A CHARACTERIZATION OF THE FINITE SIMPLE GROUPS $PSp_4(3^m)$, m ODD

BY

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Introduction

The aim of this paper is to characterize the finite simple groups PSp_4- (3^m), m odd, in terms of the structure of the centralizer of an element of order 3. The groups $PSp_4(3^m)$ belong to the family of all projective symplectic groups of dimension 4 over a finite field of $q = p^n$ elements where p is an arbitrary prime. For odd characteristic these groups have order $\frac{1}{2}q^4(q^2 + 1)\cdot(q^2 - 1)^2$ and a Sylow p-subgroup has order q^4. The center of a Sylow p-subgroup is elementary abelian of order q and the centralizer in $PSp_4(q)$ of each of the nonunit central p-elements of a p-Sylow subgroup is a group $\mathfrak{C}$ of order $q^4(q^2 - 1)$.

Although this paper deals extensively with $q = 3^m$, m odd, $m > 1$, Sections 1 and 2 obtain results for arbitrary odd characteristic. This study is a continuation of a work by the author in [7] and is very similar in nature to results obtained for even characteristic by Suzuki [10]. The main result of this paper is the following proposition:

THEOREM 3. *Let $\mathfrak{C}$ be the centralizer in $PSp_4(3^m)$, m odd, $m > 1$, of an element of order 3 lying in the center of some Sylow 3-subgroup. Let G be a finite group satisfying:*

(a) *G contains an element α of order 3 such that $C_G(\alpha)$ is isomorphic to $\mathfrak{C}$.*

(b) *For all z in the center of $C_G(\alpha)$, $C_G(z) = C_G(\alpha)$.*

(c) *Not all central 3-elements belong to the same conjugacy class of G.*

Then one of the following cases holds:

(i) *$C_G(\alpha)$ is a normal subgroup of G.*

(ii) *G is a simple group isomorphic to $PSp_4(3^m)$.*

A similar but not identical result has been obtained for $PSp_4(3)$ in [7].

Let G be a finite group. A nontrivial proper subgroup D of G is called a CC-subgroup if D contains the centralizer of each of its nonunit elements. The methods of this paper use extensively the results on CC-subgroups which were studied by Herzog in [8]. It is shown that in the simple case of Theorem 3 a group satisfying conditions (a), (b) and (c) has a local 3-structure identical to that of $PSp_4(3^m)$, m odd. This knowledge is then used to determine the structure of the centralizer of a central involution and the results of Wong [11] are applied to conclude that G is isomorphic to $PSp_4(3^m)$. In the nonsimple case it is found that the center of a Sylow 3-group P of G is weakly closed in P. This is enough information to determine that a Sylow 2-group of G is quater-

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nion or semi-dihedral. The results of Gorenstein and Walter [6], Alperin, Brauer, Gorenstein [1], and Glauberman [3] are then used to determine the structure of G.

A more general study of the groups $PSp_4(q)$, q odd, is found in Section 2. This section characterizes the local p-structure of groups satisfying the following hypothesis:

HYPOTHESIS A. *Let $\mathfrak{C}$ be the centralizer in $PSPp_4(p^n)$ of an element of order $p > 2$ in the center of some Sylow p-subgroup. Let G be a finite group satisfying:*

(a) *G contains an element α of order p such that $C_G(\alpha)$ is isomorphic to $\mathfrak{C}$.*

(b) *For all z in the center of $C_G(\alpha)$, $C_G(z) = C_G(\alpha)$.*

(c) *Not all central p-elements belong to the same conjugacy class of G.*

It will be shown that groups satisfying Hypothesis A have a Sylow p-group P of order q^4 and P has a unique elementary abelian subgroup M of order q^3. We prove the following proposition.

THEOREM 1. *Let G satisfy hypothesis A. If $N_G(M)$ in not p-closed, $N_G(M) = MJ$, $M \cap J = 1$ where $J = F \times D$, $F \cong PGL(2, q)$ and D is a cyclic group of order $(q - 1)/2$.*

The author feels that Theorem 3 can be extended to include the entire family $PSp_4(p^n)$ and that Theorem 1 is the basic foundation for such an extension. At the moment the proof seems to be limited by the character theory results on CC-subgroups. A slight modification of the methods in this paper together with theorems similar to those found in [8] is thought to be sufficient for such an extension. Moreover, the methods of this paper could possibly be used to investigate the larger dimensional classical groups.

1. Structure of $\mathfrak{C}$

Let q be a power of an odd prime number p.

Setting

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

we may take $PSp_4(q)$ as the group of all matrices A of degree 4 with coefficients in F_q such that $A'JA = J$, where A' denotes the transpose of A and we identify two such matrices if they are negatives of each other. Let $\mathfrak{C}$ be the centralizer in $PSp_4(q)$ of the element α of order p given by,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

It is calculated that $\mathfrak{C}$ consists of all matrices

$$(1)\qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & \multicolumn{2}{c}{\multirow{2}{*}{S}} & 0 \\ b & & & 0 \\ c & e & f & 1 \end{bmatrix},$$

where $S = (s_{ij})$ is a 2-dimensional matrix of determinant 1, $e = bs_{11} - as_{21}$, $f = bs_{12} - as_{22}$.

Let us define L to be the subgroup of $\mathfrak{C}$ consisting of all matrices (1) for which $a = b = c = 0$. In particular,

$$L = \left\langle \begin{bmatrix} 1 & & \\ & S & \\ & & 1 \end{bmatrix} \middle| S \in SL(2, q) \right\rangle$$

and is a subgroup of $\mathfrak{C}$ isomorphic to $SL(2, q)$.

The mapping of $\mathfrak{C}$ which sends every element of $\mathfrak{C}$ to the corresponding element of L is a homomorphism of $\mathfrak{C}$ whose kernel is a p-group of order q^3 and exponent p. Denote by U the kernel of this homomorphism so that

$$U = \left\langle \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & b & -a & 1 \end{bmatrix} \right\rangle.$$

It follows that $\mathfrak{C} = UL$, $U \cap L = 1$ and that $|\mathfrak{C}| = q^4(q^2 - 1)$. The index of $\mathfrak{C}$ in $PSp_4(q)$ is $\frac{1}{2}(q^2 + 1)(q^2 - 1)$ and is a number relatively prime to p so that $\mathfrak{C}$ contains a Sylow p-group of $PSp_4(q)$. In fact, a Sylow p-group of $\mathfrak{C}$ has order q^4 and consists of all matrices,

$$P = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & f & 1 \end{bmatrix}, e = b - ad, f = -a \right\rangle.$$

Several subgroups of $\mathfrak{C}$ will be used in the following sections so they are listed here for convenience. Define M to be the subgroup of P consisting of all matrices with $a = 0$. It is easily verified that M is the unique elementary abelian subgroup of P of order q^3 and is thus characteristic in P. The center of P is elementary abelian of order q and is the subgroup of M with $b = d = 0$. Denote the center of P by Z. Since $\mathfrak{C} = UL$ and L is isomorphic to $SL(2, q)$, Z coincides with the center of $\mathfrak{C}$.

Define K_1 to be the subgroup of $\mathfrak{C}$ given by,

$$K_1 = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle,$$

where ϵ is generator of the multiplicative group of F_q. Clearly K_1 is a cyclic group of order $q - 1$ and contains a unique involution t given by

$$t = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

It is calculated that PK_1 is the normalizer of P in $\mathfrak{C}$ and that $C_P(t)$ is a subgroup T of M of order q^2 consisting of all matrices of M with $b = 0$.

2. Structure of $N_G(M)$

Throughout this section let G be a finite group satisfying the following hypothesis.

HYPOTHESIS A. *Let $\mathfrak{C}$ be the centralizer in $PSp_4(p^n)$ of an element of order $p > 2$ in the center of some Sylow p-subgroup. Let G be a finite group satisfying:*

(a) *G contains an element α of order p such that $C_G(\alpha)$ is isomorphic to $\mathfrak{C}$.*

(b) *For all z in the center of $\mathfrak{C}_G(\alpha)$, $C_G(z) = C_G(\alpha)$.*

(c) *Not all central p-elements belong to the same conjugacy class of G.*

We will use the properties of $\mathfrak{C}$ discussed in Section 1 and identify the subgroups of $\mathfrak{C}$ with subgroups of G retaining the same notation given earlier.

(2.1) *P is a Sylow p-subgroup of G.*

Proof. Let S be a Sylow p-subgroup of G containing P and let x be an element in the center of S. The element x centralizes P and thus $x \in S \cap C_G(\alpha) = P$. This implies that x is in the center of P and that $S \subseteq C_G(x) = C_G(\alpha)$. Therefore $S = P$ and P is a Sylow p-subgroup of G.

(2.2) $N_G(M) = MJ, M \cap J = 1.$

Proof. From (2.1), P is a Sylow p-group of $N_G(M)$. Furthermore, $P = AM, A \cap M = 1$, where A is a complement for M of order q. It follows by a theorem of Gaschütz that $N_G(M)$ splits over M.

Let $N_G(P) = PK$, $P \cap K = 1$ and choose K so that $C_G(Z) \cap K = K_1$. Then K_1 is a normal subgroup of K and the involution t is in the center of K. Throughout the remainder of this section we will keep this same notation so that t is a central involution of K.

(2.3) *The group K contains no four-group $\langle t, \tau\rangle$ such that t and τ are conjugate in $N_G(M)$.*

Proof. Suppose that $\langle t, \tau\rangle$ is a four-group contained in K and that $\tau = t^y$ for some $y \in N_G(M)$. Then τ is an involution of $K - K_1$ and thus acts fixed-point-free on the nontrivial elements of Z. Furthermore, $C_M(\tau) = T^y$ is an elementary abelian subgroup of M of order q^2.

Let I be the subgroup of U containing all elements inverted by t. It is calculated that $|I| = q^2$ and that $I \cap M$ is a subgroup of P' of order q. In fact, $P' = (I \cap M)Z$ and since $I \cap M$ is left invariant by $\langle t, \tau\rangle$, τ centralizes no element of P' not contained in $I \cap M$. However, P' and T^y are subgroups of M of order q^2 so that $| P' \cap T^y | \geq q$. It follows that $P' \cap T^y = I \cap M$.

On the other hand, $I = C_I(\tau) \times C_I(t\tau)$. If τ centralized an element x of $I - (I \cap M)$, τ would centralize $[x, I \cap M] = Z$ which is not the case. Hence $C_I(t\tau)$ is a subgroup of I of order q and $P = C_I(t\tau)M$. It follows that $t\tau$ centralizes P/M. We have seen that $P' \cap T^y = I \cap M$ is a subgroup of order q so that $M = T^yP'$. Thus τ centralizes M/P' and, since t centralizes M/P', $t\tau$ centralizes M/P'.

This implies that $t\tau$ stabilizes the normal series $P/P' \supset M/P' \supset \bar{1}$ of P/P' so that $t\tau$ centralizes P/P'. From the structure of P, $P' = \Phi(P)$ and we conclude that $t\tau$ centralizes P. This is impossible and the proposition (2.3) follows.

(2.4) *No element of $P' - Z$ is conjugate to an element of Z.*

Proof. Let z be an element of Z and suppose that z is conjugate to an element of $P' - Z$. From the structure of PK_1, $C_{PK_1}(x) = M$ for all $x \in P' - Z$ so that x has $q(q-1)$ conjugates in $P' - Z$. This implies that all elements of $P' - Z$ are conjugate and that z is conjugate to an element v of $P' - Z$ which is inverted by t. The group M is the unique elementary abelian subgroup of P of order q^3 so that x and v are conjugate in $N_G(M)$. Let $z^y = v$, $y \in N_G(M)$. Then t normalizes $C_G(Z^y)$ and the involution yty^{-1} normalizes $C_G(Z)$ and M. It follows that yty^{-1} normalizes $O_P(C_G(Z)) = U$ and hence leaves $P = UM$ invariant.

