 2, 3 and for any cohomology theory.

Some results can be obtained in the case $p > 3$, and will be consigned elsewhere.

2. Application to topology

We begin with the definition of the Wu classes. First recall Atiyah-Hirzebruch [1, p. 162] construction of a multiplicative functor $\boldsymbol{\lambda}$ associated to a cohomology automorphism λ. We need to know that corresponding to the total Steenrod square $\lambda = Sq$ (for $p = 2$) or to the reduced power $\lambda = \rho_p$, working in characteristic 2 or (resp.) p (p an odd prime) there exists an operation $\boldsymbol{\lambda}$ which associates to every vector bundle ξ an element $\boldsymbol{\lambda}(\xi) \in H^{**}(B_\xi)$ (direct product of Z_p-cohomology groups of the base space) which is natural with respect to bundle maps and behaves multiplicatively with respect to Whitney sums of bundles. This $\boldsymbol{\lambda}$ is given by Thom formula: $\boldsymbol{\lambda}(\xi) = \phi^{-1}\lambda\phi(1)$ where ϕ is the Thom isomorphism. This gives Wu classes in the classical case for $p > 2$.

A similar construction can be given working with cobordism characteristic classes. See, for instance, Novikov [6] or Landweber [3]. We sketch the method, following Ricabarra [7].

Let F be either the real field R or the complex field C. Let h^* be a reduced cohomology theory which is assumed to have Stiefel-Whitney classes (for $F = R$) or Chern classes (for $F = C$). Let $FP_+(\infty)$ be the infinite projective space over F and FB_+ the classifying space (i.e., either BO_+ or BU_+ as $F = R$ or $F = C$; every space is provided with an extra-point). Let $\gamma \in h^c(FP_+(\infty))$ denote a fundamental class, where $c = 1$ or $c = 2$ according as $F = R$ or $F = C$. Given a semi-group of indices (usually $Z_+ = \{0, 1, 2, \cdots\}$ or $Z_+ \times Z_+$; elements denoted m or (m, n), with a positive valuation $|m| = m$, $|(m, n)| = m + n$) one defines a *universal multiplicative functor* $s_m(\gamma) \in h^{c|m|}(FB_+)$ by functoriality with respect to γ, the normalization condition $s_0(\gamma) = 1$, and the multiplicative condition represented by the commutativity of the diagram

$$\begin{array}{ccc} FB_+ \wedge FB_+ & \xrightarrow{\nabla} & FB_+ \\ \downarrow \Sigma & & \downarrow s_m \\ X \wedge X & \xrightarrow{\nabla} & X \end{array} \tag{2.1}$$

where X is a spectrum representing h, the ∇'s are the product-structures, and $\Sigma = \sum_{i+j=m} s_i \wedge s_j$

The *generating sequence* $(g_m(\gamma))$ for $(s_m(\gamma))$ is defined by the relation

$$s_m(\gamma) \circ j_B = (-1)^{|m|} g_m(\gamma) \tag{2.2}$$

where $j_B : FP_+(\infty) \to FB_+$ is the canonical inclusion. The classical Hirzebruch (decomposition) principle, takes the following form:

LEMMA. *A universal multiplicative functor and the generating sequence determine each other through relation* (2.2).

The proof uses the classifying map

$$FP_+(\infty) \wedge \cdots \wedge FP_+(\infty) \xrightarrow{\varphi} FB_+(i).$$

One defines s_m by means of $\varphi^*(s_m) = (-1)^{|m|} \sum g_{m_1} \wedge g_{m_2} \wedge \cdots \wedge g_{m_i}$ for all sequences $(m_1, \cdots, m_i)$ with $m = \sum m_j$.

Examples. (a) Taking $F = R$ and $(g_m(\gamma)) = (1, \gamma, 0, 0, \cdots)$ we get the Stiefel-Whitney classes.

(b) Taking $F = C$ and $(g_m(\gamma)) = (1, 0, 0, \gamma^q, 0, 0, \cdots)$ we get the Chern classes for $q = 1$ and the Wu universal classes for $q > 1$.

When $q = p - 1$, p a prime, and we work mod p we identify the ordinary Wu classes as elementary symmetric functions of γ_j^{p-1} (see [1, §2.14]). Notation: we shall denote, in the universal situation

$$s_n(\gamma) = c_n^q, \qquad q = 1, 2, \cdots; \quad n = 0, 1, 2, \cdots. \tag{2.3}$$

For $q = 1$, one can identify $c_n^1 = W_n$ (Stiefel-Whitney) or $c_n^1 = C_n$ (Chern) when (resp.) $F = R$ or $F = C$.

