THE ALGEBRAIC STRUCTURE OF CERTAIN Ω-SPECTRA

BY

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Introduction

An Ω-spectrum $SP(\mu)$ for cohomology with coefficients in a module bundle μ is shown to be of the form $SP(\mu) = T \otimes \mu$, where T is a homotopy-commutative tensor algebra of group bundles. Furthermore, T is shown to be the image of the sphere spectrum under a suitable imbedding of the category of finite cell-complexes in an appropriate category of group bundles. Finally, an interpretation of the cohomology modules of a fibre space in terms of this imbedding is given.

1. Compactly generated bundles

Let CG be the category of compactly generated spaces in the sense of [3, §2]. For $B \in CG$ let $C_B = CG \downarrow B$ be the category of spaces over B (see [2, p. 46]) and C_B^* be the category of "sectioned" spaces over B (an object of C_B^* consists of a $\xi \in C_B$ together with a continuous section S_ξ of ξ; a morphism of C_B^* is a section preserving morphism of C_B). For $\xi, \zeta \in C_B^*$ denote by $\xi\zeta$ and $\xi \wedge \zeta$ the fibre product and fibre smash product respectively. (The fibre of $\xi \wedge \zeta$ over $b \in B$ is the smash product of the fibres of ξ and ζ over b with respect to the base points $S_\xi(b)$ and $S_\zeta(b)$ respectively. Give $\xi \wedge \zeta$ the quotient topology defined by the canonical quotient map $q : \xi\zeta \to \xi \wedge \zeta$. Since q is a relative homeomorphism, $\xi \wedge \zeta \in C_B^*$ by 2.5 [3].)

A *short exact sequence* in C_B^* is a sequence of the form

$$\beta \xrightarrow{i_1} \xi_1 \xrightarrow{i_2} \xi_2 \xrightarrow{i_3} \xi_3 \xrightarrow{i_4} \beta$$

where $\beta = \mathrm{id}_B$, i_1 and i_4 are induced by S_{ξ_1} and the projection of ξ_3 respectively, i_2 is a closed injection, i_3 is a proclusion (see [3, p. 276]) and

$$\text{image } i_n = i_{n+1}^{-1} (\text{image } S_{\xi_{n+1}}) \quad (n = 1, 2).$$

Exactness in G_B, the category of compactly generated abelian group bundles over B ($\gamma \in G_B$ means $\gamma \in C_B^*$, the fibres have an abelian group structure for which addition and inversion are globally continuous, and S_γ is the 0-section) is similarly defined. A homotopy h_t in C_B^* (respectively G_B) is required to be a map in C_B^* (respectively G_B) for each $t \in I$.

Let C be the category of finite cell-complexes with base point (typically denoted by *) and base point preserving maps. For $X \in C$ the assignment $X \to (X_B, S_{X_B})$, where X_B denotes the product space over B with fibre X and $S_{X_B}(b) = (b, {}^*)$, defines a covariant functor $C \to C_B^*$ for which $(X \wedge Y)_B$

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$= X_B \wedge Y_B$ where the first $\wedge$ denotes smash product in C. In view of Proposition 2.2 [3], $(\quad)_B$ is an exact functor.

2. Free group bundles

For $\xi \in C_B^*$ define $F(\xi)$, the free abelian group bundle generated by ξ, as follows: the fibre of $F(\xi)$ over $b \in B$ is the free abelian group (no topology yet) generated by the fibre ξ_b of ξ over b, where $S_\xi(b)$ (identified with $0x$ for $0 \in Z =$ the integers, $x \in \xi_b$) is understood to be the zero of the group. For $m \geq 0$ define ϕ_m: $(Z_B\, \xi)^m \to F(\xi)$ by

$$\phi_m((n_1, x_1), \cdots, (n_m, x_m)) = \sum_i n_i\, x_i$$

($(Z_B\, \xi)^0$ is the image of S_ξ). Clearly the ϕ_m's extend to a map ϕ on the topological sum $\bigcup_{m=0}^{\infty} (Z_B\, \xi)^m \to F(\xi)$. $F(\xi)$ is then given the quotient topology of ϕ. That addition, inversion, the projection and the 0-section of $F(\xi)$ are continuous is seen by factoring through the proclusion ϕ and applying the fibred version of 2.2 [3]. One shows $F(\xi)$ is compactly generated by showing, as in §6 [3], that $F(\xi)$ has the topology of the union of the compactly generated spaces Image ϕ_m, $m \geq 0$. Further, for $f : \zeta \to \xi$ in C_B^*, $F(f)$ defined by $\sum n_i\, x_i \to \sum n_i f(x_i)$ is clearly a morphism $F(\zeta) \to F(\xi)$ in G_B.