Thus the involutions yty^{-1} and t belong to PK and for some $x \in PK$, $(xy)t(xy)^{-1}$ and t are involutions of K conjugate in $N_G(M)$. Using (2.3), $(xy)t(xy)^{-1} = t$ and $xy \in C_G(t)$. Then $(Z)^{xy} = Z^y$ is a subgroup of M centralized by t. This is impossible as Z^y contains an element v inverted by t. The proposition now follows.

From (2.2), $N_G(M) = MJ$, $M \cap J = 1$ and J may be chosen to contain K. The next proposition begins the investigation of the structure of J.

(2.5) *If $N_G(M)$ is not p-closed, $C_J(t) \neq K$.*

Proof. Using (2.1), P is a Sylow p-group of $N_G(M)$ and thus $P \cap J$ is a Sylow p-group of J with Sylow p-normalizer $(P \cap J)K$. The group $N_J(Z)$ leaves $O_P(C_G(Z))$ invariant and hence normalizes $UM = P$. Therefore, $N_J(Z) = (P \cap J)K$.

If $N_J(Z)$ coincides with J, $J = (P \cap J)K$ and P is a normal subgroup of $N_G(M)$ contrary to hypothesis. It follows that J contains an element y such that Z^y is a subgroup of M different from Z. From (2.4) we conclude that for some $z \in Z$, $z^y \in M - P'$ and, as every element in $M - P'$ is conjugate in PK_1 to an element of $T - Z$, we may assume $z^y \in T - Z$, $y \in J$.

Then t centralizes z^y and (b) of Hypothesis A implies that t centralizes Z^y.

Therefore yty^{-1} is an involution of J which centralizes Z and we have

$$yty^{-1} \in (P \cap J)K_1.$$

Using (2.3), there exists $x \in P \cap J$ for which $(xy)t(xy)^{-1} = t$ so that $xy \in C_J(t)$. Then $Z^{xy} = Z^y$ and we conclude that xy does not normalize Z and, consequently, $C_J(t) \neq K$.

The next proposition is most critical in our discussion of the structure of J. The group TK_1 is a subgroup of $C_G(Z)$ so that we will describe the action of K_1 on T in such a way that T is identified with a group of matrices as given in section 1.

(2.6) *If $N_G(M)$ is not p-closed,* $[C_J(t) : K] = 2$. *Moroever, for* $\tau \in C_J(t) - K$, $\quad Z^\tau = \{x \in T \mid x$ *has* (4.1) *entry zero*$\}$.

Proof. Let Z_1 and Z_2 be any two conjugates of Z in G and suppose $Z_1 \cap Z_2 \neq 1$. For $x \in Z_1 \cap Z_2$, condition (b) of Hypothesis A implies that Z_1 is the center of $C_G(x)$. Similarly, Z_2 must be the center of $C_G(x)$ and thus $Z_1 = Z_2$. We conclude that distinct conjugates of Z intersect trivially.

By (2.5), $C_J(t)$ contains an element τ which does not normalize Z. Thus $Z^\tau \subseteq T - Z$ and $T = ZZ^\tau$. Let k_1 be a generator of K_1 and assume that k_1 does not normalize Z^τ. From the above remarks, $Z^{\tau k_1} \cap Z^\tau = 1$ and $T = Z^\tau Z^{\tau k_1}$. Let $c \in F_q$ and let z be an element of Z with (4.1) entry c. Then $z = xy$ for some $x \in Z^\tau$, $y \in Z^{\tau k_1}$. Since $y = w^{k_1}$ for some $w \in Z^\tau$ and k_1 leaves the (4, 1) entry of w fixed, xw is an element of Z^τ ith (4, 1) entry c. We conclude that every element of F_q appears as a (4, 1) entry of an element of Z^τ and, by a comparison of the number of elements, every element of Z^τ has a different (4, 1) entry. Furthermore, for $x \in Z^\tau$, $x^{k_1^i}$, $1 \leq i \leq (q-1)/2$ is an element of $Z^{\tau k_1^i}$ with the same (4, 1) entry as x but (3, 2) entry multiplied by ε^{2i}. Thus $x^{k_1^i}$ can not be an element of Z^τ. We conclude that $Z^{\tau k_1^i}$, $1 \leq i \leq (q-1)/2$ are distinct conjugates of Z in $T - Z$.

For purposes of contradiction let us now assume that the conjugates $Z^{\tau k_1^i}$, $1 \leq i \leq (q-1)/2$ are all of the conjugates of Z in $T - Z$. Foı any conjugate Z_1 of Z in M, (2.4) implies that $Z_1 = Z$ or $Z_1 \subseteq M - P'$. Furthermore, every element of $M - P'$ is conjugate in P to an element of $T - Z$ so, for a conjugate Z_1 of Z in $M - P'$, there exists an element $a \in P \cap J$ such that $Z_1^a \cap T \neq 1$. Let $x \in Z_1^a \cap T$. Then t centralizes x and must centralize Z_1^a which is the center of $C_G(x)$. This implies that $Z_1^a \subseteq T - Z$ and must be one of the $(q-1)/2$ conjugates $Z^{\tau k_1^i}$. It follows that Z_1 is a conjugate of Z^τ via the action of PK_1. It is calculated that no element of $(P \cap J)K_1$ whose order is divisible by p can normalize Z^τ so that $N_{PK_1}(Z^\tau) = M\langle t\rangle$. Hence Z^τ has exactly $q(q-1)/2$ conjugates in $M - Z$ via the action of PK_1. We conclude that M contains exactly $q(q-1)/2 + 1$ conjugates of Z and calculate $|J| = q(q^2 - q + 2)|K|/2$. The number of conjugates of Z in $T - Z$ is $(q-1)/2$ and all occur via the action of $C_J(t)$. Thus,

$$|C_J(t)| = |K|(q+1)/2,$$

and we conclude that $(q+1)/2$ divides $q(q^2-q+2)/2$. The integers $(q+1)/2$ and q are relatively prime so $(q+1)/2$ divides $(q^2-q+2)/2$. However,

$$q(q+1)/2-(q^2-q+2)/2=q-1$$

so that $(q+1)/2$ divides $q-1$. This occurs if and only if $q=3$. The case $q=3$ is exceptional and has been investigated in (2.4) of [7].

As a consequence of the preceding paragraph, we may now assume that $T-Z$ contains, in addition to the $(q-1)/2$ conjugates $Z^{\tau k_1^i}$, another conjugate Z_1 of Z.

First of all, consider the subgroup V of M containing all elements of M inverted by t. It is calculated that V is a subgroup of P' of order q and, as τ leaves V invariant, every element of the coset Vz, $z \in Z$ is conjugate to an element of the set Vz^τ. Let x, y be two elements of Z^τ with the same $(3,2)$ entry. Then $xy^{-1} \in Z \cap Z^\tau = 1$ so that $x=y$. Thus the elements of Z^τ have distinct $(3,2)$ entries and every element of F_q appears as a $(3,2)$ entry of some element in Z^τ. Let z^τ be an element of Z^τ with $(3,2)$ entry 1 and $(4,1)$ entry c so that a typical element of the coset Vz^τ is given by,

$$(2) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 1 & 1 & 0 \\ c & b & 0 & 1 \end{pmatrix}, \quad b \in Fq.$$

It is calculated that an element of the form (2) is conjugate in PK_1 to the $(q-1)/2$ elements,

$$(3) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \varepsilon^{2i} & 1 & 0 \\ c-b^2 & 0 & 0 & 1 \end{pmatrix}, \quad 1 \le i \le (q-1)/2.$$

In fact, as b is an arbitrary element of F_q, there are $(q-1)^2/4$ elements of the form (3) each of which is conjugate to an element of the coset Vz. Replacing the $(3,2)$ entry of (2) by ε, the same argument shows that $T-Z$ contains $(q-1)^2/4$ elements with $(3,2)$ entries ε^{2i+1}, $1 \le i \le (q-1)/2$ and each of these elements are conjugate to elements of $P'-Z$. Counting the elements of $T-Z$ which belong to the $(q-1)/2$ conjugate $Z^{\tau k_1^i}$, there are exactly $q-1$ elements of $T-Z$ which do not belong to one of the conjugates $Z^{\tau k_1^i}$ or are not conjugate to an element of $P'-Z$. It follows that Z_1 must be the only conjugate of Z in $T-Z$ distinct from $Z^{\tau k_1^i}$, $1 \le i \le (q-1)/2$.

We have shown that $T-Z$ contains $(q+1)/2$ conjugates of Z and that the conjugate Z_1 does not belong to the orbit of Z^τ under the action of K_1. Hence K_1 leaves Z_1 fixed. Furthermore, an earlier argument shows that any conjugate of Z in $M-P'$ is conjugate in PK_1 to Z^τ or Z_1. Using the structure of P, $N_{PK_1}(Z_1)=MK_1$ so that Z_1 has q conjugates in $M-P'$ via the action of PK_1 while Z^τ has $q(q-1)/2$ such conjugates. Thus Z_1 and Z^τ are not conjugate in PK_1 and $M - P'$ contains exactly $q(q+1)/2$ conjugates of Z. This implies that M contains exactly $(q^2+q+2)/2$ conjugates of Z and we calculate $|J| = q(q^2+q+2)|K|/2$.

If $C_J(t)$ acts intransitively on the set $\{Z, Z^{\tau k_1}, Z_1\}$, $|C_J(t)| = (q+1)|K|/2$ and $(q+1)/2$ divides $(q^2+q+2)/2$ which is impossible. It follows that $C_J(t)$ acts transitively and that $|C_J(t)| = (q+3)|K|/2$. This implies that $(q+3)/2$ divides $q(q^2+q+2)/2$ which occurs only in the cases $q = 3, 5, 9$. We shall now investigate each of these cases and show that we have an impossible situation in each case.

Let $q = 3$. The structure of J given in (2.4) of [7] shows that $T - Z$ contains exactly one conjugate of Z, a contradiction to the existence of the $(q+1)/2$ conjugates $Z_1, Z^{\tau k_1}$. If $q = 9$, $|C_J(t)| = 6|K|$ and t centralizes an element of order three. Thus there exists $y \in J$ such that t^y centralizes an element of $P \cap J$. However, $P \cap J$ is a T.I. subset of J. Indeed, let S be a Sylow 3-subgroup of $N_G(M)$ such that $S \cap P$ contains an element x not contained in M. From the structure of P, $C_M(x) = Z$ so that Z must coincide with the center of S. Using the structure of $C_G(Z)$, S contains both U and M so that $S = P$. It now follows that $t^y \in (P \cap J)K$ and, for some $a \in P \cap J$, $t^{ya} \in K$. Using (2.3), $t^{ya} = t$. This is impossible as t^{ya} centralizes an element of $P \cap J$ while t inverts the nontrivial elements of $P \cap J$.

For $q = 5$, $|C_J(t)| = 4|K|$ and $|J| = 2^4 5|K|$. Furthermore, K/K_1 induces a group of automorphisms of the cyclic group Z so that K/K_1 is a 2-group with $|K/K_1| \leq 4$. The conjugate Z_1 of Z is a subgroup of T of order 5 and is left invariant by K_1 so, letting k_1 be a generator of K_1 and noticing that k_1 centralizes no element of $T - Z$, k_1 induces an automorphism of Z_1 of order 2. It follows that a generator z of Z is conjugate in K to z^{-1} and that $|K/K_1| \neq 1$. Hence $|K| = 2^3$ or 2^4. In the latter case J has a Sylow 2-group of order 2^8 which is impossible as J is isomorphic to a subgroup of $GL(3,5)$. Thus $|K| = 2^3$ and $|J| = 5 \cdot 2^7$. This implies that J has a Sylow 2-group S which is isomorphic to a Sylow 2-group of $GL(3,5)$ and we may assume that S contains the 2-group $C_J(t)$. Using [2] and computing the centralizers in S of non-central involutions, we must have that $|C_J(t)| = 2^4$ or 2^6. However, $C_S(t) = C_J(t)$ and is a 2-group of order 2^5. We conclude that the case $q = 5$ is impossible.

We have finally shown that for no value of q is it possible for the $(q+1)/2$ conjugates $Z^{\tau k_1}, Z_1$ to exist. It follows that Z^τ is a conjugate of Z normalized by K_1 and, from the structure of TK_1, $Z^\tau = \{x \in T \mid x \text{ has } (4,1) \text{ entry zero}\}$. More important, τ was an arbitrary element of $C_J(t) - K$ so we conclude that $[C_J(t) : K] = 2$.