Thom isomorphism in the universal level identifies $h^*(FB_+) \approx h^*(FM)$ where FM is the cobordism spectrum (i.e., $FM = MO$ or $FM = MU$ according as $F = R$ or $F = C$) and transforms an universal multiplicative functor $(s_m(\gamma))$ into a multiplicative (as in (2.1)) sequence $(\hat{s}_m(\gamma))$ with $\hat{s}_m(\gamma) \in h^{c|m|}(FM)$ and $\hat{s}_0(\gamma)$ the unique multiplicative morphism of spectra $FM \to X$ (X representing h). When γ is the fundamental class of cobordism, say $\gamma = \gamma_c$, and we put $S_n^q = \hat{c}_n^q(\gamma)$, then

$$S_n^q \circ c_m^q(\gamma_c) = S_{n,m}^q(\gamma) \tag{2.4}$$

has a meaning for any γ and can be computed, as we shall see, by taking $\gamma = \gamma_c$. The meaning in this case of the composition in the left of (2.4) is the following: $c^q_m(\gamma_c)$ is the characteristic universal class, which goes into Wu clases under the unique multiplicative morphisms $CM \to K(Z) \to K(Z_{q+1})$, while $S^q_n(\gamma_c)$ is a cobordism operation going into Steenrod operation ρ^n in $K(Z_{q+1})$ when $q + 1$ is an odd prime (similarly for $F = R$, and $q = 1$).

To compute $(S^q_{n,m})$ we notice that it has a structure of universal multiplicative functor, then it can be obtained by the lemma, computing first the generating sequence (i.e. using (2.2)) and then coming back, by the decomposition principle, as in the proof of the lemma. We get easily the generating sequence:

$$g_{n,m}(\gamma) = 0 \quad \text{if} \quad (n, m) \neq (rq, q), \qquad 1 \leq r \leq q,$$

$$g_{rq,q}(\gamma) = C(q, r)\gamma^{rq+q}.$$

Thus the expression for $S^q_{n,m}$ follows.

To state the result we use some definitions.

Let $P_{s,q}(k) = \{w \in P(k)/p(w) \leq q \text{ and } m(w) \leq s\}$ where $k, s \geq 0$ and $q \geq 1$ are integers.

If $w = (i_1, \cdots, i_m) \in P_{s,q}(k)$, we define

(a) $\bar{w} = (j_1, \cdots, j_s) \in P(q(k + s))$ where $j_r = q(i_r + 1)$ if $1 \leq r \leq m$ and $j_r = q$ if $m < r \leq s$,

(b) $w^* = (u_1, \cdots, u_s) \in P(k + s)$ where $u_r = i_r + 1$ if $1 \leq r \leq m$ and $u_r = 1$ if $m < r \leq s$, and

(c) $C_w = C(q, i_1)C(q, i_2) \cdots C(q, i_m)$.

Then given $n, m \geq 0$ and $q \geq 1$, $S^q_{n,m}$ is given by

$$S^q_{n,m} = 0 \quad \text{if} \quad n \neq \dot{q} \quad \text{or} \quad m \neq \dot{q},$$

$$S^q_{kq,sq} = \textstyle\sum_{w \in P_{s,q}(k)} C_w \, S(\bar{w})$$

and of course c^q_m is given by

$$c^q_m = 0 \quad \text{if} \quad m \neq \dot{q},$$

$$c^q_{kq} = S[a_q, \cdots, a_1] \quad \text{with} \quad a_q = k \quad \text{and} \quad a_j = 0 \quad \text{if} \quad 1 \leq j < q.$$

Milnor has shown (see [5]) that in ordinary cohomology mod the prime $q + 1$, the $S^q_{n,m}$ are polynomials in the c^q_r (or rather their Z_{q+1} versions). Ricabarra [7] shows that for general cohomology the $S^1_{n,m}$ are degree two polynomials in the c^1_r and obtains their explicit expressions. Notice that at the unitary cobordism level all coefficients are integers; in particular, Wu relations are a special case of stronger relations (see (2.7)) involving integral coefficients and valid in the universal level. The conjecture (in [7]) that the $S^q_{n,m}$ are degree $q + 1$ polynomials in the c^q_r, also holding at the universal level, is proved next.