2.1. LEMMA. *$F : C_B^* \to G_B$ is a covariant, homotopy preserving, exact functor.*

Proof. That F is covariant and homotopy preserving is trivial. Exactness for the most part is direct. That F preserves the properties "closed injection" and "proclusion" is proved as in 6.7 [3].

3. Tensor products

Define the tensor product $\nu \otimes \gamma$ of $\nu, \gamma \in G_B$ as follows: the fibre of $\nu \otimes \gamma$ over b is the tensor product (no topology yet) of the corresponding fibres of ν and γ. Give $\nu \otimes \gamma$ the quotient topology of $\theta : F(\nu\gamma) \to \nu \otimes \gamma$, the canonical epimorphism defined by the standard construction of tensor product. That addition, inversion, the projection and the 0-section are continuous is proved as in the $F(\xi)$ case above. In general $\nu \otimes \gamma$ may not be in G_B but the following lemma shows it is in G_B when ν is free.

3.1. LEMMA. *If $\xi, \zeta \in C_B^*$ and $\gamma \in G_B$ then* (a) $F(\xi) \otimes \gamma \in G_B$ (b) $F(\xi \wedge \zeta)$ *and $F(\xi) \otimes F(\zeta)$ are naturally isomorphic in G_B.*

Proof. (a) Let $F(\xi\,; \gamma)$ be the bundle obtained by replacing Z_B by γ in the construction of $F(\xi)$, §2. That $F(\xi\,;\gamma)$ is in G_B follows as in the $F(\xi)$ case. Part (a) will follow if it is shown that $F(\xi\,;\gamma)$ and $F(\xi) \otimes \gamma$ are isomorphic. To this end consider the diagram where

$$\alpha(\sum g_i\, x_i) = \sum x_i \otimes g_i, \qquad \beta((\sum (n_i\, x_i)) \otimes g) = \sum (n_i\, g) x_i,$$

$\bar{\alpha} = \bigcup_m \alpha^m$ where

$\alpha^m(\cdots, (g_i, x_i), \cdots) = (\cdots, (1, 1, x_i, g_i), \cdots) : (\gamma\xi)^m \to (Z_B(Z_B\,\xi)^1\gamma)^m,$

$\bar{\beta} = \bigcup_{m,p} \beta^{p,m}$ where, for $i = 1, \cdots, m$ and $j = 1, \cdots, p$,

$$\beta^{p,m}(\cdots ; (a_i, \cdots, (n_{ij}, x_{ij}), \cdots, g_i); \cdots) = (\cdots, ((a_i n_{ij})g_i, x_{ij}), \cdots) : (Z_B(Z_B\,\xi)^p\gamma)^m \to (\xi\gamma)^{pm},$$

$\bar{\phi} = \bigcup_m ((\mathrm{id})\phi_1(\mathrm{id}))^m$ where ϕ_1 (and ϕ_2, ϕ_3) is the map ϕ of §2 associated to ξ (resp. $\gamma\xi$, $F(\xi)\gamma$) and θ is the canonical map associated to the tensor product. Clearly $\bar{\alpha}$, $\bar{\beta}$ are continuous and form a commutative diagram with α, β respectively. Since the horizontal maps are quotient maps, α and β are seen to be continuous mutual inverses.

$$\begin{array}{ccc}
\bigcup_{m=0}^{\infty} (\gamma\xi)^m & \xrightarrow{\quad\phi_2\quad} & F(\xi;\gamma) \\
\bar{\beta}\uparrow\downarrow\bar{\alpha} & & \beta\uparrow\downarrow\alpha \\
\bigcup_{m,p=0}^{\infty} (Z_B(Z_B\,\xi)^p\gamma)^m = \bigcup_{m=0}^{\infty} (Z_B\,(\bigcup_{p=0}^{\infty} (Z_B\,\xi)^p)\gamma)^m \xrightarrow{\bar{\phi}} & & \\
\bigcup_{m=0}^{\infty} (Z_B(F(\xi)\gamma))^m \xrightarrow{\phi_3} F(F(\xi)\gamma) & \xrightarrow{\quad\theta\quad} & F(\xi)\otimes\gamma
\end{array}$$

(b) Letting $\gamma = F(\zeta)$ in the above argument shows $F(\xi; F(\zeta))$ and $F(\xi) \otimes F(\zeta)$ are naturally G_B-isomorphic. Similarly one shows $F(\xi; F(\zeta))$ and $F(\xi \wedge \zeta)$ are naturally G_B-isomorphic (see 6.13 [3]).