(2.7) *Assume $N_G(M)$ is not p-closed. Then $K = K_1 K_1^\tau$, $[K_1, K_1^\tau] = 1$, $K_1 \cap K_1^\tau = \langle t \rangle$ where τ is some element of $C_J(t) - K$.*

Proof. As a consequence of (2.6), K is a normal subgroup of $C_J(t)$ of index 2 so that $C_J(t) = K\langle\tau\rangle$ for some $\tau \in C_J(t) - K$. This implies that K_1^τ

is a normal subgroup of K which centralizes Z^τ. Letting k_1 be a generator of K_1, k_1^τ leaves $C(Z, t) = Z \times L$ invariant and induces an automorphism of L which centralizes the Sylow p-subgroup Z^τ of L and leaves K_1 invariant. Hence k_1^τ induces the trivial autmorphism of L (see [9]) and, consequently, centralizes K_1. No element of K_1 distinct from t centralizes an element of $T - Z$ so we conclude that K contains the normal abelian group $K_1 K_1^\tau$ with $K_1 \cap K_1^\tau = t$.

Furthermore, K/K_1 acts as a regular group of automorphisms on the nontrivial elements of Z which implies that $|K/K_1| \leq q - 1$. Hence $|K| \leq (q-1)^2$ and $K_1 K_1^\tau$ is a subgroup of K of index at most 2. If $|K| = (q-1)^2$, K/K_1 acts transitively on the nontrivial elements of Z. This violates (c) of Hypothesis A and we conclude $K = K_1 K_1^\tau$.

Notice that the structure of K given by (2.7) coincides with the structure of K in $PSp_4(q)$. Without condition (c) of Hypothesis A, it is possible for $[K : K_1 K_1] = 2$. Indeed, consider the semi-inner automorphism θ of $PSp_4(q)$ which interchanges the two central classes of p-elements in $PSp_4(q)$ (see [9]). The extension of $PSp_4(q)$ by θ is a group which satisfies (a) and (b) of Hypothesis A with $|K| = (q-1)^2$.

By (2.7), K is an abelian group which acts irreducibly on $P \cap J$. Hence K contains a subgroup K_0 which centralizes $P \cap J$ with K/K_0 cyclic. Since $K_1 \cap K_0 = 1$, the structure of K forces

$$|K/K_0| = q - 1 \quad \text{and} \quad K = C_K(P \cap J) \times K_1.$$

We are now able to prove the main proposition of this section.

Theorem 1. *Let G be a finite group satisfying Hypothesis* A. *If $N_G(M)$ is not p-closed, $N_G(M) = MJ$, $M \cap J = 1$ and $J = F \times D$ where $F \cong PGL(2, q)$ and D is cyclic of order $(q-1)/2$.*

Proof. Let $x \in J$ such that Z^x is a conjugate of Z which belongs to $T - Z$. Then xtx^{-1} centralizes Z and is an involution of $J \cap C_G(Z) = (P \cap J)K_1$. For some $a \in P \cap J$, $(ax)t(ax)^{-1} = t$ so that $ax \in C_J(t)$. However, $C_J(t)$ has K as a subgroup of index 2 which implies that $Z^{ax} = Z^\tau$. Since $Z^{ax} = Z^x$, $Z^x = Z^\tau$ and we conclude that $T - Z$ contains the unique conjugate Z^τ. Furthermore, for any conjugate Z_1 of Z which belongs to $M - T$, there exists $a \in P \cap J$ for which $Z_1^a \cap T \neq 1$. Let $x \in Z_1^a \cap T$ and notice that t centralizes x and must consequently centralize Z_1^a which is the center of $C_G(x)$. Hence $Z_1^a \subseteq T - Z$ and we have $Z_1^a = Z^\tau$. This proves that Z, $Z^{\tau a}$, where a ranges over the q elements of $P \cap J$, are the distinct conjugates of Z in M. Thus, $[J : (P \cap J)K] = q + 1$ and it follows that $P \cap J$, $(P \cap J)^{\tau a}$, $a \in P \cap J$ are the $q + 1$ Sylow p-groups of J.

Let us consider the representation of J as a permutation group of its $q + 1$ Sylow p-subgroups. Clearly $(P \cap J)K$ is the subgroup of J fixing a letter and K fixes both $P \cap J$ and $(P \cap J)^\tau$. For $1 \leq i \leq q - 1$, $a \in P \cap J$, $k_1^{-i} a k_1^i = b$, $(P \cap J)^{\tau a k_1^i} = Z^{\tau b}$. Hence, as k_1 acts transitively on the nontrivial elements of

$P \cap J$, K acts transitively on the remaining $q - 1$ letters $(P \cap J)^{\tau a}$, $a \in P \cap J$, $a \neq 1$.

Let D be the subgroup of J fixing three Sylow p-groups. Assuming D fixes $P \cap J$ and $(P \cap J)^{\tau}$, D is a subgroup of K. Let $d \in D$ and let $(P \cap J)^{\tau a}$, $a \neq 1$ be a third conjugate of $P \cap J$ fixed by D. Then

$$(\tau a)d(\tau a)^{-1} \in (P \cap J)K$$

so that $ada^{-1} \in (P \cap J)^{\tau}K$. Letting $d^{-1}ad = b$, $b \in P \cap J$, $ada^{-1} = dba^{-1}$ and we conclude that $ba^{-1} \in (P \cap J) \cap (P \cap J)^{\tau}$. Since $P \cap J$ is a T.I. subset of J (see case $q = 9$ of (2.6)), $a = b$ and we have that d centralizes a. We have seen that $K = K_1 \times C_K(P \cap J)$ so that $C_K(a) = C_K(P \cap J)$. This implies $D \subseteq C_K(P \cap J)$. Clearly $C_K(P \cap J) \subseteq D$ so that $D = C_K(P \cap J)$. This proves that D fixes all Sylow p-subgroups of J and must therefore coincide with the kernel of the representation of J on its Sylow p-groups. Consequently J/D may be viewed as a triply transitive permutation group on $q + 1$ letters for which the subgroup fixing 3 letters is trivial. A theorem of Zassenhaus [12] now applies and we have that $J/D \cong PGL(2, q)$.

We have seen that $K = K_1 \times D$ so that $| D | = (q - 1)/2$. Furthermore, $K = K_1^{\tau} \times D$ as τ normalizes K and D. Hence $k_1^{\tau} k_1^{i} \in D$ for some integer i satisfying $(i, q - 1) = 1$. From the structure of K, $k_1^{\tau} k_1^{i}$ has order $(q - 1)/2$ and we conclude that D is a cyclic group of order $(q - 1)/2$.

We now claim that J splits over D. It is computed that

$$| J | = q(q + 1)(q - 1)^2/2$$

so that if r is an odd prime divisor of $| D |$, a Sylow r-subgroup of K is a Sylow r-subgroup of J. Since K is abelian, a Sylow r-subgroup R of J splits over $R \cap D$ for all odd prime divisors r of $| D |$. It remains to consider a Sylow 2-group S of J. If $q \equiv -1 \bmod 4$, $(q - 1)/2$ is odd so that $S \cap D = 1$. If $q \equiv 1 \bmod 4$, let $q - 1 = 2^w e$, $(2, e) = 1$. The element τ normalizes K and $\tau^2 \in K$ so that we may assume τ has 2-power order. Consider the 2-group $\langle k_1^{e}, k_1^{e\tau} \rangle$ which is a Sylow 2-group of K of order 2^{2w-1}. Since τ interchanges $k_1^{e}, k_1^{e\tau}$, $S = \langle k_1^{e}, \tau \rangle$ is a 2-group of order 2^{2w} and a comparison of orders shows that S is a Sylow 2-group of J. Now J is generated by $(P \cap J)K$ and $(P \cap J)^{\tau}$ and we have seen that D centralizes $(P \cap J)K$. Furthermore, as D is normal in J, D centralizes $(P \cap J)^{\tau}$ and we conclude that D is a subgroup of the center of J. In particular, $S \cap D \subseteq C_S(\tau) = \langle (k_1 k_1^{\tau})^e, t \rangle$. A comparison of orders shows that $S \cap D$ is generated by $(k_1 k_1^{\tau})^e$ or $t(k_1 k_1^{\tau})^e$. Let

$$E = \langle (k_1^{-1} k_1^{\tau})^e, t \rangle.$$

It is seen that $| E | = 2^w$ and that E is normalized by τ. Thus $E\langle \tau \rangle$ is a 2-group of order 2^{w+1} and satisfies $E\langle \tau \rangle \cap (S \cap D) = 1$. Thus $S = E\langle \tau \rangle \times (S \cap D)$ and S splits over $S \cap D$. A theorem of Gaschütz now applies and we conclude that $J = FD$, $F \cap D = 1$. As D is in the center of J and

$$J/D \cong PGL(2, q), \qquad J = F \times D$$

and the statement of Theorem 1 follows.

3. The nonsimple case

Throughout this section let G be a finite group satisfying Hypothesis B.

HYPOTHESIS B. *G satisfies* (a), (b) *of Hypothesis* A *for* $p = 3$, $q = 3^m$, *m odd, $m > 1$. The subgroup $N_G(M)$ has P as a normal Sylow* 3*-subgroup.*

It will be shown that a group satisfying Hypothesis B is an extension of $C_G(Z)$ by a group of automorphisms of order less than or equal to $q - 1$. After a series of lemmas, it is shown that a group satisfying Hypothesis B has a quaternion or semidihedral Sylow 2-group. This gives enough information to establish the structure of G.

It is an immediate consequence of Hypothesis B that $N_G(M) \subseteq N_G(P)$. This fact, together with the fact that M is a characteristic subgroup of P, implies that $N_G(M) = N_G(P)$. Hence $N_G(M) = PK$, $P \cap K = 1$ and K can be chosen to be a complement for P containing t in its center. Because $C_M(t) = T$ and T is an elementary abelian 3-group of order q^2 containing Z, $C_G(T)$ is a subgroup of $C_G(Z)$. It is calculated that $C_G(T) = M\langle t\rangle$ and we have that M is a characteristic subgroup of $C_G(T)$. Therefore M is normal in $N_G(T)$ and $N_G(T) = MK$. This implies that T is a Sylow 3-subgroup of $C_G(t)$.

For all nontrivial elements $z \in Z$, $C(z, t) = Z \times L$ where L is isomorphic to $SL(2, q)$. Let us identify L with its 2-dimensional matrix representation over F_q in such a way that $T \cap L$ corresponds to the collection of all matrices

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \lambda \in F_q$$

and $K_1 = C_K(Z)$ corresponds to

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

where ε is a primitive element of F_q. Then any automorphism of L is given by $B \to A^{-1}\beta^{\phi}A$ where A is a nonsingular 2-dimensional matrix over F_q and ϕ is an automorphism of F_q (see [9]). It is easily calculated that any automorphism of L which centralizes the quaternion group generated by

$$x = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a field automorphism. In the following series of lemmas let Q be the quaternion group generated by x and y. As q is an odd power of 3, 4 is the highest power of 2 dividing $q + 1$ so that Q is a Sylow 2-group of L.

(3.1) *If $C_G(Q) \neq ZC_K(Q)$, $C_G(Q)/\langle t\rangle$ is isomorphic to $PSL(2, q)$.*

Proof. The involution t is contained in the center of Q so that $C_G(Q)$ is a subgroup of $C_G(t)$. Also, $C_G(t)$ has an abelian Sylow 3-group so a Sylow 3-group of $C_G(Q)$ containing Z must centralize Z. Since

$$C_G(Q) \cap C_G(Z) = Z\langle t\rangle,$$

Z is a Sylow 3-group of $C_G(Q)$.

Suppose that $y \in N_G(Z) \cap C_G(Q)$. Then y induces an automorphism of L which centralizes Q. From the remark preceding (3.2), y normalizes $T \cap L$ and K_1. As $T = (T \cap L)Z$, y normalizes T so $y \in TK$. Since

$$TK \cap N_G(K_1) = ZK, \qquad y \in ZK.$$

Therefore, $N_G(Z) \cap C_G(Q) = ZC_K(Q)$.