THEOREM 2.1.

$(2.5)_q \quad S^q_{n,m} = \sum_{0 \leq r_0 \leq \cdots \leq r_q, q(r_0+\cdots+r_q)=n+m} W_m(r_0, \cdots, r_q) c^q_{qr_0} \cdots c^q_{qr_q}$

where the $W_m(r_0, \cdots, r_q)$ *are integers.*

Proof. Let $n = kq$ and $m = sq$ (if not theorem is obvious).

If $I_q : Z[t^q_1, \cdots, t^q_r] \to Z[T_1, \cdots, T_r]$ denotes the Z-algebra isomorphism such that $I_q(t^q_j) = T_j$, we have $I_q(F(\overline{w})) = F(w^*)$ and $I_q(c^q_{iq}) = s_i$. Then

$$I_q(S^q_{n,m}) = T_q(k, s) = \sum_{w \in P_{s,q}(k)} C_w\, S(w^*).$$

As $p(w^*) \leq q + 1$ for each $w \in P_{s,q}(k)$, it follows from $(1.2)_{q+1}$ that

$(2.6)_q \quad T_q(k, s) = \sum_{0 \leq r_0 \leq \cdots \leq r_q, q(r_0+\cdots+r_q)=n+m} W_m(r_0, \cdots, r_q) s_{r_0} \cdots s_{r_q}$

where the $W_m(r_0, \cdots, r_q)$ are integers.

The theorem follows applying I_q^{-1} to $(2.6)_q$.

It is clear from the proof of Theorem 2.1 that in order to make explicit $(2.5)_q$ it is enough to give an explicit formula for $(1.2)_{q+1}$. In particular if $q = 1$ both problems coincide and from $(1.2)_2$ one deduces Ricabarra's formula:

$(2.7) \qquad S^1_{n,m} = S[n, m - n] = \sum_{h=0}^{n} A(m - n, h) c^1_{n-h}\, c^1_{m+h}.$

Since

$$A(m - n, h) \equiv E(m - n - 1, h) \pmod 2$$

from (2.7) one gets the classical Wu formula [8], [9]

$$Sq^n c^1_m = S^1_{n,m} = \sum_{h=0}^{n} \binom{m-n+h-1}{h} c^1_{n-h}\, c^1_{m+h}$$

This formula can be generalized in the case $q = 2$ as follows:

THEOREM 2.2.

$$S^2_{2k,2s} = \sum_{0 \leq r_0 \leq r_1 \leq r_2, r_0+r_1+r_2=k+s} W_s(r_0, r_1, r_2) c^2_{2r_0}\, c^2_{2r_1}\, c^2_{2r_2}$$

where

$$W_s(r_0, r_1, r_2) = \sum_{c=\max\{0,k-s\}}^{[k/2]} 2^c (W_s[r_0] \oplus W_s[r_1])$$

and

$$W_s[r_i] = \sum_{H=0}^{c-r_i} (-1)^H B(r_{1-i}, r_2, s + H) D(s - k + c, k - 2c, c - r_i, H) \qquad (i = 0, 1).$$

Proof. Theorem 2.2 follows easily from Theorems 2.1 and 1.1 noting that

$$T_2(k, s) = \sum_{c=\max\{0,k-s\}}^{[k/2]} 2^c S[c, k - 2c, s - k + c]$$

and that the polynomials $S[a_3, a_2, a_1]$ which appear in the above formula satisfy $m = a_3 + a_2 + a_1 = s$, $n = 3a_3 + 2a_2 + a_1 = k + s$, and $2a_3 + a_2 = k$.

A table illustrating Theorem 2.2 for $k + s \leq 4$ is given in Section 6.

3. Two lemmas

Let $(c; b; a)$ be a triple of non-negative integers, $n = 3c + 2b + a$ and $m = c + b + a$.