4. Principal bundles

An exact sequence

$$0 \to \gamma \xrightarrow{\ i\ } \nu \xrightarrow{\ j\ } \gamma_1 \to 0$$

in G_B is *universal* if ν is G_B-shrinkable (there is a homotopy $h_t : \nu \to \nu$ in G_B with $h_0 = \mathrm{id}$, $h_1 = 0$) and is *locally split* (*numerably split*) if there is an open (numerable) cover $\{U_\alpha\}$ of γ_1 and continuous maps $s_\alpha : U_\alpha \to \nu$ such that $js_\alpha = \mathrm{id}_{U_\alpha}$. This defines $j : \nu \to \gamma_1$ as a universal (local or numerable) principal γ-bundle in the sense of [4], [5] and [6]. Recall $\gamma \in G_B$ is an (L)NDR if (locally) the 0-section is a deformation retract of an open neighborhood (see [4]). Write F_B for the composition $F \circ (\ \ )_B : C \to C_B^*$. If the unit interval I and its boundary $S^0 = \{0, 1\}$ have base point 0, then $0 \to S^0 \to I \to I/S^0 = S^1 \to *$ is exact in C.

4.1. LEMMA. *If ν is an $(L)NDR$ then* (a) $F_B(S^1) \otimes \nu$ *is an* $(L)NDR$ *and* (b) *the sequence*

$$0 \to F_B(S^0) \otimes \nu \to F_B(I) \otimes \nu \to F_B(S^1) \otimes \nu \to 0$$

is a universal, numerably (locally) split exact sequence.

Proof. (a) The proof of 3.1 (a) shows $F_B(S^1) \otimes \nu$ and $F(S_B^1; \nu)$ are isomorphic. This latter bundle can be described by "fibring" the construction of §9 [3] or §5 [7]. This is done in [4] and part (a) follows from Theorem 1 [4].

(b) The sequence is exact since $(\ )_B$, F, and $\otimes\ \nu$ are exact (2.1 and the fact that $F_B(S^1)$ is free). Universality follows by noting that if h_t is a G_B-shrinking of $F_B(I)$ (h_t exists by 2.1) then $h_t \otimes \text{id}$ is a G_B-shrinking of $F_B(I) \otimes \nu$. Finally, note that the sequence in question is essentially sequence 3.1 of [5]. The results of [4] (in particular 3.3 [5]) then imply the remainder of (b).

5. Ω-spectra and tensor algebras

An *acyclic resolution* of $\gamma \in G_B$ is an exact sequence in G_B,

$$0 \to \gamma \xrightarrow{i_0} \nu_0 \xrightarrow{i_1} \nu_1 \to \cdots,$$

for which $0 \to \text{Image } i_n \to \nu_n \to \text{Image } i_{n+1} \to 0$ is a universal, locally split exact sequence for $n \geq 0$. The *spectrum* of the resolution is the family $\{\text{Image } i_n, n \geq 1\}$.

The result of applying $-\wedge S^1$ to the exact sequence

$$0 \to S^0 \xrightarrow{i_0} I \xrightarrow{j_1} I/S^0 = S^1 \to *$$

is the exact sequence

$$* \to S^1 \xrightarrow{k_1} I\wedge S^1 \xrightarrow{j_2} S^1\wedge S^1 \to *$$

(note $S^0 \wedge X = X \wedge S^0 = X$). Iterating this operation generates a family of exact sequences. These sequences can be joined to produce the *canonical resolution* of S^0:

$$0 \to S^0 \xrightarrow{i_0} I \xrightarrow{i_1} I\wedge S^1 \xrightarrow{i_2} I\wedge S^1\wedge S^1 \to \cdots$$

where $i_n = k_n j_n$, $n \geq 1$.
The family $\{\text{Image } i_n, n \geq 1\}$ associated to this resolution defines the sphere spectrum SP (recall $S^n = S^{n-1} \wedge S^1$) [1, pp. 10–11].

5.1. PROPOSITION. (a) *The image of the canonical resolution of S^0 under the functor F_B is an acyclic resolution of Z_B with spectrum $F_B(SP)$.*

(b) *For γ an LNDR, the sequence obtained by applying $-\otimes\gamma$ to the resolution in* (a) *is an acyclic resolution of γ with spectrum $SP(\gamma) = F_B(SP) \otimes \gamma$.*

Proof. Since Z_B is LNDR, part (a) is part (b) when $\gamma = Z_B$. To show (b) note that $0 \to F_B(S^0) \otimes \gamma \to F_B(I) \otimes \gamma \to F_B(S^1) \otimes \gamma \to 0$ is a universal, locally split exact sequence by 4.1 (b) with $\nu = \gamma$. Let S be the sequence of 4.1 (b) with $\nu = F_B(S^1) \otimes \gamma$. S is a universal, locally split exact sequence since ν is LNDR by 4.1 (a). Further, S is isomorphic to

$$0 \to F_B(S^1) \otimes \gamma \to F_B(I \wedge S^1) \otimes \gamma \to F_B(S^1 \wedge S^1) \otimes \gamma \to 0$$

in view of 3.1 (b). Iterating this argument produces the result.