Let $X = C_G(Q)/\langle t\rangle$ and let $\bar{Z}$ be the image of Z in X. For all nontrivial $z \in Z$,

$$C_G(Q) \cap C_G(z) = Z\langle t\rangle$$

so that $C_X(\bar{z}) = \bar{Z}$. Hence $\bar{Z}$ is a CC-subgroup of X of order q. If

$$C_K(Q)^- = \bar{1}, \qquad N_G(Z) \cap C_G(Q) = Z\langle t\rangle$$

and $C_G(Q)$ has a normal 3-complement E. Then

$$E = \prod\nolimits_{z \in Z^\#} C_E(z)$$

which implies that $E \subseteq Z\langle t\rangle$. Hence $C_G(Q) = Z\langle t\rangle$ contrary to the hypothesis of (3.1). On the other hand, suppose $|C_K(Q)^-| = q - 1$ so that $|C_K(Q)| = 2(q-1)$. Because $q - 1$ is not divisible by 4, a Sylow 2-group of $C_K(Q)$ has order 4 and contains the central involution t. Let us suppose that $C_K(Q)$ contains an element β of order 4. As $\beta^2 = t$, β acts in a fixed-point-free manner on the $q - 1$ elements of M inverted by t. This implies that 4 divides $q - 1$ which is not the case. We may therefore assume that $C_K(Q)$ contains a four-group $\langle t, \tau\rangle$ and the involution τ induces an automorphism of L sending

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 \\ \lambda^{\phi} & 1 \end{pmatrix}$$

where ϕ is an automorphism of F_q. Because $q = 3^m$, m odd, L admits no field automorphism of order 2. We conclude $|C_K(Q)^-| \neq q - 1$.

We have now shown that X contains $\bar{Z}$ as a CC-subgroup of order q and $[N_X(\bar{Z}) : \bar{Z}] \neq 1$ or $q - 1$. In addition, $\bar{Z}$ is not a normal subgroup of X as otherwise $C_G(Q) = ZC_K(Q)$ contrary to hypothesis. By (5.1) of [8], $X \cong PSL(2, q)$.

As a result of (3.1) we will now investigate the structure of groups satisfying Hypothesis C.

HYPOTHESIS 3. *G satisfies Hypothesis* B *and* $C_G(Q) \neq ZC_K(Q)$.

Groups satisfying Hypothesis C require a rather detailed discussion of their 2-structure. After a series of propositions, it will be shown that no such groups can exist. This, of course, implies that a group satisfying Hypothesis B must have $C_G(Q) = ZC_K(Q)$.

(3.2) *Let G satisfy Hypothesis* C. *Then*

$$N_G(Q) = \langle x\rangle QC_G(Q) \quad \text{or} \quad \langle x, \sigma\rangle C_G(Q)$$

where x is an element of $T \cap L$ which normalizes Q and σ is an involution of $N_G(Z)$ with $Q\langle\sigma\rangle$ a semi-dihedral group of order 16.

Proof. It is seen that $N_L(Q)$ is the subgroup of L generated by

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and Q. The group $N_G(Z) \cap N_G(Q)$ induces a group of automorphisms of Z with kernel $Z\langle x\rangle Q$ and, since $N_G(Z) \cap N_G(Q)/Z\langle x\rangle Q$ acts regularly on Z,

$$[N_G(Z) \cap N_G(Q) : Z\langle x\rangle Q] \leq q - 1.$$

By (3.1), $| C_K(Q) | = q - 1$ so $| (Z\langle x\rangle Q)C_K(Q) | = | Z\langle x\rangle Q | (q - 1)/2$ and we conclude $(Z\langle x\rangle Q)C_K(Q)$ has index at most 2 in $N_G(Z) \cap N_G(Q)$.

Let us suppose $[N_G(Z) \cap N_G(Q) : (Z\langle x\rangle Q)C_K(Q)] = 2$. Then K/K_1 acts transitively on Z so that $| K | = (q - 1)^2$ and K has a Sylow 2-group of order 4. This implies that K contains an involution τ such that $\langle t, \tau\rangle$ is a four group. Because τ does not centralize Q, τ induces an automorphism of L which comes from the natural action of $GL(2, q)$ on $SL(2, q)$ so that $L\langle\tau\rangle$ is isomorphic to a subgroup of $GL(2, q)$. Let σ be an appropriate involution of $L\langle\tau\rangle$ such that $Q\langle\sigma\rangle$ is a Sylow 2-group of $L\langle\tau\rangle$. Then $Q\langle\sigma\rangle$ has order 16 and, by a comparison of orders, is isomorphic to a Sylow 2-group of $GL(2, q)$. By [2], $Q\langle\sigma\rangle$ is semi-dihedral and

$$N_G(Z) \cap N_G(Q) = \langle x, \sigma\rangle QC_K(Q).$$

For $y \in N_G(Q)$, Z^y is a Sylow 3-group of $C_G(Q)$ so for some $w \in C_G(Q)$, $Z^{yw} = Z$. Hence $yw \in N_G(Z) \cap N_G(Q)$ and $y \in (N_G(Z) \cap N_G(Q))C_G(Q)$. We conclude that $N_G(Q) = \langle x\rangle QC_G(Q)$ or $\langle x, \sigma\rangle QC_G(Q)$.

(3.3) *Let G satisfy Hypothesis* C *and W be a Sylow* 2*-group of $C_G(Q)$. Then W is quaternion or elementary abelian of order* 8.

Proof. Let W be a Sylow 2-group of $C_G(Q)$. Since a Sylow 2-group of $PSL(2, q)$ is a four-group, $W/\langle t\rangle$ is a four-group and $| W | = 8$. If W contains no element of order 4, W is elementary abelian as desired. We may therefore assume that W contains an element a of order 4. Furthermore, the involutions of $W/\langle t\rangle$ are conjugate in $C_G(Q)$ so W must contain another element b of order 4 with $\langle a\rangle \cap \langle b\rangle = \langle t\rangle$. Now a and b do not commute as otherwise ab is an involution conjugate to a or at in $C_G(Q)$. Thus W is nonabelian and must be quaternion.

We have seen that $C_G(Q)/\langle t\rangle \cong PSL(2, q)$ so that $C_K(Q)/\langle t\rangle$ is cyclic of order $(q - 1)/2$. By hypothesis $(q - 1)/2$ is odd and it follows that $C_K(Q)$

contains an element of order $q - 1$. Let k_2 be a generator of $C_K(Q)$ and consider the action of k_2 as an automorphism of L. Since k_2 centralizes Q, k_2 sends $B \longrightarrow B^{\phi}$ where B is any 2-dimensional matrix of L and ϕ is an automorphism of F_q. In particular, we have identified $T \cap L$ with the collection

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \lambda \epsilon Fq$$

so k_2 centralizes

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

On the other hand, x normalizes Q and induces an automorphism of $C_G(Q)/\langle t \rangle$ which centralizes the Sylow 3-normalizer $ZC_K(Q)/\langle t \rangle$ and must consequently induce the trivial automorphism. Let W be any Sylow 2-group of $C_G(Q)$. Then x centralizes $W/\langle t \rangle$ and therefore W.

If the involution σ exists, $Q\langle \sigma \rangle$ is a semi-dihedral group by (3.2). In addition, σ induces an automorphism of $C_G(Q)$ inverting the nontrivial elements of the Sylow 3-group Z. If W is elementary abelian, $W = \langle t, \tau, \mu \rangle$ for some involutions τ and μ. Applying a theorem of Gaschütz, $C_G(Q)$ splits over $\langle t \rangle$ so that $C_G(Q) = \langle t \rangle \times Y$, $Y \cong PSL(2, q)$ and we may assume $W \cap Y = \langle \tau, \mu \rangle$. Then σ normalizes Y and inverts Z so that $Y\langle \sigma \rangle$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ as a subgroup of index 2. Choosing W appropriately, $\langle \tau, \mu, \sigma \rangle$ is a dihedral group of order 8. If W is quaternion, $W\langle \sigma \rangle/\langle t \rangle$ is isomorphic to a Sylow 2-group of $P\Gamma L\ (2, q)$ and is dihedral of order 8. Let s be an element of $W\langle \sigma \rangle$ such that s has order 4 in the factor group $W\langle \sigma \rangle/\langle t \rangle$. If $s^4 = 1$, s^2 is an involution of W and we must have $s^2 = t$. This contradicts our choice of s and we have $s^4 = t$. Hence $W\langle \sigma \rangle$ has a maximal cyclic subgroup of order 8 and since W is quaternion, $W\langle \sigma \rangle$ is not dihedral or generalized quaternion. A characterization of such groups [5, p. 193] implies that $W\langle \sigma \rangle$ is semi-dihedral.

We are now in a position to determine a Sylow 2-group of G.

(3.4) *Let G satisfy Hypothesis* C. *If a Sylow 2-group W of $C_G(Q)$ is elementary abelian, QW or $QW\langle \sigma \rangle$ is a Sylow 2-group of G.*

Proof. In the preceding paragraph we noticed that

$$C_G(Q) = \langle t \rangle \times Y, \quad Y \cong PSL(2, q)$$

where $W = \langle t, \tau, \mu \rangle$ and $W \cap Y = \langle \tau, \mu \rangle$. All involutions of Y are conjugate and there exists an element of order 3 which cyclically permutes them. In fact, Z is a Sylow 3-group of Y so that for suitable choice of W, we may assume $\langle \tau, \mu \rangle$ is normalized by an element z of Z. Furthermore,

$$N_G(Q) \cap N_G(QW) = \langle x, z \rangle QW \quad \text{or} \quad \langle x, z, \sigma \rangle QW$$

where $x \epsilon T \cap L$, permutes the subgroups of Q of order 4, and centralizes W.

Let $Q = \langle a, b \mid a^2 = b^2 = t, ab = ba^{-1} \rangle$. Consider $y \epsilon N_G(QW)$ and suppose $a^y = f\beta$, $b^y = h\gamma$ for $f, h \in Q$ and $\beta, \gamma \in \langle \tau, \mu \rangle$. Because a^y inverts b^y, f and h do not commute. Hence x or x^2 interchanges $\langle f \rangle$ and $\langle h \rangle$ so that $a^{yx^i} = h\beta$ or $th\beta$, $i \in \{1, 2\}$. In the case $\beta \neq \gamma$, $\beta^{z^j} = \gamma$ for $j \in \{1, 2\}$ and $a^{yx^iz^j} = h\gamma$ or $th\gamma$. That is, $a^{yx^iz^j} = b^y$ or tb^y and $yx^iz^jy^{-1} \in N_G(Q)$. In particular, x^iz^j normalizes Q^y. In the case $\beta = \gamma$, $a^{yx^i} = h\gamma$ or $th\gamma$ with $i \in \{1, 2\}$ and $a^{yx^i} = b^y$ or tb^y. Hence x^i leaves Q^y invariant. In any case, Q^y is normalized by an element of the form x^iz^j, $i \neq 0$.

It is computed that QW contains 7 quaternion groups normalized by an element x^iz^j, $i \neq 0$. Indeed, let the action of z on $\langle \tau, \mu \rangle$ and x on Q be given as follows:

$$z\colon \tau \longrightarrow \mu \longrightarrow \tau\mu; \quad x\colon a \longrightarrow b \longrightarrow ab.$$

Then xz normalizes each of the quaternion groups $Q_1 = Q$, $Q_2 = \langle a\tau, b\mu \rangle$, $Q_3 = \langle a\mu, b\tau\mu \rangle$ and $Q_4 = \langle a\tau\mu, b\tau \rangle$ while x^2z leaves $Q_5 = \langle a\tau, ab\mu \rangle$, $Q_6 = \langle a\tau\mu, ab\tau \rangle$ and $Q_7 = \langle a\mu, ab\mu\tau \rangle$ invariant. Notice that a quaternion group containing an element of Q or order 4 and normalized by x^iz^j, $i \neq 0$ must coincide with Q and Q is the only quaternion group normalized by $\langle x \rangle$. This proves that the 7 quaternion groups Q_i are exactly the quaternion subgroups left invariant by an element x^iz^j, $i \neq 0$.