If $b > 0$, let $r = \min\{b, c\}$ and $s = \min\{a, c - h\}$ with $0 \le h \le r$. Developing the product $F[b, a].s_c$ we get

$$F[b, a].s_c = \sum_{h=0}^{r} \sum_{k=0}^{s} G(a, b, c, h, k)F[h, b + k - h, a + c - h - 2k]$$

where

$$G(a, b, c, h, k) = C(b + k - h, k)C(a + c - h - 2k, a - k).$$

Then

$$S[b, a].s_c = \sum_{h=0}^{r} \sum_{k=0}^{s} G(a, b, c, h, k)S[h, b + k - h, a + c - h - 2k]. \tag{3.1}$$

If $c > 0$, (3.1) implies

$$\begin{aligned} &S[c + b, a].s_c \\ &\quad = \sum_{h=0}^{c} \sum_{k=0}^{h} J(a, b, h, k)S[c - h, b + h + k, a + h - 2k] \end{aligned} \tag{3.2}$$

where

$$J(a, b, h, k) = C(b + h + k, k)C(a + h - 2k, a - k).$$

LEMMA 3.1. *Let H and K be non-negative integers with $H \le K$ and $K \ge 1$. If*

$$X_1 = \sum_{h=0}^{K} \sum_{k=0}^{h} (-1)^{K-h+1} J(a, b, h, k) \cdot \frac{b + 2H + h - k}{b + H + h} E(b, H, h, -k)E(a, K - H, -k, k - h)$$

and

$$X_2 = \sum_{h=0}^{K} \sum_{k=}^{h} (-1)^{K-h+1}J(a, b, h, k)E(b, H - 1, h, -k) \cdot E(a, K - H - 1, -k, k - h)$$

then

$$X = X_1 + X_2 = 0.$$

Proof. The terms in X_1 with $k > H$ or $h > K - H + k$ are zero, and therefore if we sum first over $h - k$ and then over k we get

$$X_1 = \sum_{k=0}^{H} \sum_{h=0}^{K-H} (-1)^{K-h-k+1}M_1(b, H, h, k)M_2(a, H, K, h, k)$$

where

$$\begin{aligned} M_1(b, H, h, k) &= \frac{b + 2H + h}{b + H + h + k} E(b + h + k, k)\, E(b, H, h + k, -k) \\ &= \frac{b + 2H + h}{b + H + h + k} E(b + h + k, H)C(H, k) \end{aligned}$$

and

$$M_2(a, H, K, h, k) = E(h, a - k)E(a, K - H, -k, -h) = E(a - k, K - H)C(K - H, h)$$

Then

$$X_1 = (-1)^{K+1}\sum_{k=0}^{H} (-1)^k C(H, k)E(a - k, K - H)M_3(b, H, K, k)$$

where

$$M_3(b, H, K, k) = \sum_{h=0}^{K-H} (-1)^h C(K - H, h) \cdot \frac{b + 2H + h}{b + H + k + h} E(b + k + h, H) = \frac{1}{H!}\sum_{t=0}^{H-1} b_t(b, H, K)k^t.$$

Since

$$E(a - k, K - H) = \frac{1}{(K - H)!}\sum_{s=0}^{K-H} a_s(a, H, K)k^s$$

it follows that

$$X_1 = \frac{(-1)^{K+1}}{(K - H)!H!}\sum_{k=0}^{H} (-1)^k C(H, k) \sum_{u=0}^{K-1} g_u(a, b, H, K)k^u. \tag{3.3}$$

Since for each $t = 0, 1, \cdots, H - 1$ we have

$$b_t(b, H, K) = \sum_{h=0}^{K-H} (-1)^h C(K - H, h) \sum_{i=0}^{H-t} n_{ti}(b, H)h^i$$

it follows that $b_t = 0$ if $t > 2H - K$. Then if

$$T = \min\{2H - K, H - 1\} = H - 1 \quad \text{if} \quad H = K$$
$$= 2H - K \quad \text{if} \quad H < K,$$

$b_t = 0$ if $t > T$. Therefore $g_u = 0$ if

$$u > K - H + T = H - 1 \quad \text{if} \quad H = K$$
$$= H \quad \text{if} \quad H < K.$$

From (3.3) we get $X_1 = 0$ if $H = K$, and

$$X_1 = \frac{(-1)^{K+1}}{(K - H)!H!}(-1)^H H!\, g_H = \frac{(-1)^{K+H+1}}{(K - H)!} a_{K-H}\, b_{2H-K}$$
$$= \frac{(-1)^{K+H+1}}{(K - H)!}(-1)^{K-H}(-1)^{K-H}(K - H)!\, n_{2H-K,K-H}$$
$$= (-1)^{K+H+1}C(H - 1, 2H - K)$$

if $H < K$.