For $\xi \in C_B$, $\gamma \in G_B$ let $H^n(\xi; \gamma)$ be the nth cohomology group of ξ with coefficients in γ (see §4 [5]). An Ω-spectrum for $H^*(-;\gamma)$ on a subcategory C' of C_B is a family $\{\gamma_n\}$, $n \geq 1$ of $\gamma_n \in G_B$ such that (1) $H^n(-;\gamma)$ and $[-, \gamma_n]$ ([ ] denotes fibre homotopy classes) are naturally equivalent functors on C', $n \geq 1$ and (2) there is a fibre homotopy equivalence $g_n : \gamma_n \to \Omega\gamma_{n+1}$ ($\Omega\gamma_{n+1}$ is the vertical loop space of γ_{n+1}, see §6 [6]).

For a free bundle $\nu \in G_B$, let $T(\nu)$ be the (positively graded) tensor algebra of ν; i.e., $T_1(\nu) = \nu$, $T_{n+1}(\nu) = T_n(\nu) \otimes \nu$. For $\gamma \in G_B$ denote $T(\nu) \otimes \gamma$ by $T(\nu;\gamma)$. Let P_B be the full subcategory of C_B consisting of those ξ with paracompact total space.

5.2. Theorem. *If $\gamma \in G_B$ is an LNDR then $T(F_B(S^1);\gamma)$ is an Ω-spectrum for $H^*(-;\gamma)$ on P_B.*

Proof. By §3 [5] and 6.1 [6] the spectrum of an acyclic resolution of γ is an Ω-spectrum for $H^*(-;\gamma)$ on P_B. The result now follows from 5.1 (b) in view of 3.1 (b) and the definition of SP.

If G is a discrete abelian group then, as in 10.6 [3], $T_n(F(S^1);G)$ is an Eilenberg-MacLane space $K(G, n)$. With the understanding that $K(G, n)$ is represented by $T_n(F(S^1);G)$ one has:

5.3. Corollary. *For G a discrete abelian group,*

$$K(G, n) = (\otimes_n K(Z, 1)) \otimes G.$$

5.4. *Remark.* Since S^1 is a topological group, it is not difficult to see that $F(S^1)$ can be identified with the group ring, $Z(S^1)$, of S^1. Thus $T_n(F_B(S^1))$ is the n-fold tensor product of group ring bundles. Note, however, that the multiplication of $F(S^1)$ is homotopically trivial in view of the homotopy type of the spaces involved.

The above results can be extended to include more general coefficients (see [6]). If, for example, Λ is a compactly generated, commutative ring bundle with unit then for any compactly generated Λ-module bundle μ, $F(\xi) \otimes \mu$ ($\otimes$ over Z_B) has an obvious Λ-module bundle structure. In particular if μ is LNDR, $T(F_B(S^1);\mu)$ is a graded Λ-module bundle and is an Ω-spectrum on P_B for cohomology with coefficients in μ ($H^n(\xi;\mu)$ is now a module over the ring Hom (ξ, Λ) (see §4 [6]). Further, if Λ is LNDR then $T(F_B(S^1);\Lambda)$ is a graded Λ-algebra. As in 11.11 [3] one sees that the Z_B-algebra $T = T(F_B(S^1))$ is homotopy commutative (π_{mn} and $\pi_{nm} \circ \tau$ are homotopic where $\pi_{mn} : T_m \otimes T_n \to T_{m+n}$ is the product (isomorphism) in T and $\tau : T_m \otimes T_n \to T_n \otimes T_m$ is the twist map $\tau(x \otimes y) = (-1)^{mn} y \otimes x$) and consequently $T(F_B(S^1);\Lambda)$ is a homotopy commutative graded Λ-algebra. Summing up gives:

5.5. Theorem. (a) *If μ is an LNDR Λ-module bundle then the graded Λ-module $T(F_B(S^1);\mu)$ is an Ω-spectrum for $H^*(-;\mu)$ on P_B.*

(b) *If Λ is LNDR then $T(F_B(S^1);\Lambda)$ is a homotopy commutative Λ-algebra and $H^*(-;\Lambda)$ has the structure of a commutative* Hom $(-, \Lambda)$*-algebra.*

In view of 5.4 [6], $H^*(-;\mu)$ has the following interpretation:

5.6. COROLLARY. *For $\xi \in P_B$ and μ an LNDR Λ-module bundle, $H^n(\xi;\mu)$ is in bijection with the set of isomorphism classes of principal $T_{n-1}(F_B(S^1);\mu)$-bundles on ξ, $n \geq 1$ (here $T_0(F_B(S^1);\mu) = \mu$).*

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