Now $N_G(QW)$ induces a permutation group on the Q_i with $\langle x, z \rangle QW$ or $\langle x, z, \sigma \rangle QW$ the subgroup fixing the letter Q. Hence, if $N_G(QW)$ acts transitively on the Q_i, $N_G(QW)$ has a subgroup of index 7. However $(QW)' = \langle t \rangle$ and W coincides with the center of QW. Therefore any automorphism of QW of order 7 centralizes W, QW/W and consequently QW. As $C_G(QW) = W$, $N_G(QW)$ can contain no element of order 7 and we conclude that $N_G(QW)$ acts intransitively on the Q_i.

Let us suppose $N_G(QW) \neq QW$ or $\langle x, z, \sigma \rangle QW$. Without loss of generality assume Q and Q_2 are conjugate. Since z transitively permutes Q_2, Q_3 and Q_4 and x permutes Q_5, Q_6 and Q_7 transitively, Q and Q_5 can not be conjugate. Otherwise $N_G(QW)$ acts transitively on the Q_i in contradiction to the previous paragraph. We see that $N_G(QW)$ induces a transitive permutation group on the orbit $\{Q, Q_2, Q_3, Q_4\}$ with kernel $QW\langle xz \rangle$. However, let us consider $C_G(xz, t)$. We have seen that $N_G(T) = MK$ and that TK is a Sylow 3-normalizer of $C_G(t)$. In addition $|K/K_1| \leq q - 1$ so that $K = K_1K_2$ or $[K : K_1K_2] = 2$. In both cases K leaves Z and $T \cap L$ invariant. Now K_1 acts $\frac{1}{2}$-transitively on the nontrivial elements of $T \cap L$ and $C_K(z) = K_1$ so it follows that $C_K(xz) = \langle t \rangle$ and $T\langle t \rangle$ is a Sylow 3-normalizer of $C_G(xz, t)$. Hence $C_G(xz, t)$ has a normal 3-complement E which is left invariant by Z. We conclude that

$$E \subseteq (T \cap L)\langle t \rangle \quad \text{and} \quad C_G(xz, t) = T\langle t \rangle.$$

Finally, for $y \in N_G(QW)$, $(xz)^y \in QW\langle xz \rangle$ so that $(xz)^{ys} = xz$ or $(xz)^2$ for some $s \in QW$. In the first case $ys \in T\langle t \rangle \cap N_G(QW) = \langle x, z, t \rangle$ and $y \in \langle x, z \rangle QW$. In the second case (xz) and $(xz)^2$ are conjugate so that σ must exist. Then $(xz)^{ys\sigma} = xz$ as σ inverts the nontrivial elements of Z and $T \cap L$. We have that $ys\sigma \in \langle x, z, t\rangle$ and $y \in \langle x, z, \sigma\rangle QW$. In both cases y normalizes Q a contradiction to hypothesis. We conclude $N_G(QW) = \langle x, z\rangle QW$ or $\langle x, z\rangle QW\langle\sigma\rangle$.

If σ does not exist, $N_G(QW) = \langle x, z\rangle QW$ and QW is a Sylow 2-group of G as desired. We may therefore assume that σ exists and consider $S = Q\langle\tau, \mu, \sigma\rangle$ where $Q\langle\sigma\rangle$ is semi-dihedral and $\langle\tau, \mu, \sigma\rangle$ is dihedral. Let τ be the central involution of $W\langle\sigma\rangle$. Then $S' = \langle a, \tau\rangle$ where a is an element of Q or order 4. Furthermore, $C_S(S') = \langle s\rangle W$ where s is an element of $Q\langle\sigma\rangle$ of order 8. Since every involution of $\langle s\rangle W$ belongs to W, W is a characteristic subgroup of $\langle s\rangle W$ and consequently is characteristic in S. We conclude that $C_S(W) = QW$ is left invariant by $N_G(S)$ and the conclusion of the preceding paragraph yields that $N_G(S) = QW\langle\sigma\rangle$. Therefore $QW\langle\sigma\rangle$ is a Sylow 2-group of G.

(3.5) *Let G satisfy Hypothesis* C. *If a Sylow 2-group of $C_G(Q)$ is quaternion, QW or $QW\langle\sigma\rangle$ is a Sylow 2-group of G.*

Proof. Let us first assume that QW is a Sylow 2-group of $N_G(Q)$. We show that no element of Q of order 4 is conjugate in G to an element of W. Indeed, suppose y is an element of Q of order 4 and there exists $g \in G$ with $y^g \in W$. Then $C_G(y^g)$ contains Z^g and the remarks preceding (3.4) imply that y^g is centralized by an element $x \in T \cap L$. A Sylow 3-group of $C_G(y)$ which contains Z is abelian and is thus a subgroup of $C_G(y, Z) = Z\langle d\rangle$ where d is an element of L of order $q + 1$. Hence Z is a Sylow 3-group of $C_G(y)$. It follows that x is conjugate to an element $z \in Z$. However, x and z are elements of M where M is the unique abelian subgroup of P of order q^3. This implies that y and z are conjugate in $N_G(M) = PK$. Since Z is a characteristic subgroup of P, this is impossible. The elements of QW of order 4 belong to Q or W so that $N_G(QW)$ permutes the elements of Q of order 4 and leaves Q invariant. Therefore QW is a Sylow 2-group of $N_G(QW)$ and is a Sylow 2-group of G.

On the other hand suppose that $QW\langle\sigma\rangle$ is a Sylow 2-group of $N_G(Q)$. Let $S = QW\langle\sigma\rangle$ and notice that $Q\langle\sigma\rangle$ and $W\langle\sigma\rangle$ are semi-dihedral groups. Let $x \in Q$, $y \in W$ and suppose $xy\sigma$ is an element of order 4. Then $(x\sigma)^4(y\sigma)^4 = 1$ so that $(x\sigma)^4 \in Q \cap W = \langle t\rangle$. If $(x\sigma)^4 = t$, $(y\sigma)^4 = t$ and $(x\sigma)^2$, $(y\sigma)^2$ are elements of order 4 belonging to Q and W respectively. Hence $(xy\sigma)^2 = (x\sigma)^2(y\sigma)^2$ is an involution of QW different from t. When $(x\sigma)^4 = 1$, $(y\sigma)^4 = 1$, one of the elements $x\sigma$ and $y\sigma$ has order 4 which, without loss of generality, we may assume to be $y\sigma$. Because $W\langle\sigma\rangle$ is semi-dihedral, W contains all elements of $W\langle\sigma\rangle$ of order 4. In particular, $y\sigma \in W$ which is not the case. We have therefore shown that an element of order 4 which belongs to $S - QW$ has as its square an involution of QW different from t. The center of S is generated by t so that no element of Q of order 4 is conjugate in $N_G(S)$ to an element of $S - QW$. Thus $N_G(S)$ permutes the elements of QW of order 4 and, as no element of order 4 in Q is conjugate to an element of W, $N_G(S)$ leaves Q invariant. This implies $N_G(S) = S$ and we conclude that $S = QW\langle\sigma\rangle$ is a Sylow 2-group of G.

(3.6) *If G satisfies Hypothesis* B, $C_G(Q) = ZC_K(Q)$. *In particular, no group satisfying Hypothesis* C *can exist.*

Proof. If G satisfies Hypothesis C, the structure of a Sylow 2-group of G is given by (3.4) and (3.5). We will show that in any of these cases t is conjugate to no involution of a Sylow 2-group of G other than itself. This implies that $G = C_G(t)O(G)$. Because $C_G(t)$ has a Sylow 3-group of order q^2 and a Sylow 3-group P of G containing Z has order q^4, $P \cap O(G)$ is a nontrivial normal subgroup of P and $Z \cap O(G) \neq 1$. Consequently, $C_G(Q)$ has a nontrivial normal subgroup of odd order which is impossible unless

$$C_G(Q) = ZC_K(Q).$$

In order to complete the proof of (3.6) it is therefore sufficient to show that in a group G satisfying Hypothesis C, t is conjugate to no involution of QW or $QW\langle\sigma\rangle$ other than itself.

Let us assume that W is elementary abelian and that QW is a Sylow 2-group of G. As W coincides with the center of QW and contains all involutions, fusion of involutions is controlled by $N_G(QW)$. However, $N_G(QW) = \langle x, z\rangle QW$ as seen in (3.4) so that t is conjugate to no involution of $QW - \langle t\rangle$.

We may now let $QW\langle\sigma\rangle$ be a Sylow 2-group of G with W elementary abelian. Let $W = \langle t, \tau, \mu\rangle$ so that $\langle\tau, \mu, \sigma\rangle$ is dihedral of order 8. Every involution $xy\sigma$, $x \in Q$, $y \in \langle\tau, \mu\rangle$ satisfies $(x\sigma)^2(y\sigma)^2 = 1$ so that $x\sigma$ and $y\sigma$ are involutions of $Q\langle\sigma\rangle$ and $\langle\tau, \mu, \sigma\rangle$ respectively. Since $\langle\tau, \mu, \sigma\rangle$ is dihedral and $y\sigma \notin \langle\tau, \mu\rangle$, $y\sigma$ is conjugate in $\langle\tau, \mu, \sigma\rangle$ to σ. Hence $xy\sigma$ is conjugate to $a\sigma$, $a \in Q$. Now $Q\langle\sigma\rangle$ is semi-dihedral so has two classes of involutions with representatives t and σ. We conclude that $xy\sigma$ is conjugate to σ and every involution of $QW\langle\sigma\rangle - QW$ is conjugate to σ. Let τ be the central involution of $\langle\tau, \mu, \sigma\rangle$ so that $\langle t, \tau\rangle$ is the center of $QW\langle\sigma\rangle$. Every involution of $QW - \langle t\rangle$ is conjugate in $C_G(Q)$ to τ or $t\tau$, and by (3.4), $N_G(QW\langle\sigma\rangle) = QW\langle\sigma\rangle$. Hence t, τ, $t\tau$ belong to distinct conjugacy classes of G.

Let us assume that t is conjugate to an involution of $QW\langle\sigma\rangle - \langle t\rangle$ so that t is conjugate to σ and let R be a Sylow 2-group of $C_G(\sigma)$ containing $\langle t, \tau, \sigma\rangle$. As σ and t are conjugate, G has exactly three classes of involutions K_i, $i = 1, 2, 3$ with representatives t, τ and $t\tau$. The involutions of $\langle t, \tau, \sigma\rangle$ are partitioned among these classes such that $K_1 = \{t, \sigma, t\sigma, t\tau\sigma, \tau\sigma\}$, $K_2 = \{\tau\}$, $K_3 = \{t\tau\}$. Letting $E = N_R\langle t, \tau, \sigma\rangle$, E centralizes $\langle\tau, t\rangle$ and, since $QW\langle\sigma\rangle$ is a Sylow 2-group of $C_G\langle t, \tau\rangle$, there exists $g \in C_G(t, \tau)$ for which $E^g \subseteq QW\langle\sigma\rangle$. By our choice of E, E^g centralizes $\langle t, \tau, \sigma^g\rangle$ and σ^g is an involution of $QW\langle\sigma\rangle$ different from t. Now σ^g is not conjugate to τ or $t\tau$ so $\sigma^g \notin QW$. Hence σ^g is conjugate in $QW\langle\sigma\rangle$ to σ. However, $C_G(\sigma) \cap QW\langle\sigma\rangle = \langle t, \tau, \sigma\rangle$ so that $|E^g| = 8$. Hence $E = \langle t, \tau, \sigma\rangle$ is a Sylow 2-group of $C_G(\sigma)$, a contradiction to our assumption that t and σ are conjugate. We conclude that t is conjugate to no involution of $QW\langle\sigma\rangle - \langle t\rangle$.

We now consider W to be quaternion. Let Q be generated by elements a_1 and b_1 of order 4 and assume W is generated by corresponding elements a_2, b_2.