A similar argument on X_2 shows that

$$\begin{aligned} X_2 &= 0 && \text{if } H = K \\ &= (-1)^{K+H} C(H-1, 2H-K) && \text{if } H < K \end{aligned}$$

and the lemma follows from comparison of the expressions for X_1 and X_2.

LEMMA 3.2.

$$\begin{aligned} S[c, b, a] = \textstyle\sum_{K=0}^{c} (-1)^K \{ \sum_{H=0}^{K} (-1)^H D(a, b, K, K-H) \\ \cdot S[c+b+H, a+K-2H] \} s_{c-K} \end{aligned}$$

Proof. We prove the lemma by induction on c. If $c = 0$ the lemma is trivial. Assume that it is true for any triple $(c'; b; a)$ with $c' < c > 1$.

It follows from (3.2) that

$$S[c, b, a] = S[c+b, a]s_c + L(c, b, a) \tag{3.4}$$

where

$$\begin{aligned} &L(c, b, a) \\ &\quad = -\textstyle\sum_{h=1}^{c} \sum_{k=0}^{h} J(a, b, h, k) S[c-h, b+h+k, a+h-2k]. \end{aligned} \tag{3.5}$$

From the inductive hypothesis it follows for $h = 1, \cdots, c$, that

$$\begin{aligned} &S[c-h, b+h+k, a+h-2k] \\ &K = \textstyle\sum_{r=0}^{c-h} (-1)^r \{ \sum_{s=0}^{r} (-1)^s D(a+h-2k, b+h+k, r, r-s) \\ &\qquad \cdot S[c+b+k+s, a+h-2k+r-2s] \} s_{c-h-r}. \end{aligned}$$

If we substitute this expression in (3.5) and we sum over $H = k + s$ and $K = h + r$ we get

$$L(c, b, a) = \textstyle\sum_{K=1}^{c} \sum_{H=0}^{K} N(a, b, H, K) S[c+b+H, a+K-2H] s_{c-K} \tag{3.6}$$

where

$$\begin{aligned} N(a, b, H, K) &= -\sum_{h=1}^{K} \sum_{k=0}^{h} (-1)^{K-h-H+k} J(a, b, h, k) \\ &\qquad \cdot D(a+h-2k, b+h+k, K-h, K-h-H+k) \\ &= \sum_{h=1}^{K} \sum_{k=0}^{h} (-1)^{K-h+1} J(a, b, h, k) \frac{b+2H+h-k}{b+H+h} \\ &\qquad \cdot E(b, H, h, -k) E(a, K-H, -k, k-h) \\ &\quad + \sum_{h=1}^{K} \sum_{k=0}^{h} (-1)^{K-h+1} J(a, b, h, k) E(b, H-1, h, -k) \\ &\qquad \cdot E(a, K-H-1, -k, k-h); \end{aligned}$$

then by Lemma 3.1

$$
\begin{aligned}
N(a, b, H, K) &= -(-1)^{K+1} J(a, b, 0, 0)[(b + 2H)/(b + H)] \\
&\qquad \cdot E(b, H)E(a, K - H) + E(b, H - 1)E(a, K - H - 1)] \\
&= (-1)^K[(-1)^H A(b, H)E(a, K - H) + E(b, H - 1) \\
&\qquad \cdot E(a, K - H - 1)] \\
&= (-1)^K[(-1)^H D(a, b, K, K - H)]
\end{aligned}
$$

If we substitute the above expression in (3.6) and then in (3.4) we conclude that Lemma 3.2 is verified for any triple $(c; b; a)$.

4. Proof of Theorem 1.1

We want to give an explicit expression for

$$(1.2)_3 \qquad S[c, b, a] = \textstyle\sum_{0 \le r_1 \le r_2 \le r_3, r_1 + r_2 + r_3 = n} V_w(r_1, r_2, r_3) s_{r_1} s_{r_2} s_{r_3}.$$

Clearly $V_w(r_1, r_2, r_3) = 0$ if $r_1 > c$ or $r_3 < m$, and therefore if $r_2 > n - m - r_1$. On the other hand $r_2 \le r_3 = n - r_1 - r_2$ implies $r_2 \le [(n - r_1)/2]$. Then $(1.2)_3$ can be written as follows:

$$(4.1) \qquad S[c, b, a] = \textstyle\sum_{r_1=0}^{c} \sum_{r_2=r_1}^{d(r_1)} V_w(r_1, r_2, r_3) s_{r_1} s_{r_2} s_{r_3}.$$