Every involution of QW different from t is given by xy, $x \in Q$, $y \in W$ where x and y have order 4. Furthermore $\langle x, z\rangle$ normalizes QW such that z transitively permutes the subgroups of W of order 4 and x centralizes W. Hence every involution of $QW - \langle t\rangle$ is conjugate to $v = a_1 a_2$.

In the case that QW is a Sylow 2-group of G and t is conjugate to an involution of $QW - \langle t\rangle$, t and v are conjugate. Let V be a Sylow 2-group of $C_G(v)$ containing $R = \langle a_1, a_2, b_1 b_2\rangle$. Since R has index 2 in QW, R and R' are normal subgroups of V. It is computed that $R' = \langle t\rangle$ so that $\langle t\rangle$ is left invariant by V. This implies $\langle t, v\rangle$ is contained in the center of V which is impossible as QW and V are isomorphic. We conclude that t is conjugate to no involution of $QW - \langle t\rangle$.

Finally, let us consider the case when W is quaternion and $QW\langle\sigma\rangle$ is a Sylow 2-group of G. An involution $yw\sigma$, $x \in Q, w \in W$ satisfies $(y\sigma)^2(w\sigma)^2 = 1$ so that $y\sigma$ and $w\sigma$ are both elements of order 4 or both involutions. If $w\sigma$ has order 4, $w\sigma \in W$ because W contains the elements of order 4 in $W\langle\sigma\rangle$. Thus $w\sigma$ is an involution of $W\langle\sigma\rangle$ different from t and is conjugate in $W\langle\sigma\rangle$ to σ. This implies $yw\sigma$ is conjugate to an involution of $Q\langle\sigma\rangle$ different from t. But $Q\langle\sigma\rangle$ is also semi-dihedral so that $yw\sigma$ is conjugate to σ. This proves that every involution of $QW\langle\sigma\rangle - \langle t\rangle$ is conjugate to σ or v. The centralizer of v in $QW\langle\sigma\rangle$ is $R = \langle a_1, a_2, b_1 b_2, \sigma\rangle$ which has index 2 in $QW\langle\sigma\rangle$. Furthermore, $R' = \langle t\rangle$ so that the argument of the preceding paragraph implies t and v are not conjugate.

We may assume, therefore, that t and σ are conjugate. Then G has two classes of involutions with representatives t and v. The involutions of $\langle t, v, \sigma\rangle$ are partitioned in such a way that $t, \sigma, v\sigma, t\sigma$ and $vt\sigma$ belong to one class while v, vt belong to the second class. Let S be a Sylow 2-group of $C_G(\sigma)$ containing $\langle t, v, \sigma\rangle$ and consider $E = N_S\langle t, v, \sigma\rangle$. Clearly E permutes v and vt so leaves $\langle v, t\rangle$ invariant and, since $QW\langle\sigma\rangle$ is a Sylow 2-group of $N_G\langle v, t\rangle$, $E^g \subseteq QW\langle\sigma\rangle$ for some $g \in G$. Now $\langle t, \tau, \sigma^g\rangle \subseteq QW\langle\sigma\rangle$ so that σ^g is an involution different from t. Furthermore, v and σ are not conjugate and we conclude that σ^g is conjugate to σ in $QW\langle\sigma\rangle$. Also, $C_G(\sigma) \cap QW\langle\sigma\rangle = \langle t, v, \sigma\rangle$ which implies E^g is conjugate to a subgroup of $\langle t, v, \sigma\rangle$. Therefore $E = \langle t, v, \sigma\rangle$ is a Sylow 2-group of $C_G(\sigma)$. We conclude that t and σ are not conjugate.

(3.8) *If G satisfies Hypothesis* B, *a Sylow* 2-*group of G is quaternion of order* 8 *or semi-dihedral of order* 16.

Proof. From the beginning remarks of this section, $C_G(z, t) = ZL$ for $z \in Z$ and $N_G(P) = PK$ where K is a complement of P containing t in its center. Consider $y \in N_G(Z) \cap C_G(t)$. For $z \in Z$, z^y and z are conjugate in K so there exists $k \in K$ such that $z^{yk} = z$. Hence $yk \in ZL$ and $y \in ZLK$. We conclude that $N_G(Z) \cap C_G(t) = ZLK$.

Let us suppose $[ZLK: ZL]$ is odd. It follows that Q is a Sylow 2-group of ZLK and by (3.6), $C_G(Q) = ZC_K(Q)$. Hence $N_G(Q) \subseteq ZLK$ and Q must be a Sylow 2-group of G.

If $[ZLK: ZL]$ is even, K/K_1 induces a regular group of permutations of the nontrivial elements of Z with $|K/K_1|$ even. Since $q - 1$ is not divisible by 4, K has a Sylow 2-group of order 4. Thus K contains an involution τ different from t and τ induces a nontrivial outer automorphism of L of order 2. Hence $L\langle\tau\rangle$ is isomorphic to a subgroup of $GL(2, q)$ containing $SL(2, q)$ as a subgroup of index 2. Letting σ be an appropriate involution of $GL(2, q)$ which normalizes Q, $Q\langle\sigma\rangle$ is isomorphic to a Sylow 2-group of $GL(2, q)$. In particular, $Q\langle\sigma\rangle$ is semi-dihedral of order 16 and is a Sylow 2-group of ZLK. Now Q is a characteristic subgroup of $Q\langle\sigma\rangle$ so that $N_G(Q\langle\sigma\rangle) \subseteq N_G(Q)$. However, $N_G(Q) \subseteq ZLK$ and we conclude $Q\langle\sigma\rangle$ is a Sylow 2-group of G.

(3.9) *If G satisfies Hypothesis* B, $C_G(t) = ZLK$.

Proof. By (3.8), Q or $Q\langle\sigma\rangle$ is a Sylow 2-group of $C_G(t)$. Hence $X = C_G(t)/\langle t\rangle$ has a dihedral Sylow 2-group. As X involves $PSL(2, q)$, [6] implies $X/O(X)$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$. Let D be the largest normal subgroup of $C_G(t)$ of odd order. Then $C_G(t)/D$ has a Sylow 3-group of order q so $|T \cap D| = q$. Now L has no normal subgroup of odd order which implies $T \cap D$ contains no element of $T \cap L$. Furthermore, K_1 acts $\frac{1}{2}$-transitively on the nontrivial elements of $T \cap L$ so that D can contain no element of $T - Z$. Otherwise D would contain an element of $T \cap L$ which is not the case. We conclude that $D \cap T = Z$ and that Z is a Sylow 3-group of D. Moreover, $C_D(Z) = Z$ so that Z is a CC-subgroup of D. If Z is not a normal subgroup of D, (4.4) of [8] implies $|D|$ is even. We conclude that Z is characteristic in D and consequently normal in $C_G(t)$. Hence $C_G(t) = ZLK$.

THEOREM 2. *If G satisfies Hypothesis* B, $G = C_G(Z)K$.

Proof. If Q is a Sylow 2-group of G, t is the unique involution of Q and [3] implies $G = C_G(t)O(G)$. Hence $P \cap O(G) \neq 1$ and $|O(G)|$ is divisible by 3. Let A be a minimal characteristic subgroup of $O(G)$ and assume A is a $3'$-group. As A is left-invariant by Z, A is a subgroup of $C_G(Z)$ which is impossible. This implies that A is a normal subgroup of P and $A \cap Z \neq 1$. Let $z \in A \cap Z$ and consider $g \in G$. Then $z^g \in A \cap Z$ as z is conjugate to no element of $P - Z$ and there exists $k \in K$ such that $z^{gk} = z$. This implies $g \in C_G(Z)K$ and $G = C_G(Z)K$ as desired.

We may now assume that $Q\langle\sigma\rangle$ is a Sylow 2-group of G. If σ and t are not conjugate, $G = C_G(t)O(G)$ and the argument of the preceding paragraph applies. Thus, let σ and t be conjugate. We may further assume $O(G) = 1$ for otherwise G contains a normal 3-subgroup A such that $A \cap Z \neq 1$. Proposition 2, p. 15 of [1] may now be used to conclude that G contains a normal subgroup X of odd index with X a simple group with Sylow 2-group $Q\langle\sigma\rangle$. Clearly $C_X(t)$ contains $L\langle\sigma\rangle$ so that $P \cap X$ is a nontrivial normal subgroup of P. Consequently, $Z \cap X \neq 1$ and since $C_X(t) \subseteq ZLK$, $Z \cap X$ is a normal subgroup of $C_X(t)$. An application of the first main theorem of [1] implies

$Z \cap X$ is a subgroup of the center of $C_X(t)$ contrary to the fact that σ inverts the nontrivial elements of $Z \cap X$. We conclude that σ and t can not be conjugate and $G = C_G(Z)K$ as desired.

4. The structure of $C_G(t)$

Throughout this section let G be a group satisfying Hypothesis A for the prime $p = 3$ and $q = 3^m$, m odd, $m > 1$. We will prove the following main proposition.

THEOREM 3. *Let $\mathfrak{C}$ be the centralizer in $PSp_4(3^m)$, m odd, $m > 1$ of an element of order 3 lying in the center of some Sylow 3-subgroup. Let G be a finite group satisfying:*

(a) *G contains an element α of order 3 such that $C_G(\alpha)$ is isomorphic to $\mathfrak{C}$.*

(b) *For all z in the center of $C_G(\alpha)$, $C_G(z) = C_G(\alpha)$.*

(c) *Not all central 3-elements belong to the same conjugacy class of G.*

Then one of the following cases occurs:

(i) *$C_G(\alpha)$ is a normal subgroup of G.*

(ii) *G is a simple group isomorphic to $PSp_4(3^m)$.*

Let G be a group satisfying (a), (b) of Theorem 3. The results of Section 3, particularly Theorem 2, imply that we may assume $N_G(M)$ is not 3-closed. Otherwise $C_G(\alpha)$ is normal in G and we are in case (i) of the above theorem. Hence the results of Section 2 are valid and the structure of $N_G(M)$ is given by Theorem 1. Since q is an odd power of 3, $q + 1$ is divisible by 4 with $q + 1 = 4e$, $(2, e) = 1$. We retain this notation throughout the section.

(4.1) *$N_G(T) \cap C_G(t) = TK\langle\tau\rangle$ where τ is an involution of $N_G(M)$.*

Proof. We have seen that T is an elementary abelian group of order q^2 centralized by t which contains Z. Hence $C_G(T) = M\langle t\rangle$. Clearly M is characteristic in $C_G(T)$ so that $N_G(T) \subseteq N_G(M)$. From (2.6),

$$[N_G(T) \cap N_G(M) : MK] = 2$$

so that $N_G(T) \cap N_G(M) = MK\langle\tau\rangle$ where τ is an involution which centralizes t but does not normalize Z. Thus

$$N_G(T) = MK\langle\tau\rangle \quad \text{and} \quad N_G(T) \cap C_G(t) = TK\langle\tau\rangle.$$

As a result of (2.7), the structure of K is given by $K = K_1 K_1^\tau$, $[K_1, K_1^\tau] = 1$, $K_1 \cap K_1^\tau = \langle t\rangle$ and τ is the same involution which appears in the statement of (4.1). Now Z^τ is the unique conjugate of Z in $T - Z$ and is normalized by K_1. Since a generator k_1 of K_1 has order $q - 1$, $k_1^{(q-1)/2} = t$. Furthermore, no element of K_1 different from t centralizes an element of $T - Z$ so that K_1 induces a group of automorphisms of Z^τ which partitions the nontrivial elements into two orbits of length $(q - 1)/2$ with representatives z^τ, $z^{-\tau}$ for some $z \in Z$. Similarly, K_1^τ partitions the nontrivial elements of Z into two orbits of length $(q - 1)/2$ such that z and z^{-1} lie in different orbits. Thus the action of K is determined on T and we compute for any $z \in Z$, $z \neq 1$,

$$C_G(z^{-1}z^\tau) \cap TK\langle\tau\rangle = T\langle t\rangle, \quad C_G(zz^\tau) \cap TK\langle\tau\rangle = T\langle t, \tau\rangle.$$

(4.2) *For any* $z \in Z$, $C_G(z^{-1}z^\tau, t) = T\langle t\rangle$.