From Lemma 3.2 and $(1.2)_2$ we get

$$
\begin{aligned}
(4.2) \quad S[c, b, a] &= \textstyle\sum_{K=0}^{c} (-1)^K \{ \sum_{H=0}^{K} (-1)^H D(a, b, K, K - H) \\
&\qquad \cdot \textstyle\sum_{h=0}^{c+b+H} A(a + K - 2H, h) s_{c+b+H-h} s_{m+K-H+h} \} s_{c-K}
\end{aligned}
$$

Now given a triple $(r_1; r_2; r_3)$ $(0 \le r_1 \le c,\ r_1 \le r_2 \le d(r_1),\ r_3 = n - r_1 - r_2)$, we get the corresponding $V_w(r_1, r_2, r_3)$ from (4.2) as a symmetric sum

$$(4.3) \qquad V_w(r_1, r_2, r_3) = V_{r_1} \oplus V_{r_2}$$

where V_{r_i} $(i = 1, 2)$ is the coefficient of $s_{c-K} s_{c+b+H-h} s_{m+K-H+h}$ for $r_i = c - K$ and $r_{3-i} = c + b + H - h$, i.e. for $K = c - r_i$ and $h = c + b + H - r_{3-i}$. Therefore

$$
\begin{aligned}
V_{r_i} &= (-1)^{c-r_i} \textstyle\sum_{H=0}^{c-r_i} (-1)^H D(a, b, c - r_i, c - r_i - H) \\
&\qquad \cdot A(a + c - r_i - 2H, c + b + H - r_{3-i}).
\end{aligned}
$$

If in the above sum we add with respect to $c - r_i - H$ and observe (see (1.1)) that

$$
\begin{aligned}
A(a - c + r_i + 2H, 2c + b - r_i - r_{3-i} - H) \\
&= A(2(m + H) - r_{3-i} - r_3, r_3 - (m + H)) \\
&= B(r_{3-i}, r_3, m + H)
\end{aligned}
$$

we obtain

(4.4) $V_{r_i} = \sum_{H=0}^{c-r_i} (-1)^H B(r_{3-i}, r_3, m + H) D(a, b, c - r_i, H).$

Theorem 1.1 follows from (4.1), (4.3) and (4.4).

5. Proof of Theorem 1.2

From $(1.3)_2$ we get

(5.1) $$s_{i_1} s_{i_2} s_{i_3} = \sum_{k=1}^{i_1} E(i_2 - k, i_1 - k) S[k, i_1 + i_2 - 2k] s_{i_3} + E(i_2, i_1) s_{i_1+i_2} s_{i_3}.$$

For each $k = 1, \cdots, i_1$, (3.1) implies

(5.2) $$S[k, i_1 + i_2 - 2k] . s_{i_3} = \sum_{c=0}^{k} \sum_{t=0}^{T} G(i_1 + i_2 - 2k, k, i_3, c, t) \cdot S[c, k + t - c, n - 2k - i_3 - 2t$$

where

$$T = \min\{i_1 + i_2 - 2k, i_3 - c\}.$$

From $(1.3)_2$ we get

(5.3) $$s_{i_1+i_2} s_{i_3} = \sum_{b=0}^{i} C(n - 2b, i_1 + i_2 - b) S[b, n - 2b].$$

Substituting (5.2) and (5.3) in (5.1) and changing the order of the sum we get

(5.4) $$s_{i_1} s_{i_2} s_{i_3} = L_1 + L_0$$

where

$$L_1 = \sum_{c=1}^{i_1} \sum_{k=c}^{i_1} \sum_{t=0}^{T} E(i_2 - k, i_1 - k) E(k - c, t) \cdot E(n - c, -2k - t, -t, i_1 + i_2) S[c, k + t - c, n - 2k - c - 2t]$$

and

$$L_0 = \sum_{k=1}^{i_1} \sum_{t=0}^{T} E(i_2 - k, i_1 - k) E(k, t) E(n, -2k - t, -t, i_1 + i_2) \cdot S[k + t, n - 2k - 2t] + \sum_{b=0}^{i} E(i_2, i_1) C(n - 2b, i_1 + i_2 - b) S[b, n - 2b]$$

If in L_1 we sum with respect to $b = k + t - c$ and t, and notice that $i_1 + i_2 - 2b - 2c = a + c - i_3$, we get

(5.5) $$L_1 = \sum_{c=1}^{i_1} \sum_{b=0}^{i-c} V'_w(c, b, a) S[c, b, a].$$

Now let

$$P(c, b, a, t) = \binom{a+c-i_3+2t}{i_1-b-c+t} \binom{b}{t} \binom{a}{i_3-c-t}$$

so that

$$V'_w(c, b, a) = \sum_{t=0}^{b} P(c, b, a, t).$$

If in the first term of L_0 we sum with respect to $b = k + t$ and t, and we take account that $c = 0$, $a = n - 2b$ and $i_1 + i_2 - 2b = a - i_3$, we conclude that

$$L_0 = \sum_{b=1}^{i} \sum_{t=0}^{b-1} P(0, b, a, t) S[b, a] + \sum_{b=0}^{i} P(0, b, a, b) S[b, a]$$
$$= \sum_{b=0}^{i} \sum_{t=0}^{b} P(0, b, a, t) S[b, a],$$

i.e.,

$$L_0 = \sum_{b=0}^{i} V'_w(0, b, a) S[0, b, a]. \tag{5.6}$$

Substituting (5.5) and (5.6) in (5.4), the theorem follows.

6. Table for Theorem 2.2

In the following table, explicit expressions of the $S^2_{2k,2s}$ as polynomials in the c^2_r for $k + s \leq 4$ are given. For simplicity we denote by (r_0, r_1, r_2) the monomial $c^2_{2r_0}\, c^2_{2r_1}\, c^2_{2r_2}$.

k	s	$S^2_{2k,2s}$
1	s	$-(s+1)\,(0, 0, s+1) + (0, 1, s)$
2	1	6 (0, 0, 3) − 6 (0, 1, 2) + 2 (1, 1, 1)
2	2	10 (0, 0, 4) − 4 (0, 1, 3) − 3 (0, 2, 2) + 2 (1, 1, 2)
2	3	15 (0, 0, 5) − 5 (0, 1, 4) − 3 (0, 2, 3) + 2 (1, 1, 3)
2	4	21 (0, 0, 6) − 6 (0, 1, 5) − 3 (0, 2, 4) + 2 (1, 1, 4)
2	5	28 (0, 0, 7) − 7 (0, 1, 6) − 3 (0, 2, 5) + 2 (1, 1, 5)
2	6	36 (0, 0, 8) − 8 (0, 1, 7) − 3 (0, 2, 6) + 2 (1, 1, 6)
3	2	−10 (0, 0, 5) + 10 (0, 1, 4) − 2 (0, 2, 3) − 4 (1, 1, 3) + 2 (1, 2, 2)
3	3	− 26 (0, 0, 6) + 16 (0, 1, 5) + 6 (0, 2, 4) − 5 (0, 3, 3) − 6 (1, 1, 4) + 2 (1, 2, 3)
3	4	− 49 (0, 0, 7) + 23 (0, 1, 6) + 9 (0, 2, 5) − 5 (0, 3, 4) − 8 (1, 1, 5) + 2 (1, 2, 4)
3	5	− 80 (0, 0, 8) + 31 (0, 1, 7) + 12 (0, 2, 6) − 5 (0, 3, 5) − 10 (1, 1, 6) + 2 (1, 2, 5)
4	2	12 (0, 0, 6) − 12 (0, 1, 5) − 12 (0, 2, 4) + 12 (0, 3, 3) + 12 (1, 1, 4) − 12 (1, 2, 3) + 4 (2, 2, 2)
4	3	42 (0, 0, 7) − 30 (0, 1, 6) − 22 (0, 2, 5) + 18 (0, 3, 4) + 20 (1, 1, 5) − 8 (1, 2, 4) − 6 (1, 3, 3) + 4 (2, 2, 3)
4	4	98 (0, 0, 8) − 56 (0, 1, 7) − 34 (0, 2, 6) + 22 (0, 3, 5) + 1 (0, 4, 4) + 30 (1, 1, 6) − 10 (1, 2, 5) − 6 (1, 3, 4) + 4 (2, 2, 4)
5	3	− 32 (0, 0, 8) + 32 (0, 1, 7) + 8 (0, 2, 6) − 28 (0, 3, 5) + 16 (0, 4, 4) − 20 (1, 1, 6) + 20 (1, 2, 5) − 4 (1, 3, 4) − 8 (2, 2, 4) + 4 (2, 3, 3)

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