Proof. The above remarks imply that $T\langle t\rangle$ is a Sylow 3-normalizer of $C_G(z^{-1}z^\tau, t)$ so that $C_G(z^{-1}z^\tau, t)$ has a normal 3-complement B such that $C_G(z^{-1}z^\tau, t) = TB$, $T \cap B = 1$. By hypothesis, Z is a noncyclic elementary abelian 3-group so that $B = \prod_{z \in Z^\#} C_B(z)$. However, $C_B(z) \subseteq Z \times L$ and it is seen that $C_B(z) = \langle t\rangle$ for all $z \in Z^\#$. Hence $B = \langle t\rangle$ and we have

$$C_G(z^{-1}z^\tau, t) = T\langle t\rangle.$$

(4.3) *For* $z \in Z$, $C_G(zz^\tau, t) = T\langle t, \tau\rangle$.

Proof. We have seen that $T\langle t, \tau\rangle$ is a Sylow 3-normalizer of $C_G(zz^\tau, t)$ and it is calculated that $\langle zz^\tau \mid z \in Z\rangle$ is in the center of $T\langle t, \tau\rangle$. Applying a theorem of Grün, $C_G(zz^\tau, t)$ has a normal subgroup R of index q such that

$$R \cap T = \langle z^{-1}z^\tau \mid z \in Z\rangle.$$

Let $X = R/\langle t\rangle$ and consider the image $(R \cap T)^-$ of $R \cap T$ in X. Since $C_R(y) = (R \cap T)\langle t\rangle$ for all $y \in R \cap T$, $(R \cap T)^-$ is a CC-subgroup of X. Furthermore, $C_G(zz^\tau, t, R \cap T) = C_G(z, zz^\tau, t) = T\langle t\rangle$ so that

$$N_G(R \cap T) \cap C_G(zz^\tau, t) = T\langle t, \tau\rangle.$$

We conclude that $N_R(R \cap T) = (R \cap T)\langle t, \tau\rangle$ and consequently,

$$N_X(R \cap T)^- = ((R \cap T)\langle\tau\rangle)^-.$$

Thus $(R \cap T)^-$ has index 2 in $N_X(R \cap T)^-$ and Theorem 4.4 of [8] now applies. We conclude that $(R \cap T)^-$ is a normal subgroup of X so that

$$R = (R \cap T)\langle t, \tau\rangle \quad \text{and} \quad C_G(zz^\tau, t) = T\langle t, \tau\rangle.$$

Section 1 of [11] shows that the structure of the centralizer of a central involution in $PSp_4(q)$ to have (as a normal subgroup of index 2) a subgroup which is the central product of two copies of $SL(2, q)$. In fact, if C is the centralizer of a central involution t_0 of $PSp_4(q)$, $C = L_1 L_2\langle\tau_0\rangle$ where $[L_1, L_2] = 1$, $L_i \cong SL(2, q)$, $L_1 \cap L_2 = \langle t_0\rangle$ and τ_0 is an involution which interchanges L_1 and L_2. In the following proposition we will show that $C_G(t)$ has a subgroup isomorphic to $L_1 L_2 \langle \tau_0\rangle$. The remaining part of the section will be devoted to showing that this subgroup coincides with $C_G(t)$.

(4.4) $C_G(t)$ *contains a subgroup* L *and an involution* τ *for which*

$$L \cong SL(2, q), \qquad L \cap L^\tau = \langle t\rangle, \qquad [L, L^\tau] = 1.$$

Proof. We have seen that $C_G(z, t) = Z \times L$ with $L \cong SL(2, q)$ and that τ is an involution of $C_G(t)$ with Z^τ a Sylow 3-subgroup of L. Let c be an element of order 4 in L which inverts a generator of K_1. Then L is the union of the two double cosets $Z^\tau K_1$ and $Z^\tau cK_1$.

Since K_1^τ leaves $Z \times L$ invariant, K_1^τ induces a group of automorphisms of L which centralizes the Sylow 3-normalizer $Z^\tau K$. Consequently, K_1^τ induces a trivial group of automorphisms and $[K_1^\tau, L] = 1$. This implies $[K_1, L^\tau] = 1$. Furthermore, $[Z, L] = 1$ so that $[Z^\tau, L^\tau] = 1$ and we conclude that $Z^\tau K_1$ centralizes L^τ. Because L^τ is the union of the cosets ZK_1^τ and $Zc^\tau K_1^\tau$, it now follows that $[L, L^\tau] = [Z^\tau cK_1, Zc^\tau K_1^\tau]$. We now apply elementary commutator relations (see [5, p. 18]) to conclude $[L, L^\tau] = [c, c^\tau]$. In addition, c inverts K_1 with $c^2 = t$ so that K centralizes $[c, c^\tau]$.

Let $A = [c, c^\tau]$ and notice that A is a normal subgroup of $\langle L, L^\tau\rangle$. Thus, if $|A|$ is divisible by 3, $T \cap A \neq 1$. But K centralizes no element of T other than the identity. Hence A is a $3'$-group. Now A is left invariant by Z and we conclude that $A \subseteq L$. This implies $A \subseteq \langle t\rangle$ and $c^{-\tau}cc^\tau = c$ or ct. In either case c^τ induces an automorphism of L which centralizes $Z^\tau K_1$. Therefore c^τ is the trivial automorphism and $A = [L, L^\tau] = 1$. Finally, $L \cap L^\tau$ is a subgroup of the center of L so that $L \cap L^\tau = \langle t\rangle$.

As a result of (4.4), $[L, L^\tau] = 1$ and $H = \langle ll^\tau \mid l \in H\rangle$ is a subgroup of LL^τ isomorphic to $PSL(2, q)$. Retaining this notation, we are now able to compute $C_G(t, \tau)$.

(4.5) $C_G(t, \tau) = \langle t, \tau\rangle \times H$, *where* $H = \langle ll^\tau \mid l \in L\rangle$.

Proof. Let $D = \langle zz^\tau \mid z \in Z\rangle$. From (4.3), $C_G(D, t) = T\langle t, \tau\rangle$ and

$$N_G(D) \cap C_G(t, \tau) \subseteq N_G(T) = TK\langle\tau\rangle.$$

Hence $D\langle k_1^\tau k_1, t, \tau\rangle$ is a Sylow 3-normalizer of $C_G(t, \tau)$. Let

$$X = C_G(t, \tau)/\langle t, \tau\rangle$$

and consider the image $\bar{D}$ of D in X. For $y \in D$, $x \in C_G(t, \tau)$ such that $(xy)^- = (yx)^-$ we see that x normalizes $\langle y\rangle$. In addition, x leaves

$$C_G(t, \tau, y) = D\langle t, \tau\rangle$$

invariant and must leave D fixed. However, no element of $(k_1 k_1^\tau)^-$ centralizes $\bar{y}$ so that $\bar{x} \in \bar{D}$. This proves that $\bar{D}$ is a CC-subgroup of X of order q with $|N_X(\bar{D})| = q(q-1)/2$. Theorem 5.1 of [8] applies and we conclude that $X \cong PSL(2, q)$.

Now $H = \langle ll^\tau \mid l \in H\rangle$ is a subgroup of $C_G(t, \tau)$ isomorphic to $PSL(2, q)$ with $H \cap \langle t, \tau\rangle = 1$. A comparison of orders implies $C_G(t, \tau) = \langle t, \tau\rangle \times H$ as desired.

A Sylow 2-group of L is a quaternion group Q of order 8. Let Q be generated by elements a and b of order 4 which satisfy $a^2 = b^2 = t$, $ab = ba^{-1}$. Then every involution of LL^τ different from t has the form xy where x and y are elements of order 4 in L and L^τ respectively and, as all elements of order 4 in $SL(2, q)$ are conjugate, every involution of $LL^\tau - \langle t\rangle$ is conjugate to $v = aa^\tau$.

On the other hand, for $x \in L$, $y \in L^\tau$ and $(xy\tau)^2 = 1$, $(xy^\tau)(yx^\tau) = 1$ so that $xy^\tau \in L \cap L^\tau = \langle t \rangle$. Hence $xy\tau = (x\tau)y^\tau = x\tau x^{-1}$ or $xt\tau x^{-1}$. We conclude that every involution of $LL^\tau\langle\tau\rangle - LL^\tau$ is conjugate to τ or $t\tau$.

(4.6) *$QQ^\tau\langle\tau\rangle$ is a Sylow 2-group of G.*

Proof. Let S be a Sylow 2-group of G which contains the 2-group $QQ\langle\tau\rangle$ and consider $y \in N_S(QQ^\tau\langle\tau\rangle)$. The center of $QQ^\tau\langle\tau\rangle$ is generated by t so that $y^{-1}\tau y$ is an involution of $QQ^\tau\langle\tau\rangle$ different from t. Furthermore, if $y^{-1}\tau y \in QQ^\tau$, the remarks preceding (4.6) imply that τ and v are conjugate. However, $\langle a, bb^\tau, \tau\rangle$ centralizes v while $\langle aa^\tau, bb^\tau\rangle \times \langle t, \tau\rangle$ is a Sylow 2-group of $C_G(t, \tau)$. A comparison shows that τ and v are not conjugate in $C_G(t)$. Hence $y^{-1}\tau y$ is an involution of $QQ^\tau\langle\tau\rangle - QQ^\tau$. For $x \in LL^\tau$, $(yx)^{-1}\tau(yx) \in \langle\tau, t\rangle$ and we conclude that yx leaves $\langle t, \tau\rangle$ and $C_G(t, \tau)$ invariant. Now

$$C_G(t, \tau) = \langle t, \tau\rangle \times H$$

where $H \cong PSL(2, q)$ with $H = \langle ll^\tau \mid l \in L\rangle$. Therefore, if β is an element of order 3 in $\langle zz^\tau \mid z \in Z\rangle$, $\beta^{yx} \in H$ and for some $h \in H$, $\beta^{yxh} \in \langle\beta\rangle$. By (4.3), $C_G(\beta, t) = T\langle t, \tau\rangle$ and we conclude that yxh leaves T fixed. Hence

$$yxh \in TK\langle\tau\rangle \quad \text{and} \quad y \in LL^\tau\langle\tau\rangle.$$

This implies $N_S(QQ^\tau\langle\tau\rangle) = QQ^\tau\langle\tau\rangle$ and $S = QQ^\tau\langle\tau\rangle$.

We will retain the notation introduced in (4.6) for the Sylow 2-group $S = QQ^\tau\langle\tau\rangle$. In particular, the involutions $v = aa^\tau$ and $w = bb^\tau$ are of importance in the following discussion.

(4.7) *The involutions v and t are not conjugate in G.*

Proof. Let $E = \langle t, \tau, v, w\rangle$ and notice $C_G(E) \subseteq C_G(t, \tau) = \langle t, \tau\rangle \times H$ so that $C_G(E) = E$. The proof of (4.7) is now identical to (2.2) of [11].

(4.8) *The involution t is conjugate to τ or $t\tau$.*

Proof. If t is conjugate to no involution of $S - QQ^\tau$, $G = C_G(t)O(G)$ by [3]. In this case $P \cap O(G)$ is a nontrivial normal subgroup of P and we conclude $Z \cap O(G) \neq 1$. Consequently, $O(C_G(t))$ contains a nontrivial element $z \in Z$ and we conclude that $zz^\tau \in O(C_G(t))$. This implies that $\langle ll^\tau \mid l \in L\rangle$ has a normal subgroup of odd order which is impossible. Hence t is conjugate to an involution of $S - \langle t\rangle$. The remarks preceding (4.6) and the fact that v and t are not conjugate imply that t is conjugate to τ or $t\tau$.

For the remainder of this section let us assume that t is conjugate in G to the involution $t\tau$ rather than τ. This assumption can be made without loss of generality because the arguments which follow are symmetric in τ and $t\tau$. Particularly important is the fact that τ and $t\tau$ are in different conjugacy classes of G. Indeed, we have the following proposition.

(4.9) *G has exactly two classes of involutions K_1 and K_2 such that $K_1 \cap C_G(t)$*

consists of classes in $C_G(t)$ *represented by* t *and* $t\tau$ *and* $K_2 \cap C_G(t)$ *consists of classes represented by* τ *and* v.

Proof. This is identical to (2.4) of [11] with the notational change $\tau = u$. The structure of $SL(2, q)$ together with the fact that $q + 1$ is divisible by 4 implies $C_L(a) = \langle d \rangle$ where d is an element of L of order $q + 1$. Using this notation we compute $C_G(a)$.

(4.10) $C_G(a) = \langle d \rangle L^\tau$, $\langle d \rangle \cap L^\tau = \langle t \rangle$, $[\langle d \rangle, L^\tau] = 1$.

Proof. The 2-group $F = \langle a, a^\tau, b^\tau \rangle$ is a subgroup of S of order 16 which centralizes a. Let W be a Sylow 2-group of $C_G(a)$ containing F and assume $W \neq F$. Since the center of S is generated by t, $[W: F] = 2$ and $W \subseteq N_G(F)$. Now $\langle a, a^\tau, b, b^\tau \rangle$ is a 2-group of $N_G(F)$ and S contains no normal cyclic subgroup of order 4. This implies that $\langle a, a^\tau, b, b^\tau \rangle$ is a Sylow 2-group of $N_G(F)$. Comparing orders, W and $\langle a, a^\tau, b, b^\tau \rangle$ are isomorphic. This is impossible as the centers of these groups have orders 4 and 2 respectively. We conclude that F is a Sylow 2-group of $C_G(a)$.

Let A be the largest normal subgroup of $C_G(a)$ of order relatively prime to 3. Clearly A contains a. In addition, $C_G(a)$ is a subgroup of $C_G(t)$ so has an abelian Sylow 3-subgroup containing Z. This fact, together with $C_G(a, Z) = \langle d \rangle \times Z$, implies Z is a Sylow 3-group of $C_G(a)$. Furthermore. A is Z-invariant so that $A \subseteq \langle d \rangle$.

Let $X = C_G(a)/A$ and notice that $\bar{F} = \langle a, a^\tau, b^\tau \rangle^- = \langle a^\tau, b^\tau \rangle^-$ is a Sylow 2-group of X. Because $L^\tau \cap A = \langle t \rangle$, $\bar{F}$ is a four-group and

$$(L^\tau)^- \cong PSL(2, q).$$

Let D be a subgroup of $C_G(a)$ for which $O(X) = D/A$. Then $Z \cap D = 1$ as otherwise $O(X) \cap (L^\tau)^- \neq 1$. We conclude that D is a $3'$-group and $D \subseteq A$. Therefore, $O(X) = 1$ and, using [6], X is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$. It follows that $L^\tau A$ is a normal subgroup of $C_G(a)$. Finally, $N_G(Z) = C_G(Z)K_1^\tau$ with $C_G(a) \cap N_G(Z) = Z\langle d \rangle K_1^\tau$. Applying the Frattini argument, $C_G(a) = \langle d \rangle L^\tau$ as desired.

The involution $v = aa^\tau$ is centralized by d, τ and $w = bb^\tau$. In particular, $C_G(v, t) \cap LL^\tau \langle \tau \rangle = \langle d, \tau, w \rangle$. Let $R = \langle a, a^\tau, t\tau, w \rangle$. It is computed that $\langle d, \tau, w \rangle = R\langle d^4, (d^\tau)^4 \rangle$ where $\langle d^4, (d^\tau)^4 \rangle$ is a normal 2-complement for $\langle d, \tau, w \rangle$. Keeping this same notation we are able to determine $C_G(v, t)$.

(4.11) $C_G(t, v) = \langle d, \tau, w \rangle$.

Proof. Let $R = \langle a, a^\tau, t\tau, w \rangle$. As $|R| = 32$ and a Sylow 2-group of G has order 64 with center of order 2, R is a Sylow 2-group of $C_G(t, v)$. We first determine $N = N_G(R) \cap C_G(t, v)$. For $y \in N$, $y^{-1}(t\tau)y \in R - \langle a, a^\tau, w \rangle$. Furthermore, τ and $t\tau$ are not conjugate so there exists $x \in LL^\tau \langle \tau \rangle$ for which $(yx)^{-1}t\tau(yx) = t\tau$. Hence $yx \in C_G(t\tau, t) \subseteq LL^\tau \langle \tau \rangle$ and $y \in LL^\tau \langle \tau \rangle$. Consequently, $N \subseteq \langle d, \tau, w \rangle$ and we have that N has a normal 2-complement B with $N = R \times B$. This implies $N' \cap R = R'$. Using a theorem of Grün, the focal subgroup R^* is the subgroup of R generated by all elements of R which are conjugate in $C_G(t, v)$ to elements of R'. However, $R' = \langle t, v\rangle$ and we conclude $R^* = R'$. This implies $C_G(t, v)$ has a normal subgroup X of index 8 with $X \cap R = \langle t, v\rangle$ and, by a theorem of Burnside, X has a normal 2-complement E. Clearly E is a normal 2-complement for $C_G(t, v)$ and we have $C_G(t, v) = RE, R \cap E = 1$.

To complete (4.11) it remains to show $E \subseteq \langle \tau, d, w\rangle$. Indeed, the four-group $\langle \tau, w\rangle$ leaves E invariant so that $E = C_E(\tau)C_E(\tau w)C_E(w)$. Now

$$C_G(t, \tau, v) = \langle t, \tau\rangle \times \langle dd^{\tau}, w\rangle$$

so that $C_E(\tau) = \langle (dd^{\tau})^4\rangle$. On the other hand, $b^{-1}(\tau w)b = t\tau$ so b interchanges $C_G(t, v, \tau w)$ and $C_G(t, v, \tau)$. It follows that

$$C_G(t, v, \tau w) = \langle t, \tau w\rangle \times \langle d^{-1}d^{\tau}, w\rangle$$

and we compute $C_E(\tau w) = \langle (d^{-1}d^{\tau})^4\rangle$.

We shall now show $C_E(w) = 1$. Because a interchanges w and wt, a leaves $C_E(w)$ fixed. Furthermore, $C_G(a) = \langle d\rangle L^{\tau}$ so $C_G(t, v, a) = \langle d, d^{\tau}\rangle$ and we compute $C_G(t, v, a, w) = \langle t, v\rangle$. This implies that a induces a fixed-point-free auotmorphism of $C_E(w)$ which inverts the nontrivial elements. Similarly, b interchanges v and vt so leaves $\langle t, v\rangle$ and $C_G(t, v)$ invariant. In particular, b leaves E invariant. But $[w, b] = 1$ so b induces an automorphism of $C_E(w)$. Now a and b are conjugate by $x \in T \cap L$ so that $C_G(b) = \langle d^x\rangle L^{\tau}$. It follows that $C_G(t, v, b) = \langle d^{\tau}\rangle$ and $C_G(t, v, b, w) = \langle t\rangle$. We conclude that b induces a fixed-point-free automorphism of $C_E(w)$ which inverts the nontrivial elements. Consequently, ab centralizes $C_E(w)$. However, $C_G(t, ab, v) = \langle d^{\tau}\rangle$ and $C_G(t, ab, v, w) = \langle t\rangle$. Thus $C_E(w) = 1, E = C_E(\tau)C_E(\tau w) = \langle d^4, (d^{\tau})^4\rangle$ and (4.11) now follows.

(4.12) $C_G(t) = LL^{\tau}\langle\tau\rangle, L \cap L^{\tau} = \langle t\rangle, [L, L^{\tau}] = 1, L \cong SL(2, q)$.

Proof. We have seen (4.4) that $C_G(t)$ contains a subgroup $LL^{\tau}\langle\tau\rangle$ with the properties (4.12). We must show that $LL^{\tau}\langle\tau\rangle$ coincides with $C_G(t)$. To accomplish this we show that $LL^{\tau}\langle\tau\rangle$ contains all involutions of $C_G(t)$ and then apply a Frattini argument.

Let u be an involution of $C_G(t)$ different from t and consider the image $\bar{u}$ of u in the factor group $C_G(t)/\langle t\rangle$. Because u is conjugate in $C_G(t)$ to τ, $t\tau$ or v, $\bar{u}$ is conjugate to $\bar{\tau}$ or $\bar{v}$ in $C_G(t)/\langle t\rangle$. In fact, $C_G(t, v)$ and $C_G(t, \tau)$ are not isomorphic so $\bar{\tau}$ and $\bar{v}$ belong to different conjugacy classes of $C_G(t)/\langle t\rangle$.

Let us assume $\bar{u}$ and $\bar{v}$ are conjugate in $C_G(t)/\langle t\rangle$. Then $\langle u, \tau\rangle^-$ is a dihedral group with a nontrivial central involution $\bar{x}$, $x \in C_G(t)$. Thus

$$x^{-1}\tau x \in \langle t, \tau\rangle$$

and since τ and $t\tau$ belong to different conjugacy classes of G, $x^{-1}\tau x = \tau$. We conclude that $x \in C_G(t, \tau)$ and is conjugate to an involution of the 2-group $\langle t, \tau, v, w\rangle$. From $(ux)^- = (xu)^-$, $xu = ux$ or uxt. In the first case $u \in C_G(x, t) \subseteq LL^\tau\langle\tau\rangle$ by (4.5) and (4.11). We may therefore assume $uxu = xt$. Should x be conjugate to τ or τt in $C_G(t)$, $uxu = xt$ implies τ and $t\tau$ are conjugate. We conclude that x is conjugate in $C_G(t, \tau)$ to an involution of $\langle t, v, w\rangle$. In particular, x and xt are conjugate in $LL^\tau\langle\tau\rangle$. Hence for some $y \in LL^\tau\langle\tau\rangle$, $(uy)^{-1}x(uy) = x$ so that $uy \in C_G(x, t) \subseteq LL^\tau\langle\tau\rangle$. This implies $u \in LL^\tau\langle\tau\rangle$ in this case as well.

Now let us assume $\bar{u}$ and $\bar{\tau}$ are conjugate so that $\langle u, v\rangle^-$ contains an involution $\bar{x}$ in its center. Then x leaves the four-group $\langle t, v\rangle$ invariant and (4.11) implies $x \in \langle d, \tau, w, b\rangle$. If x is an element of order 4 with $x^2 \neq t$, u centralizes x^2 and $u \in C_G(x^2, t) \subseteq LL^\tau\langle\tau\rangle$. If $x^2 = t$, x is conjugate in $LL^\tau\langle\tau\rangle$ to a. But $C_G(a) = \langle d\rangle L^\tau$ so $N_G\langle a\rangle = \langle d, b\rangle L^\tau$ and because u centralizes or inverts x, $u \in N_G\langle x\rangle \subseteq LL^\tau\langle\tau\rangle$. Consequently we may assume x is an involution of $\langle d, \tau, w, b\rangle$ different from t. In addition we may assume $uxu = xt$ as otherwise $u \in LL^\tau\langle\tau\rangle$ as desired. The argument of the preceding paragraph now applies and we have $u \in LL^\tau\langle\tau\rangle$ in all cases.

We have shown $LL^\tau\langle\tau\rangle$ contains all involutions of $C_G(t)$ and since $LL^\tau\langle\tau\rangle$ is generated by involutions, $LL^\tau\langle\tau\rangle$ is a normal subgroup of $C_G(t)$. Finally, T is a Sylow 3-subgroup of $LL^\tau\langle\tau\rangle$ and a Frattini argument can be applied to conclude

$$C_G(t) = LL^\tau\langle\tau\rangle(C_G(t) \cap N_G(T)).$$

But $C_G(t) \cap N_G(T) = TK\langle\tau\rangle$ which yields $C_G(t) = LL^\tau\langle\tau\rangle$.

A consequence of (4.12) is the fact that G is a finite group satisfying the hypothesis of the main theorem of [11]. We conclude $G = C_G(t)O(G)$ or $G \cong PSp_4(q)$. In the first case $P \cap O(G)$ is a nontrivial normal subgroup of P and thus $Z \cap O(G) \neq 1$. However $C_G(t)$ contains no normal subgroup of odd order. Hence $G \cong PSp_4(q)$ and the proof of Theorem 3 is completed.

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